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A NONLINEAR NONLOCAL PROBLEM FOR A HYPERBOLIC EQUATION

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Abstract In this paper we study the nonlocal problem for a hyperbolic equation with nonlinearities in boundary conditions. Nonlinear conditions of this type appear when modeling the movement of a vibrating string with elastic fastening of the end which is not subject to Hooke's law. The existence and uniqueness of a generalized solution is proved. We use apriori estimates to derive energy inequalities to prove the uniqueness part. Faedo-Galerkin approximations are constructed and their convergence is established to show the existence of the generalized solution.

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1. INTRODUCTION

Problems with non-classical boundary conditions for evolution equations describe a large number of processes in thermoelasticity, plasma physics and chemical engineering. A class of problems with nonlocal integral terms in boundary conditions, so-called energy specifications, is of special research interest. Starting from the pioneering articles of Cannon [1] and Kamynin [2], a large number of papers have been devoted to the study of initial boundary value problems with nonlocal integral conditions for evolution equations. For a read-up, see [3]-[8].

Boundary value problems with nonlinear boundary conditions for PDEs have been considered by different authors [9]-[11]. We shall focus on papers devoted to problems with special type of nonlinear terms in boundary conditions. In [12], the initial boundary value problem for the wave equation with the nonlinear second-order dissipative boundary condition $\left(\frac{\partial u}{\partial \nu} + K(u)u_{tt} + |u_t|^\rho u_t\right)\Big|_\Sigma = 0$ was investigated and the existence of global generalized solutions was proved. In [13], the global solvability of the problem for the nonlinear damped wave equation with a purely nonlinear boundary condition $(u_x + K(u)u_{tt} + |u_t|^\rho u_t)(1, t) = 0$ has been established. We also mention the paper [14] devoted to the parabolic problem with a combination of nonlocal integral and nonlinear boundary conditions which arises from the modeling of a particular type of reaction-diffusion system.

In the present work, we develop further the ideas of [6], [12] and study a nonlocal hyperbolic problem with boundary conditions that include both nonlinear and nonlocal integral components. Motivated by [12], [13] we consider nonlinearity in the boundary condition of the form $|u|^p u$ that describes the case of modeling the movement of a vibrating string when the elastic fastening of the end is not subject to Hooke's law. The main aim of the paper is to establish the existence and uniqueness of the generalized solution. We note that the presence of the integral term in the nonlinear condition may complicate the applications of classical techniques [15].
The paper is organized as follows. In the section 2 we set a problem, assumptions and introduce the notion of a generalized solution. The existence and uniqueness theorem is established in the section 3.

2. PRELIMINARIES

In the domain $Q_T = \{(x,t) : 0 < x < l,\ 0 < t < T\}$ consider the equation

$$u_{tt} - (a(x,t)u_x)_x + c(x,t)u = f(x,t) \tag{1}$$

with the initial conditions

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x) \tag{2}$$

and the boundary conditions

$$u_x(0,t) = 0, \tag{3}$$

$$a(l,t)u_x(l,t) + |u(l,t)|^p u(l,t) = \int_0^l K(x,t)u(x,t)\,dx, \ \ p > 0. \tag{4}$$

In this paper, we shall assume that the following conditions are satisfied:

(A1) $f(x,t) \in L_2(Q_T)$, $\psi(x) \in L_2(0,l)$;

(A2) $c(x,t)$, $a(x,t)$, $a_t(x,t) \in C(\overline{Q}_T)$

(A3) $\varphi(x) \in W_2^1(0,l) \cap L_{p+2}(0,l)$, $\varphi'(0) = 0$,

$$a(l,t)\varphi'(l) + |\varphi(l)|^p\varphi(l) = \int_0^l K(x,0)\varphi(x)\,dx;$$

(A4) $K(x,t) \in C^1(\overline{Q}_T)$.

We introduce appropriate functional spaces that shall be used in the rest of the paper. Let W_2^1 be the usual Sobolev space. We define

$$W(Q_T) = \{u\colon u \in W_2^1(Q_T) \cap L_{p+2}(\Gamma_l)\},$$

$$\widehat{W}(Q_T) = \{v\colon v \in W(Q_T), v(x,T) = 0\},$$

$$\|u\|_{W(Q_T)} = \|u\|_{W_2^1(Q_T)} + \|u\|_{L_{p+2}(\Gamma_l)},$$

where $\Gamma_l = \{(x,t)\colon\ x = l,\ t \in (0,T)\}$. To define a solution to the problem (34)-(4), we shall apply the standard procedure ([16], p.47).

Definition 2.1. *A function $u(x,t) \in W(Q_T)$ is said to be a generalized solution to the problem (34)-(4) provided it satisfies the initial condition (2) and for any function $\eta(x,t) \in \widehat{W}(Q_T)$ the following identity holds*

$$\int_{Q_T} (-u_t\eta_t + a(x,t)u_x\eta_x + c(x,t)u\eta)\, dx\, dt+$$
$$+\int_0^T |u(l,t)|^p u(l,t)\eta(l,t)\, dt = \int_0^l \psi(x)\eta(x,0)\, dx+$$
$$+\int_{Q_T} f\eta\, dx\, dt + \int_0^T \left(\int_0^l Ku\, dx\right)\eta(l,t)\, dt. \tag{5}$$

3. MAIN RESULT

In this section, we prove the existence and uniqueness of a generalized solution exploiting apriori estimates, Faedo-Galerkin approximations and compactness arguments.

Theorem 3.1. *Let the assumptions (A1)-(A4) hold. Then there exists a unique generalized solution to the problem (34)-(4).*

Proof. **A. Uniqueness.** Assume that the problem (34)-(4) has two different solutions $u_1(x,t)$ and $u_2(x,t)$. Then the function $u = u_1 - u_2$ is an element of $W(Q_T)$ which satisfies the condition $u(x,0) = 0$ and the identity

$$\int_{Q_T} (-u_t\eta_t + a(x,t)u_x\eta_x + c(x,t)u\eta)\, dx\, dt+$$
$$+\int_0^T (|u_1(l,t)|^p u_1(l,t) - |u_2(l,t)|^p u_2(l,t))\, \eta(l,t)\, dt =$$
$$= \int_0^T \left(\int_0^l Ku\, dx\right)\eta(l,t)\, dt. \tag{6}$$

For an arbitrary $\tau \in [0,T]$, we can choose η in (6) as

$$\eta(x,t) = \begin{cases} 0, & t \in [\tau, T], \\ -\displaystyle\int_t^\tau u(x,b)\, db, & t \in [0,\tau]. \end{cases} \tag{7}$$

Using integration by parts into the first two terms of (6) we obtain

$$\frac{1}{2}\int_0^l \left(u^2(x,\tau) + a(x,0)\eta_x^2(x,0)\right) dx-$$

$$-\int_0^\tau \left(|u_1(l,t)|^p u_1(l,t) - |u_2(l,t)|^p u_2(l,t)\right) \eta(l,t)\,dt =$$

$$= \int_{Q_\tau} c(x,t)\eta_t\eta\,dx\,dt - \int_0^\tau \left(\int_0^l K(x,t)u\,dx\right)\eta(l,t)\,dt -$$

$$-\frac{1}{2}\int_{Q_\tau} a_t(x,t)\eta_x^2\,dx\,dt. \tag{8}$$

Consider the term

$$-\int_0^\tau \left(|u_1(l,t)|^p u_1(l,t) - |u_2(l,t)|^p u_2(l,t)\right) \eta(l,t)\,dt =$$

$$= \int_0^\tau \left(|u_1|^p u_1 - |u_2|^p u_2\right)|_{\Gamma_l} \int_t^\tau u(l,b)\,db\,dt.$$

Note that $\frac{d}{du}|u|^p u = (1+p)|u|^p \geq 0$ for $p > 0$. And hence,

$$\left(|u_1|^p u_1 - |u_2|^p u_2\right)(u_1 - u_2) \geq 0.$$

Therefore,

$$\int_0^\tau \left(|u_1|^p u_1 - |u_2|^p u_2\right)|_{\Gamma_l} \int_t^\tau u(l,b)\,db\,dt > 0.$$

It means that the left-hand side of (8) is positive.
The next step is to obtain estimates for the right-hand side of (8). The assumptions (A2), (A4) imply that there exist constants $c_0 > 0$, $a_1 > 0, k_1 > 0$ such that

$$\max_{\overline{Q}_T} |c(x,t)| \leq c_0, \quad \max_{\overline{Q}_T} |a_t(x,t)| \leq a_1 \tag{9}$$

and

$$k_1 = \max_{\overline{Q}_T} \int_0^l K^2(x,t)\,dx. \tag{10}$$

We also note that there exists a positive constant a_0 such that $a(x,t) > a_0$ for all $(x,t) \in Q_T$.
Applying Young's inequality and (9) to the first term on the right-hand side of (8) we obtain

$$\left|\int_{Q_\tau} c\eta_t\eta\,dx\,dt\right| \leq \frac{c_0}{2}\int_{Q_\tau} (\eta^2 + \eta_t^2)\,dx\,dt. \tag{11}$$

From the representation (7) it follows that $\eta_t = u$ and

$$\int_0^\tau \eta^2(x,t)\,dt \leq \tau^2 \int_0^\tau u^2(x,t)\,dt.$$

Therefore, the estimate (11) becomes

$$\left|\int_{Q_\tau} c\eta_t\eta\,dx\,dt\right| \le \frac{c_0}{2}(1+\tau^2)\int_{Q_\tau} u^2\,dx\,dt. \tag{12}$$

The estimate (9) implies that

$$\left|\frac{1}{2}\int_{Q_\tau} a_t(x,t)\eta_x^2\,dx\,dt\right| \le \frac{a_1}{2}\int_{Q_\tau}\eta_x^2\,dx\,dt. \tag{13}$$

Consider the second term on the right-hand side of (8). Using (10) and the Cauchy inequality we obtain

$$\left|\int_0^\tau \eta(l,t)\int_0^l K(x,t)u\,dx\,dt\right| \le$$

$$\le \frac{1}{2}\int_0^\tau \eta^2(l,t)\,dt + \frac{k_1}{2}\int_0^\tau\int_0^l u^2\,dx\,dt. \tag{14}$$

It is easy to see that the representation

$$\eta(l,t) = \int_x^l \eta_\xi\,d\xi + \eta(x,t)$$

implies the following estimate

$$\eta^2(l,t) \le 2l\int_0^l \eta_x^2\,dx + \frac{2}{l}\int_0^l \eta^2\,dx. \tag{15}$$

Therefore, we obtain by (8), (12), (13), (14), (15) that

$$\frac{1}{2}\int_0^l \left(u^2(x,\tau) + a_0\eta_x^2(x,0)\right)\,dx +$$

$$+\int_0^\tau (|u_1|^p u_1 - |u_2|^p u_2)\,|_{\Gamma_l}\int_t^\tau u(l,b)\,db\,dt \le$$

$$\le C_1\int_0^\tau\int_0^l u^2(x,t)\,dx\,dt + C_2\int_0^\tau\int_0^l \eta_x^2\,dx\,dt, \tag{16}$$

where

$$C_1 = \frac{c_0}{2}(1+\tau^2) + \frac{k_1}{2} + \frac{\tau^2}{l}, \quad C_2 = 2l + \frac{a_1}{2}.$$

Define the function $v(x,t) = \int_0^t u_x(x,b)\,db$. Using the representation (7) we obtain $\eta_x(x,t) = v(x,t) - v(x,\tau)$ and $\eta_x(x,0) = -v(x,\tau)$. Therefore, (16) implies that

$$\frac{1}{2}\int_0^l \left(u^2(x,\tau) + a_0 v^2(x,\tau)\right)\,dx \le$$

$$\leq C_1 \int_0^\tau \int_0^l u^2(x,t)\,dx\,dt + 2C_2\tau \int_0^l v^2(x,\tau)\,dx$$

$$+2C_2 \int_0^\tau \int_0^l v^2(x,t)\,dx\,dt. \tag{17}$$

By the arbitrary choice of τ, let τ be such that the inequality $a_0 - 4C_2\tau \geq \frac{a_0}{2}$ holds. Then for all $\tau \in [0, \frac{c_0}{8C_2}]$ we have

$$\int_0^l \left(u^2(x,\tau) + v^2(x,\tau)\right) dx \leq$$

$$\leq C_3 \int_0^\tau \int_0^l \left(u^2(x,t) + v^2(x,t)\right) dx\,dt,$$

where $C_3 = \dfrac{\max\{2C_1, 4C_2\}}{\min\{1, a_0 - 4C_2\tau\}}$.
Applying Gronwall's lemma we obtain that $u(x,t) = 0$ for $t \in [0, \frac{c_0}{8C_2}]$.
Next, we repeat the above arguments for $t \in [\frac{c_0}{8C_2}, \frac{c_0}{4C_2}]$ and then continue this procedure. It follows that $u(x,t) = 0$ for $t \in [0,T]$ and hence, $u_1 = u_2$ in Q_T.□
B. Existence. Consider functions $\{\rho_k\}$ such that $\rho_k \in C^2[0,l]$, $\rho_k'(0) = 0$ and $\{\rho_k\}$ is a basis in $W_2^1(0,l) \cap L_{p+2}(0,l)$. We define approximations

$$u^N(x,t) = \sum_{i=1}^N c_i(t)\rho_i(x), \tag{18}$$

where $c_i(t)$ are solutions to the Cauchy problem

$$\int_0^l \left(u_{tt}^N\rho_i + a(x,t)u_x^N\rho_i' + c(x,t)u^N\rho_i\right) dx+$$

$$+|u^N(l,t)|^p u^N(l,t)\rho_i(l) = \int_0^l f(x,t)\rho_i(x)\,dx+$$

$$+\rho_i(l)\int_0^l K(x,t)u^N\,dx, \tag{19}$$

$$c_i(0) = \alpha_i, \quad c_i'(0) = \beta_i. \tag{20}$$

In (20), α_i, β_i are coefficients of the finite sums

$$\varphi^N(x) = \sum_{i=1}^N \alpha_i(t)\rho_i(x),\ \psi^N(x) = \sum_{i=1}^N \beta_i(t)\rho_i(x). \tag{21}$$

The finite sums (21) approximate the function $\varphi(x) \in W_2^1(0,l) \cap L_{p+2}(0,l)$ and the function $\psi(x) \in L_2(0,l)$ as $m \to \infty$.

Since the functions ρ_k are linearly independent so $det(\rho_k, \rho_i) \neq 0$ and hence, the system (19) is normal. By the Caratheodory theorem, under the assumptions (A1), (A2), (A4) the Cauchy problem (19)-(20) has solutions $c_i(t)$ on $[0, t_N]$ and all the approximations (18) are defined. Thus, to show that $t_N = T$ we shall obtain apriori estimates.
We multiply (19) by $c_i'(t)$, sum up from $i = 1$ to $i = N$ and integrate the result with respect to t from 0 to τ. We have

$$\int_0^\tau \int_0^l \left(u_{tt}^N u_t^N + a u_x^N u_{xt}^N + c u^N u_t^N\right) dx + \int_0^\tau \left(|u^N(l,t)|^p u^N(l,t) u_t^N(l,t)\, dt\right) =$$

$$= \int_0^\tau \int_0^l f(x,t) u_t^N(x,t)\, dx\, dt + \int_0^\tau u_t^N(l,t) \int_0^l K(x,t) u^N(x,t)\, dx\, dt. \qquad (22)$$

Consider the last term on the right-hand side of (22). After elementary transformations we obtain

$$\int_0^\tau u_t^N(l,t) \int_0^l K(x,t) u^N(x,t)\, dx\, dt = u^N(l,t) \int_0^l K(x,t) u^N(x,t)\, dx\Big|_0^\tau -$$

$$- \int_0^\tau u^N(l,t) \int_0^l K_t(x,t) u^N(x,t)\, dx\, dt - \int_0^\tau u^N(l,t) \int_0^l K(x,t) u_t^N(x,t)\, dx\, dt.$$

Next, we note that

$$\frac{d}{dt}\left(|u^N(l,t)|^{p+2}\right) = (p+2)|u^N(l,t)|^p u^N(l,t) u_t^N(l,t).$$

And hence,

$$\frac{1}{p+2}\left(|u^N(l,\tau)|^{p+2} - |u^N(l,0)|^{p+2}\right) =$$

$$= \int_0^\tau |u^N(l,t)|^p u^N(l,t) u_t^N(l,t)\, dt.$$

Therefore, (22) becomes

$$\frac{1}{2}\int_0^l \left(\left(u_t^N(x,\tau)\right)^2 + a(x,\tau)\left(u_x^N(x,\tau)\right)^2\right) dx +$$

$$+ \frac{1}{p+2}|u^N(l,\tau)|^{p+2} = \frac{1}{2}\int_0^\tau \int_0^l a_t (u_x^N)^2\, dx\, dt +$$

$$+ \int_0^\tau \int_0^l f u_t^N\, dx\, dt - \int_0^\tau u^N(l,t) \int_0^l K_t u^N\, dx\, dt -$$

$$- \int_0^\tau \int_0^l c u^N u_t^N\, dx\, dt - \int_0^\tau u^N(l,t) \int_0^l K u_t^N\, dx\, dt +$$

$$+\frac{1}{2}\int_0^l \left(\left(u_t^N(x,0)\right)^2 + a(x,0)\left(u_x^N(x,0)\right)^2\right) dx +$$

$$+\frac{1}{p+2}(|u^N(l,0)|^{p+2} - u^N(l,0)\int_0^l K(x,0)u^N(x,0)\,dx +$$

$$+u^N(l,\tau)\int_0^l K(x,\tau)u^N(x,\tau)\,dx. \tag{23}$$

We shall estimate each term on the right-hand side of (23). Applying Young's inequality, the Cauchy inequality we obtain

$$\left|\int_0^\tau u^N(l,t)\int_0^l K u_t^N\,dx\,dt\right| \leq \frac{1}{2}\int_0^\tau \left(u^N(l,t)\right)^2 dt +$$

$$+\frac{k_1}{2}\int_{Q_\tau}\left(u_t^N(x,t)\right)^2 dx\,dt, \tag{24}$$

$$\left|\int_0^\tau u^N(l,t)\int_0^l K_t u^N\,dx\,dt\right| \leq \frac{1}{2}\int_0^\tau \left(u^N(l,t)\right)^2 dt +$$

$$+\frac{k_2}{2}\int_{Q_\tau}\left(u_t^N(x,t)\right)^2 dx\,dt, \tag{25}$$

where $k_2 = \max\limits_{\overline{Q}_T}\int_0^l K_t^2(x,t)\,dx$. We also note that Young's enquality implies

$$\left|\int_0^\tau\int_0^l c u_t^N u^N\,dx\,dt\right| \leq \frac{c_0}{2}\int_0^\tau\int_0^l \left(\left(u^N\right)^2 + \left(u_t^N\right)^2\right) dx\,dt, \tag{26}$$

$$\left|\int_0^\tau\int_0^l f_t u^N\,dx\,dt\right| \leq \frac{1}{2}\int_0^\tau\int_0^l \left(f^2 + \left(u_t^N\right)^2\right) dx\,dt \tag{27}$$

and

$$\left|u^N(l,\tau)\int_0^l K(x,\tau)u^N(x,\tau)\,dx\,dt\right| \leq$$

$$\leq \frac{1}{2}\left(u^N(l,\tau)\right)^2 + \frac{k_1}{2}\int_{Q_\tau}\left(u^N\right)^2 dx\,dt. \tag{28}$$

Moreover, for some $\varepsilon > 0$

$$\left(u^N(l,\tau)\right)^2 \leq \varepsilon\int_0^l \left(u_x^N(x,\tau)\right)^2 dx + c(\varepsilon)\int_0^l \left(u^N(x,\tau)\right)^2 dx. \tag{29}$$

To estimate the term $\int_0^l \left(u^N(x,\tau)\right)^2$, we use the representation

$$u^N(x,\tau) = \int_0^\tau u_t^N(x,t)\,dt + u^N(x,0)$$

and hence,

$$\int_0^l \left(u^N(x,\tau)\right)^2 \le 2\tau \int_{Q_\tau} (u_t^N(x,t))^2\,dx\,dt + 2\int_0^l \left(u^N(x,0)\right)^2\,dx. \qquad (30)$$

Therefore, we obtain by (23), (24), (25), (26), (27), (28), (29), (30) that

$$\int_0^l \left(\left(u^N(x,\tau)\right)^2 + \left(u_t^N(x,\tau)\right)^2 + \frac{a_0}{2}\left(u_x^N(x,\tau)\right)^2\right) dx+$$

$$+\frac{2}{p+2}(|u^N(l,\tau)|^{p+2} \le$$

$$\le R_1 \int_0^\tau \int_0^l (u_x^N)^2\,dx\,dt + R_2 \int_0^\tau \int_0^l \left(u_t^N\right)^2\,dx\,dt+$$

$$+R_3 \int_{Q_\tau} (u^N)^2\,dx\,dt + \int_{Q_\tau} f^2\,dx\,dt+$$

$$+\int_0^l \left(\left(u_t^N(x,0)\right)^2 + (a(x,0)+1)\left(u_x^N(x,0)\right)^2\right) dx+$$

$$+\frac{2}{p+2}(|u^N(l,0)|^{p+2} + R_4 \int_0^l \left(u^N(x,0)\right)^2\,dx \qquad (31)$$

where $R_1 = a_1 + 4l$, $R_2 = 1 + c_0 + k_1 + k_2 + 2T(c(\varepsilon)+1)$, $R_3 = c_0 + k_1 + \frac{4}{l}$, $R_4 = 4 + k_1 + 2c(\varepsilon_1)$.
From the assumptions (A1)-(A4) and (21) it follows that

$$\int_0^l \left(\left(u_t^N(x,0)\right)^2 + (a(x,0)+1)\left(u_x^N(x,0)\right)^2\right) dx+$$

$$+\int_{Q_\tau} f^2\,dx\,dt + \frac{2}{p+2}(|u^N(l,0)|^{p+2}+$$

$$+R_4 \int_0^l \left(u^N(x,0)\right)^2\,dx \le M_1.$$

Furthermore, from (31) we obtain

$$\int_0^l \left(\left(u^N(x,\tau)\right)^2 + \left(u_t^N(x,\tau)\right)^2 + \left(u_x^N(x,\tau)\right)^2\right) dx+$$

$$+R_5(|u^N(l,\tau)|^{p+2} \leq$$

$$\leq R \int_{Q_\tau} \left(\left(u^N\right)^2 + \left(u_t^N\right)^2 + \left(u_x^N)\right)^2 \right) dx + M_2, \tag{32}$$

where

$$R_5 = \frac{2}{\min\{1, a_0/2\}(p+2)}, \; R = \frac{\max\{R_1, R_2, R_3\}}{\min\{1, a_0/2\}},$$

$$M_2 = \frac{M_1}{\min\{1, a_0/2\}}$$

and R, M_2 do not depend on N.
The estimate (32) implies that

$$\int_0^l \left(\left(u^N(x,\tau)\right)^2 + \left(u_t^N(x,\tau)\right)^2 + \left(u_x^N(x,\tau)\right)^2 \right) dx+$$

$$\leq R \int_{Q_\tau} \left(\left(u^N\right)^2 + \left(u_t^N\right)^2 + \left(u_x^N)\right)^2 \right) dx + M_2,$$

and hence, applying Gronwall's lemma we conclude that

$$\|u^N\|_{W_2^1(Q_T)} \leq M_3. \tag{33}$$

Applying (33) to (32) we have

$$|u^N(l,\tau)|^{p+2} \leq M_4. \tag{34}$$

The combination of (33), (34) implies

$$\|u^N\|_{W(Q_T)} \leq M, \tag{35}$$

where M does not depend on N.
Therefore, there exist solutions to the Cauchy problem (19)-(20) on $[0,T]$ and hence, the approximations (18) are defined. We note that (34), (35) imply that u^N, u_t^N are bounded in $W_2^1 \cap L_{p+2}(\Gamma_l)$ and $L_2(Q_T)$ respectively as $m \to \infty$. It means that there exists a subsequence $\{u^m\}$ of $\{u^N\}$ such that u^m converges to u weakly in $W_2^1 \cap L_{p+2}(\Gamma_l)$ and u_t^m converges to u_t weakly in $L_2(Q_T)$. From (33) it follows that u^m is bounded in $W_2^1(Q_T)$ and since the embedding of $W_2^1(Q_T)$ in $L_2(Q_T)$ is compact ([16], p.25), so consider the subsequence u^m which converges in $L_2(Q_T)$ and hence, almost everywhere. Furthermore, it follows from (35) that $|u^m(l,t)|^p u^m(l,t) \in L_{\frac{p+2}{p+1}}(0,T)$. We note that boundedness of $|u^m(l,t)|^p u^m(l,t)$ in $L_{\frac{p+2}{p+1}}(0,T)$ provides weak convergence $|u^m(l,t)|^p u^m(l,t) \to \gamma(t)$ in this space. In view of the statement ([17], p. 12), $\gamma(t) = |u(l,t)|^p u(l,t)$.
The above results allow us to pass to the limit in (19). To this aim, we multiply each

of (19) by functions $h_i(t) \in C[0,T]$, $h_i(T) = 0$, take the sum from $i = 1$ to $i = m$ and integrate with respect to t from 0 to T. We have

$$\int_{Q_T} (-u_t^m \Phi_t^m + a(x,t)u_x^m \Phi_x^m + c(x,t)u^m \Phi)\, dx\, dt +$$

$$+ \int_0^T |u^m(l,t)|^p u^m(l,t)\Phi^m(l,t)\, dt =$$

$$= \int_0^l u_t^m(x,0)\Phi^m(x,0)\, dx + \int_{Q_T} f\Phi^m\, dx\, dt +$$

$$+ \int_0^T \left(\int_0^l K(x,t)u^m\, dx\right)\Phi^m(l,t)\, dt, \qquad (36)$$

where $\Phi^m(x,t) = \sum_{i=1}^m h_i(t)\rho_i(t)$.

Applying the results obtained above, we pass to the limit in (36) as $m \to \infty$ and obtain (5) for $\eta_(x,t) = \Phi(x,t)$. Note that $\int_0^T \left(\int_0^l K(x,t)u^m\, dx\right)\Phi^m(l,t)\, dt$ converges to $\int_0^T \left(\int_0^l K(x,t)u^m\, dx\right)\eta(l,t)\, dt$ under assumptions of the theorem. The set of the functions $\Phi(x,t)$ is dense in $W_2^1 \cap L_{p+2}(\Gamma_l)$ and therefore, the limit relation holds for any function $\eta(x,t) \in \widehat{W}(Q_T)$. This implies that the limit function $u(x,t)$ is the generalized solution to the problem (34)-(4). ∎

4. CONCLUSION

This work is devoted to the problem with nonlinear integral boundary conditions for the hyperbolic equation, in which a nonlinear component has a particular representation $|u|^p u$. We have shown the existence of the generalized solution by construction of convergent Faedo-Galerkin approximations. The uniqueness result has been obtained by the method of energy estimates.

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