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# $C$-METRIC SPACES

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**Abstract** We introduce the concept of a $C$-metric space and investigate some of its topological properties and the existence of fixed points for a contraction on a $C$-metric space. We also prove that a $C$-metric on a semi-linear space induces a semi-linear topology.

**Keywords:** $C$-metric, approximate metric, topology, fixed point, semi-linear space, semi-linear topology.
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## 1. INTRODUCTION

The theory of approximate metrics has important applications in problems of fixed points for single and multi-valued operators, variational and linear inequalities, optimization, approximation theory, category problems, cooperative game theory with non-transferable utility (e.g., [13], [14], [15], [16], [21], [23], [25], [26]).

In this paper we introduce the concept of a $C$-metric space and investigate some of its topological properties and the existence of fixed points for a contraction on a $C$-metric space. We also prove that a $C$-metric on a semi-linear space induces a semi-linear topology (semi-linear spaces or almost linear spaces have proved their usefulness in many theoretical and applied areas, such as computer science, theory of cluster validation, theory of best approximation, imaging processes (as in [1], [3], [6], [7], [10], [11], [12], [17], [19], [20], [22], [24], for instance).

The structure of the paper is the following: in Section 2 we present some concepts and preliminary results. In Section 3 we introduce the notion of a $C$-metric space and present some examples, topological properties and a fixed point theorem. We also prove that a $C$-metric on a semi-linear space induces a semi-linear topology. The final section contains the conclusions.

## 2. PRELIMINARIES

We denote $\mathbb{N}^* = \{1, 2, 3, ...\}$ and $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. For a non-empty set $X$, $\mathcal{P}_0(X)$ is the family of all non-empty subsets of $X$ and $\mathcal{S}(X)$ is the family of all singletons of $X$. If $X$ is a real linear space, then for every $E, F \in \mathcal{P}_0(X)$ and $\alpha \in \mathbb{R}$, denote $E + F = \{x + y; x \in E, y \in F\}$ and $\alpha E = \{\alpha x; x \in E\}$.

**Definition 2.1.** ([9]) Let $X$ be a real linear space and let $\mathcal{A}$ be a subset of $\mathcal{P}_0(X)$. A function $|\cdot| : \mathcal{A} \to [0, +\infty]$ is called a set-norm on $\mathcal{A}$ if:

(i) $|E| = 0 \Leftrightarrow E = \{0\}$, for $E \in \mathcal{A}$;
(ii) $|\alpha E| = |\alpha| \cdot |E|$, $\forall \alpha \in \mathbb{R}$, $\forall E \in \mathcal{A}$ (with the convention $0 \cdot (+\infty) = 0$);
(iii) $|E + F| \leq |E| + |F|$, $\forall E, F \in \mathcal{A}$.

**Example 2.2.** I. If $(X, \|\cdot\|)$ is a normed space, then the function defined by $|E| = \sup_{x \in E} \|x\|$ for every $E \in \mathcal{P}_0(X)$ is a set-norm on $(\mathcal{P}_0(X), \subseteq)$.

II. Let $(X, \|\cdot\|)$ be a real normed space, $\alpha \in \mathbb{R} \setminus \{0\}$ and the function defined for every $E \in \mathcal{P}_0(X)$ by

$$|E| = \begin{cases} |\alpha| \max_{x \in E} \|x\|, & \text{if } E \text{ is finite} \\ +\infty, & \text{if } E \text{ is not finite.} \end{cases}$$

Then the function $|\cdot|$ is a set-norm on $\mathcal{P}_0(X)$ (with the convention $0 \cdot (+\infty) = 0$).

In the sequel, let $T$ be a non-empty set, $\mathcal{P}(T)$ the family of all subsets of $T$ and $\mathcal{C} \subseteq \mathcal{P}(T)$ a ring of subsets of $T$ (i.e. $\mathcal{C} \neq \emptyset$, $A \cup B \in \mathcal{C}$ and $A \setminus B \in \mathcal{C}$ for every $A, B \in \mathcal{C}$).

**Definition 2.3**. If $T, Y$ are non-empty sets, then we call a set-valued function (also called a multifunction) from $T$ to $Y$ a function $F : T \to \mathcal{P}_0(Y)$.

**Definition 2.4**. ([2]) Let $L$ be a non-empty set. We say that $L$ is *a semi-linear space* if it is endowed with two operations, sum and multiplication by scalars from a field $\Gamma$ :

$$" + " \ : \ L \times L \to L$$

and

$$" \cdot " \ : \ \Gamma \times L \ \to L$$

which verify the following axioms :
(S1) $(x + y) + z = x + (y + z)$ , $\forall x, y, z \in L$;
(S2) $\exists\, \theta \in L$ such that $x + \theta = \theta + x = x$, $\forall x \in L$ ($\theta$ is called the origin of $L$);
(S3) $x + y = y \ + \ x$ , $\forall x, y \in L$;
(S4) $\lambda\,(\mu x) = (\lambda\mu)\, x$ , $\forall \lambda, \mu \in \Gamma$, $\forall x \in L$;
(S5) $1 \cdot x = x$ , $\forall\ x \in L$;
(S6) $\lambda\,(x + y) = \lambda x + \lambda y$, $\forall \lambda \in \Gamma$, $\forall\ x, y \in L$;
(S7) $0 \cdot x = \theta$, $\forall x \in L$.

**Definition 2.5**. ([4]) Let $L$ be a semi-linear-space. An application $\|\cdot\| \ : \ L \to [0, +\infty)$ is called *semi-linear norm* if it satisfies the usual axioms of a norm on a linear space.

**Definition 2.6**. ([2]) We say that a topology $\tau$ on a semi-linear space $(L, +, \cdot)$ is *a semi-linear topology on* $L$ if the operations "+" : $(\ L, \tau) \times (L, \tau) \ \to \ (L, \tau)$ and "$\cdot$" : $(\ \Gamma, |\cdot|) \times (L, \tau) \ \to (L, \tau)$ are both continuous. As well $(L, +, \cdot, \tau)$ is called *a semi-linear topological space*.

An important difference between linear and semi-linear topological spaces is that the translations are not necessarily invertible in second type of spaces. Because the elements of $(L, +)$ have not generally a symmetric, it is possible that the family $-x + \mathcal{V}(x)$ be not a system of neighborhoods of the origin for any $x \in L$ and any system of neighborhoods $\mathcal{V}(x)$ of $x$. Also $x + \mathcal{V}(\theta)$ is not a system of neighborhoods of $x$ in a semi-linear topology.

Denote by

$$\mathcal{U}(x) = \{U \subset L; \exists V \in \mathcal{V}(\theta) \text{ such that } x + V \subset U\}. \tag{1}$$

**Theorem 2.7.** ([2]) Let $(L, +, \cdot)$ be a semi-linear space, $I$ a set of indices and a collection $\{V_i; i \in I\} \subset \mathcal{P}_0(L)$.

(i) If $\{V_i; i \in I\}$ is a fundamental system of neighborhoods of the origin for a semi-linear topology $\tau$ then it satisfies the following axioms:

(V0) $\theta \in V_i$, $\forall i \in I$.

(V1) $\forall i_1, i_2 \in I$, $\exists i_3 \in I$ such that $V_{i_3} \subseteq V_{i_1} \cap V_{i_2}$.

(V2) $\forall i \in I$, $\exists j \in I$ such that $V_j + V_j \subseteq V_i$.

(V3) $\forall i \in I$, $\forall x \in L$, there exists $\delta > 0$ such that $\alpha x \in V_i$ for all $\alpha \in \mathbb{R}$ with $|\alpha| < \delta$.

(V4) $\forall i \in I$, $\exists j \in I$ such that $\bigcup_{|\alpha| \leq 1} \alpha V_j \subseteq V_i$.

(ii) If $\{V_i; i \in I\}$ satisfies (V0), (V1) and (V2) then there exists a topology $\sigma$ on $L$ such that the collection $\{U \subset L; \exists i \in I \text{ such that } x + V_i \subset U\}$ is a fundamental system of neighborhoods of $x$ in the topology $\sigma$.

## 3. C-METRIC SPACES

In this section we introduce the concept of a $C$-metric and present some of its topological properties and the existence of fixed points for a contraction on a $C$-metric space. We also prove that a $C$-metric on a semi-linear space induces a semi-linear topology.

We now introduce the following notion.

**Definition 3.1.** Let $X$ be a non-empty set. A function $d : X \times X \to \overline{\mathbb{R}}$ is called *an approximate metric* if it satisfies only a part of the metric axioms. The pair $(X, d)$ is called *an approximate metric space*.

For example, pseudometrics, generalized metrics (in the sense of Luxemburg [18], Lawvere [15] and Branciari [8]), quasimetrics and ultrametrics are approximate metrics.

In [5] we have introduced and studied another type of approximate metric, namely *A-metric*.

**Definition 3.2.** [5] Let $X$ be a non-empty set. An application $d : X \times X \to \overline{\mathbb{R}}$ is called *an A-metric* if it satisfies the following axiom:

$d(x,z) \leq d(x,y) + d(y,z), \quad \forall x,y,z \in X,$
if the addition in the right side of the above inequality makes sense.
The pair $(X,d)$ is called *an A-metric space*.

In the sequel we define and study another type of approximate metric, namely *C-metric.*

**Definition 3.3.** Let $X$ be a non-empty set. An application $d : X \times X \to \overline{\mathbb{R}}$ is called *a C-metric* if it satisfies the following properties:
(i) $d(x,y) = d(y,x), \quad \forall x,y \in X.$
(ii) $d(x,z) \leq d(x,y) + d(y,z), \quad \forall x,y,z \in X,$
if the addition in the right side of the above inequality makes sense.
The pair $(X,d)$ is called *a C-metric space*.

A $C$-metric does not generally satisfy the axioms:
(iii) $d(x,x) = 0, \forall x \in X;$
(iv) $d(x,y) = 0 \Longrightarrow x = y$ (in this case, $d$ and $(X,d)$ are called separated).
So, in a $C$-metric space we can have $d(x,x) \neq 0$ for some $x \in X$.

There exist $C$-metrics that are not metrics (see the next examples).

**Example 3.4.** I. Let $(X,\rho)$ be a metric space and let $d$ be defined by $d(A,B) = \sup_{\substack{x\in A\\ y\in B}} \rho(x,y)$, for every $A,B \in \mathcal{P}(\mathcal{X})$. Then $d$ is a $C$-metric on $\mathcal{P}(\mathcal{X})$. If $A \in \mathcal{S}(\mathcal{X})$, then $d(A,A) = 0$. If $\text{card}A \geq 2$, then $d(A,A) > 0$.

II. Let $C^+([0,1]) = \{f | f : [0,1] \to [0,+\infty), f \text{ continuous}\}$ and let $d$ be defined by

$$d(f,g) = \int_0^1 \max\{f(x), g(x)\}dx, \quad \forall f,g \in C^+([0,1]).$$

Then $d$ is a $C$-metric on $C^+([0,1])$, with $d(0,0) = 0$, that does not satisfy axiom (iii). For any constant function $f(x) = a \in (0,+\infty)$ for every $x \in [0,1]$, it holds $d(f,f) = a > 0$.

III. Let $X = \{a,b,c\}$ and let $d$ be defined by $d(a,a) = d(b,b) = d(c,c) = d(a,b) = d(b,a) = 0$, $d(a,c) = d(c,a) = d(b,c) = d(c,b) = 2$. Then $d$ is a $C$-metric on $X$ that does not satisfy axiom (iv).

**Definition 3.5.** Let $(X,d)$ be a $C$-metric space. Consider the family $\tau_d = \{\emptyset\} \cup \{D \subseteq X | D \neq \emptyset$ and there is $r \in (0,+\infty)$ such that $S(x,r) \subseteq D, \forall x \in D\}$, where $S(x,r) = \{y \in X | d(y,x) < r\} \cup \{x\}$.

**Theorem 3.6.** *Let $(X,d)$ be a C-metric space such that $d : X \times X \to [0,+\infty]$. Then the following statements hold:*

*I. $\tau_d$ is a topology on X, named the topology induced by d (for each $x \in X$, denote by $\mathcal{V}(\S)$ the family of all neighborhoods of x). Moreover, if d has real values, then $\tau_d$ is Hausdorff if and only if d is separated.*

*II. Let $V \subseteq X$ and $x \in X$. Then $V \in \mathcal{V}(\S)$ if and only if there exists $r \in (0, \infty)$ such that $S(x, r) \subseteq V$.*

*III. For every $x \in X$, it hold:*

*(III.1) $x \in V$, for every $V \in \mathcal{V}(\S)$;*

*(III.2) $V \in \mathcal{V}(\S)$, $U \supseteq V \Rightarrow U \in \mathcal{V}(\S)$;*

*(III.3) $V_1 \cap V_2 \in \mathcal{V}(\S)$, for every $V_1, V_2 \in \mathcal{V}(\S)$;*

*(III.4) for every $V \in \mathcal{V}(\S)$, there exists $W \in \mathcal{V}(\S)$ such that $V \in \mathcal{V}()$, for all $y \in W$.*

*Proof.* I. By Definition 3.5, it easily results that $\tau_d$ is a topology on $X$. Suppose now $d$ is separated. Let $x, y \in X$, $x \neq y$, $D_1 = S(x, \frac{r}{3})$, $D_2 = S(y, \frac{r}{3})$, where $r = d(x, y) \in (0, +\infty)$. Suppose by contrary that there exists $z \in D_1 \cap D_2$. There exist three situations:

(a) $z = x \neq y \Rightarrow r = d(x, y) = d(z, y) < \frac{r}{3}$, false.

(b) $z = y \neq x \Rightarrow$ the same argument as in (a).

(c) $z \neq x, z \neq y$. According to axiom (ii) it results $r = d(x, y) \leq d(x, z) + d(z, y) < \frac{2r}{3}$, false.

It follows $D_1 \cap D_2 = \emptyset$ and consequently $\tau_d$ is Hausdorff.

Now, suppose $\tau_d$ is Hausdorff and suppose by contrary that there exist $x, y \in X$, $x \neq y$ such that $d(x, y) = 0$. We observe that for every $D_1, D_2 \in \tau_d$ with $x \in D_1$, $y \in D_2$, it follows $x, y \in D_1 \cap D_2$, which contradicts the fact that $\tau_d$ is Hausdorff.

II is obvious and III is demonstrated as in the classical case. ∎

**Remark 3.7.** I. Consider $d$ from Example 3.4-I. For every $A, B \in \mathcal{P}(\mathcal{X})$ with $d(A, B) = 0$, it folows $A = B \in \mathcal{S}(\mathcal{X})$. So $\tau_d$ is Hausdorff.

II. $d$ from Example 3.4-II satisfies axiom (iv), so $\tau_d$ is a Hausdorff topology.

III. $d$ from Example 3.4-III is not separated. Consequently, $\tau_d$ is not Hausdorff.

The following notions are introduced in a standard way.

**Definition 3.8.** Let $(X, d)$ be a $C$-metric space.

I. A sequence $(x_n)_{n \in \mathbb{N}}$ in $X$ is called *d-convergent* in $X$ if there exists $a \in X$ such that $\lim\limits_{n \to \infty} d(x_n, a) = 0$, that is for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, it holds $|d(x_n, a)| < \varepsilon$ (denoted by $x_n \overset{d}{\to} a$). The set of all such points $a$ will be denoted by $\lim\limits_{n \to \infty} x_n$. If $\lim\limits_{n \to \infty} x_n$ is a singleton $\{a\}$, then we write $\lim\limits_{n \to \infty} x_n = \{a\}$ as $\lim\limits_{n \to \infty} x_n = a$. If the set $\lim\limits_{n \to \infty} x_n$ is empty, then $(x_n)_{n \in \mathbb{N}}$ is called $d$-divergent.

II. A sequence $(x_n)_{n \in \mathbb{N}}$ in $X$ is called *a d-Cauchy sequence* if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|d(x_n, x_m)| < \varepsilon$ for every $n, m \geq n_0$.

III. The $C$-metric space $(X, d)$ is said to be *d-complete* if every $d$-Cauchy sequence in $X$ is d-convergent in $X$.

IV. An application $f : X \to X$ is said to be a *d-contraction* on $X$ if there exists $L \in (0, 1)$ such that $d(f(x), f(y)) \leq Ld(x, y)$, for all $x, y \in X$.

V. An element $x \in X$ is called *a C-fixed point* of a function $f : X \to X$ if $d(f(x), x) = 0$.

**Example 3.9.** Consider $X = C^+([0,1])$. Let $(X, d)$ be the C-metric of Example 3.4-II and let $F : X \to X$ be the function defined for every $f \in X$ by $F(f) = \frac{f}{2}$. Then $F$ is a d-contraction on $X$ (with $L = \frac{1}{2}$).

The following statements result as in the classical case.

**Remark 3.10.** I. If $(X, d)$ is a separated $C$-metric space and the sequence $(x_n)_{n\in\mathbb{N}}$ in $X$ is $d$-convergent in $X$, then $\lim\limits_{n\to\infty} x_n \in \mathcal{S}(\mathcal{X})$.

II. Every $d$-convergent sequence in a $C$-metric space is $d$-Cauchy.

**Example 3.11.** I. Consider $(X, d)$ to be the $C$-metric space of Example 3.4-II. Let $(f_n)_{n\in\mathbb{N}^*} \subset C^+([0,1])$ and $f \in C^+([0,1])$ be defined by $f_n(x) = \frac{x}{n}$, $f(x) = 0$, $\forall x \in [0,1]$, $n \in \mathbb{N}^*$. Then $\lim\limits_{n\to\infty} f_n = f$.

II. Let $(X, d)$ be the $C$-metric space of Example 3.4-III and let $(x_n)_{n\in\mathbb{N}} \subset X$ be defined by $x_n = a$ for every $n \in \mathbb{N}$. Then $\lim\limits_{n\to\infty} x_n = \{a, b\}$.

**Theorem 3.12.** *Let $(X, d)$ be a d-complete C-metric space with $d : X \times X \to [0, +\infty)$ and let $f : X \to X$ be a d-contraction of $X$. Then there exists a C-fixed point of $f$. Moreover, if $(X, d)$ is separated, then $f$ has a unique fixed point.*

**Remark 3.13.** Theory of fixed points has important applications in differential equations, theory of operators, approximation problems, stability theory, measure theory etc. In our setting of $C$-metric spaces, Theorem 3.12 is useful in finding solutions of d-contractions (see for instance the nontrivial Example 3.9) in problems of variational and linear inequalities, game theory, optimization, category theory, computer science, imaging processes.

**Theorem 3.14.** *Let $d$ be a C-metric on a semi-linear space $X$ such that $d(0,0) = 0$ and $d(\alpha x, 0) = |\alpha| d(x, 0)$, for every $\alpha \in \mathbb{R}$ and every $x \in X$. Then $\tau_d$ is a semi-linear topology on $X$.*

*Proof.* For every $\varepsilon > 0$, let $V_\varepsilon = \{x \in X | d(x, 0) < \varepsilon\}$. Using the definitions, the following properties hold:

(V0) $0 \in V_\varepsilon, \forall \varepsilon > 0$.

(V1) $\forall \varepsilon_1, \varepsilon_2 > 0, \exists \varepsilon_3 = \min\{\varepsilon_1, \varepsilon_2\} > 0$ such that $V_{\varepsilon_3} \subseteq V_{\varepsilon_1} \cap V_{\varepsilon_2}$.

(V2) $\forall \varepsilon > 0, \exists \mathrm{d} = \varepsilon/2 > 0$ such that $V_{\mathrm{d}} + V_{\mathrm{d}} \subseteq V_\varepsilon$.

(V3) $\forall \varepsilon > 0, \forall x \in X$, $\exists \mathrm{d} > 0$ (arbitrary d if $d(x, 0) = 0$ and $\mathrm{d} < \varepsilon / d(x, 0)$ if $d(x, 0) > 0$ such that $\alpha x \in V_\varepsilon$ for all $|\alpha| < \mathrm{d}$.

(V4) $\forall \varepsilon > 0, \exists \mathrm{d} = \varepsilon > 0$ such that $\bigcup\limits_{|\alpha|\leq 1} \alpha V_{\mathrm{d}} \subseteq V_\varepsilon$. ∎

Let $X$ be a real linear space and $\mathcal{F} = \{F | F : T \to \mathcal{A}\}$ where $\mathcal{A}$ is a subset of $\mathcal{P}_0(X)$ satisfying the following conditions:

$$\begin{aligned} &(a)\,\{0\} \in \mathcal{A};\\ &(b) E_1 + E_2 \in \mathcal{A} \text{ for every } E_1, E_2 \in \mathcal{A};\\ &(c)\alpha E \in \mathcal{A} \text{ for every } \alpha \in \mathbb{R} \text{ and } E \in \mathcal{A}. \end{aligned} \tag{2}$$

Such subfamily $\mathcal{A} \subset \mathcal{P}_0(X)$ can be $\mathcal{P}_0(X)$ or the family $\mathcal{P}_b(X)$ of all non-empty bounded subsets of $X$ or the family $\mathcal{P}_k(X)$ of all non-empty compact subsets of $X$, where $X$ is a real normed space.

**Remark 3.15**. $\mathcal{F}$ is a semi-linear space with respect to the operations of sum and multiplication by real scalars defined by:
$(F + G)(t) = F(t) + G(t), \forall t \in T$,
$(\alpha F)(t) = \alpha F(t), \forall t \in T, \alpha \in \mathbb{R}$.
The origin is the multifunction $\theta$ defined by $\theta(t) = \{0\}$ for every $t \in T$.
Let $|\cdot|$ be a finite set-norm on $\mathcal{A}$.

**Definition 3.16**. A multifunction $F : T \to \mathcal{A}$ is called sn-bounded if there exists $\alpha \in [0, +\infty)$ such that $|F(t)| \leq \alpha$ for every $t \in T$.

**Example 3.17.** Let $\mathcal{M} = \{F; F : T \to \mathcal{A}, F \text{ is sn-bounded}\} \subseteq \mathcal{F}$. For every $F, G \in \mathcal{M}$ denote

$$\|F\| = \sup_{t \in T} |F(t)| \tag{3}$$

and

$$d(F, G) = \|F - G\|. \tag{4}$$

Then $\mathcal{M}$ is a semi-linear space, the application defined by (3) is a semi-linear norm on $\mathcal{M}$ and $d$ given by (4) satisfies the following properties:
(P1) $d(\theta, \theta) = 0$;
(P2) $d(F, G) \geq 0, \forall F, G \in \mathcal{M}$;
(P3) $d(F, G) = 0 \Longrightarrow F = G$ and $F : T \to \mathcal{S}(X)$;
(P4) $d(F, G) = d(G, F), \forall F, G \in \mathcal{M}$;
(P5) $d(F, H) \leq d(F, G) + d(G, H), \forall F, G, H \in \mathcal{M}$.
It results that $d$ is a $C$-metric and the induced topology $\tau_d$ is a Hausdorff semi-linear topology.

## 4. CONCLUSION

We have introduced the notion of $C$-metric space and investigate some of its topological properties and the existence of fixed points for a contraction on a $C$-metric space. We also demonstrated that a $C$-metric on a semi-linear space induces a semi-linear topology.

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## References

[1] Apreutesei, G. – *Hausdorff topology and some operations with subsets*, An. Şt. Univ. "Al. I. Cuza" Iaşi 44 (1998), 445-454.

[2] Apreutesei, G. – *Cauchy nets and convergent nets on semilinear topological spaces*, Topology and its Applications 159 (2012), 2922-2931.

[3] Apreutesei, G., Apreutesei, N. – *On the continuity of special set-valued functions*, An. Şt. Univ. "Dunărea de Jos" Galaţi 23 (2005), 65-68.

[4] Apreutesei, G. – *The hyperspacial topologies and and almost linear spaces*, An. Şt. Univ. "Al. I. Cuza" Iaşi 48 (2002), 3-18.

[5] Apreutesei, G., Croitoru, A. – *A-Metric spaces*, submitted for publication.

[6] Aseev, S.M. – *Quasilinear operators and their applications in the theory of multivalued mappings* (in Russian), Tr. MIAN SSSR, Nauka Moskva, 167 (1985), 25-52; Proc. Steklov Inst. Math. 2 (1986), 23-52.

[7] Borş, D.M. – *O generalizare a spaţiilor vectoriale (A generalization of vector spaces)*, Bull. Inst. Polit. Iaşi, XV (XIX), s.I, Fasc. 3-4 (1969), 17-20 (in Romanian).

[8] Branciari, A. – *A fixed point theorem of Banach-Caccippoli type on a class of generalized metric spaces*, Publ. Math. Debrecen 571-2 (2000), 31-37.

[9] Croitoru, A. – *Set-norm continuity of set multifunctions*, ROMAI Journal 6 (2010), 47-56.

[10] Croitoru, A., Borş, D.M. – *Remarks on weak linear spaces*, Bulletin of the Polytechnic Institute of Iaşi, Tomul LXI (LXV), Fasc. 2 (2015), 67-74.

[11] Edalat, A., Heckmann, R. – *A computational model for metric spaces*, Theoret. Comput. Sci., 193 (1998), 53-73.

[12] Godini, G. - *A framework for best simultaneous approximation: normed almost linear spaces*, Journal of Approximation Theory 43 (1985), 338-358.

[13] Hart, S. – *An axiomatization of the consistent non-transferable utility value*, Int. J. Game Theory 33 (2005), 355-366.

[14] Laha, A.K., Saha, M. – *Fixed point for some classes of set valued mappings on a metric space endowed with a graph*, ROMAI Journal 11, No. 1 (2015), 115-129.

[15] Lawvere, F. – *Metric spaces, generalized logic and closed categories*, Rend. del Sem. Mat. e Fis. di Milano, 43 (1973), 135-166.

[16] Liu, Y., Li, Z., Xiong, H., Gao, X., Wu, J. - *Understanding of Internal Clustering Validation Measures*, Proceedings of 2010 IEEE International Conference on Data Mining, 911-916, DOI 10.1109/ICDM.2010.35.

[17] Löhne, A. – *On convex functions with values in semi-linear spaces*, Report of the Institute of Optimization and Stochastics 07, Martin-Luther-University Halle-Wittenberg, Department of Mathematics and Computer Science, 2003.

[18] Luxemburg, W.A.J. – *On the convergences of successive approximations in the theory of ordinary differential equations*, Indag. Math., 20 (1958), 540-546.

[19] Markandeya Naidu, T., Bharathi, D. - *Reflexivity and completeness of normed almost linear spaces*, International Journal of Math. and Comp. Research 2 (2014), 703-711.

[20] Mayer, O. – *Algebraische und metrische strukturen in der intervallrechnung und einige anwendungen*, Computing 5 (1970), 144-162.

[21] Pap, E., Stajner, I. – *Generalized pseudo-convolution in the theory of probabilistic metric spaces, information, fuzzy numbers, optimization, system theory*, Fuzzy Sets and Systems 102 (1999), 393-415.

[22] Precupanu, A., Gavriluţ, A., Croitoru, A. – *A fuzzy Gould type integral*, Fuzzy Sets and Systems 161(2010), 661-680.

[23] Precupanu, T. – *Scalar minimax properties in vectorial optimization*, International Series of Numerical Mathematics, Birkhauser Verlag Basel 107 (1992), 299-306.

[24] Stamate, C., Croitoru, A. – *Convergence results for sequences of non-linear integrals*, Recent Advances In Computers Engineering Series 15 (2013), 76-81.

[25] Turinici, M. – *Functional contractions in local Branciani metric spaces*, ROMAI Journal 8, 2 (2012), 189-199.

[26] Vichkers, S. – *Localic completion of quasimetrics spaces*, Technical Report Doc 97/2, Department of Computing, Imperial College, London, 1998.