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ON THE CONSISTENCY OF A LABELED GRAPH

Beatrice Daniela Bucur
PhD Student, Department of Computer Science, University of Piteşti, Romania
ciolan_b@yahoo.com

Abstract The aim of this paper is to establish a connection between modal logics and labeled graphs, which is useful in solving the problem of *nondeterminism.* Also, we construct a canonical, maximal and consistent model of a labeled graph associated to a transition system.

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1. INTRODUCTION

The most famous theories about modal logic are based on the model built by Saul Kripke which, in a restricted sense, refers to necessity and possibility.

The semantics of modal logics consists of a non-empty set G, whose elements are called *possible worlds,* a binary relation R between the elements of G called *accessibility relation* and a labeling function which describes every situation. Modal logic makes use of the modal operators $\Box$ (necessary) and $\Diamond$ (possible), see, e.g., [3].

Definition 1 ([3])**.** *A Kripke model is a tuple (S, R, L) where S is a set of states (possible worlds), R an accessibility (transition) relation with $R \subseteq S \times S$ such that $\forall s_1 \in S\ \exists s_2 \in S$ with $(s_1, s_2) \in R$ and $L : S \to 2^{AP}$ a labeling function such that $\forall s \in S$, $L(s)$ represents all the atomic propositions true in s and AP is the set of atomic propositions.*

2. TRANSITION SYSTEMS

Transition systems are concepts used in computer sciences. They consist of states and transitions among them. The set of states can be countable or uncountable, and so can the set of transitions.

Definition 2 ([1])**.** *A transition system ST is a a tuple $(S, A, \to, P, L)$ where S is a set of states, A a set of actions, $\to \subseteq S \times A \times S$ is a transition relation, P a set of atomic propositions and $L : S \to 2^P$ a label function.*

To a transition system we can associate a set of atomic propositions which depend on the properties taken into account. Thus we can obtain a variety of choices which a logical analysis is able to predict. From the point of view of transition mechanisms, the choice is arbitrary.

3. NONDETERMINISM

What is important for modeling transition systems is the *nondeterminism,* which is more than a theoretical concept. It allows for freedom in modeling the computation systems. Intuitively, a transition system begins with an initial state and evolves to another state according to the accessibility relation. If from one state there can be more transitions, then the choice of the next state is nondeterministic. That is, the result of the selection is not a priori known, therefore one can draw no conclusion regarding the probability with which a certain transition is chosen. The same aspect is met also in the case when there is not only one state, but a set of initial states, in which case the nondeterministic factor plays a role. The analysis of these choices is useful in modeling conflict situations which may appear in case of processes executed in parallel, but also in modeling unknown interfaces (see [2]).

In a transition system, the nondeterminism can also represent implementation freedom, because one of the nondeterministic options is chosen while the other ones are ignored. The essence of nondeterminism resides in that accessibility represents, in fact, choices between alternatives.

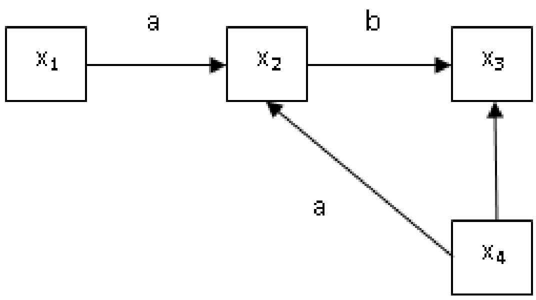

Fig. 1.: Nondeterministic transition system

However, it is useful to take into account the observable, deterministic behavior, related to various observable notions. In this way, the determinism agrees with the executable actions which are observable, or it is related to labels and relies only on the atomic propositions which take place and are observable. To simplify, the atomic propositions of the system are statements on which the binary operation yes/ no acts. If the names of the actions are not relevant, as transition represents an internal process, we can use symbols, or even omit them in certain cases.

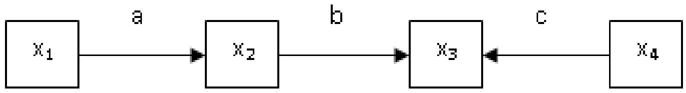

Fig. 2.: Deterministic transition system

4. LABELED GRAPHS

Definition 3 ([4]). *A labeled graph is a tuple $LG = (S, E, T, f)$ where S is a finite set of elements representing the vertices of LG, E is a set of elements used to label the edges of the graph, T is a set of binary relations on S and $f : E \to T$ a surjective function.*

Remark 1. *In the graphical representation of this structure, the vertices are drawn as boxes which contain their names. An edge from $x_i \in S$ to $x_j \in S$ is labeled by $a \in E$ if and only if $(x_i, x_j) \in f(a)$.*

5. LABELED GRAPHS AND TRANSITION SYSTEMS

We notice the following:

1 A labeled graph can be interpreted as a transition system.

2 The states of ST are the vertices of the graph.

3 The actions of ST can be associated with the labels of the graph.

We define the labeled graph associated to a transition system, and denote this by $LG(ST)$, to be a tuple (S, E, T, f), where S is the set of states, E the set of actions, T the set of atomic propositions and $f : E \to T$ a surjective function.

Definition 4. *A model M is a tuple $M = (S, E, T, f)$ where S is a set of states, T is a binary relation and $f : E \mapsto T$ is a function which assigns to an action, $a \in E$, a subset of states from S, namely those states for which a is true.*

Definition 5. *An action is true in a model M if the following conditions are satisfied:*

1 $(M, x_1) \models a \quad \forall x_1 \in f(a)$.

2 $(M, x_1) \models \neg a$ *if and only if* $(M, x_1) \not\models a$.

3 $(M, x_1) \models a \wedge b$ *if and only if* $(M, x_1) \models a$ *and* $(M, x_1) \models b$.

4 $(M, x_1) \models a \vee b$ *if and only if* $(M, x_1) \models a$ *or* $(M, x_1) \models b$.

5 $(M, x_1) \models a \to b$ *if and only if* $(M, x_1) \not\models a$ *or* $(M, x_1) \models b$.

6 $(M, x_1) \vDash \Box a$ *if and only if* $\forall x_2 \in S$ *with* $x_1 T x_2$ *we have* $(M, x_2) \vDash a$.

7 $(M, x_1) \vDash \Diamond a$ *if and only if* $\exists x_2 \in S$ *with* $x_1 T x_2$ *and* $(M, x_2) \vDash a$.

The above definition allows us to introduce the concept of a *valid* action in a model $M = (S, E, T, f)$.

Definition 6. *An action a is valid if it is true for any state* $x_i \in S$*, taht is,* $\vDash_M a$ *if and only if* $(M, x_i) \vDash a \quad \forall x_i \in M$.

Definition 7. *A framework* $\mathcal{C}$ *is a pair* $\mathcal{C} = (S, T)$ *consisting of a set of states and a binary relation on that set. If* $M = (S, T, f) = (\mathcal{C}, f)$*, we say that M is based on the framework* $\mathcal{C}$ *and* $\mathcal{C}$ *is the framework of M.*

Using the concept of framework, we can extend the definition of a valid action, by saying that an action a is valid in a framework $\mathcal{C}$ if $(M, x) \vDash a$ for every model M based on $\mathcal{C}$ and every $x \in M$.

Definition 8. *An action a is* $M-$*provable if it arises as the last element of some* $M-$*valid actions and we denote this by* $\vdash_M a$.

We say that a labeled graph $LG(ST)$ associated to a transition system is *valid with respect to a class K of frameworks* if every provable action is valid. That is, if $P \vdash_L a$ then $P \vDash_{\mathcal{C}} a$ for every $\mathcal{C} \in K$ and if $(M, x) \vDash P$ then $(M, x) \vDash a$ for all $x \in S$. (We used L to denote $LG(ST)$.)

Definition 9. *A labeled graph associated to a transition system is complete if every valid action is provable.*

The notions of correctness and completeness are relative and refer to a certain class of frameworks.

To prove the completeness, we will construct, for every L, a canonical model with the property of "validating" all provable actions. The existence of such a model is a consequence of the fact that L is complete with regards to its model.

Definition 10. *An action a is consistent if* $\nvdash_L \neg a$.

Definition 11. *A set is* $L-$*consistent if there exists no* $P' \subseteq P,\ P' = \{a_1 \ldots a_n\}$ *such that* $\vdash_L \neg(a_1 \wedge \ldots \wedge a_n)$.

Definition 12. *A set of atomic propositions P is* $L-$*consistent if* $P \vdash_L a$.

Definition 13. *A set P is* $L-$*maximal if* $P \cup \{a\}$ *is* $L-$*consistent for every* $a \notin P$.

Starting with an $L-$consistent set, Lindenbaum's lemma will allow us to find a maximal $L-$consistent set. According to this lemma, if P is a set of atomic propositions, then there is a set R such that R is maximal, $L-$consistent and $P \subseteq R$.

Rule: If $\vdash_L a \to b$ and $\vdash_L a$ then $\vdash_L b$.

Proposition 1. *If $a \notin P$ then $P \cup \{a\} \vdash_L b$.*

Proof. From $P \vdash_L a \to b$, since $\vdash_L \neg a \Rightarrow P \vdash_L \neg a \Rightarrow \neg a \in P$, i.e., every maximal $L-$consistent set of actions is complete with regards to negation. ∎

Remark 2. *For every set of actions R from L, if R is $L-$consistent, then there is an interpretation of every action such that this is true relative to that interpretation.*

We are going to construct a chain of sets whose union is the maximal set. We define recursively $P_0, P_1, P_2, \ldots$ as follows: $P_0 = R$ and, for $n \in \mathbb{N}$,

1 $P_{n+1} = P_n$ if $P_n \cup \{a_{n+1}\}$ is not $L-$consistent;

2 $P_{n+1} = P_n \cup \{a_{n+1}\}$ if $P_n \cup \{a_{n+1}\}$ is $L-$consistent.

Therefore, if P_n is $L-$consistent, then so is P_{n+1}.

Proposition 2. *Let $P = \cup_{n \in \mathbb{N}} P_n$. Then P is a maximal $L-$consistent set.*

Proof. By sake of contradiction, we prove the second claim. Assuming the contrary, it means that there exists a finite set $P' \subseteq P$ with $P' \vdash_L a$. As P' is finite and $P' \subseteq P$, it means that $\exists P_k$ so that $P' \subseteq P_k$. But then $P_k \vdash_L a$, hence P_k is not $L-$consistent, contradiction.

To prove the first claim, note that, obviously, by adding a new action to P, either P becomes $L-$inconsistent or it will equal P_n (using the previous construction). This concludes the proof. ∎

6. CONCLUSIONS

As such we constructed a canonical model $C = (S, T, f)$ with the property that it "validates" all the provable actions, with the aim of proving the completeness of a labeled graph associated to a transition system.

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