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STOCHASTIC OPTIMAL CONTROL OF A MIXING FLOW MODEL

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Abstract We consider the following controlled stochastic mixing flow model: $\dot{x}_1(t) = Gx_2(t)$, $\dot{x}_2(t) = KGx_1(t) + b_0x_2(t)u(t) + m_0x_2(t) + \sigma_0x_2(t)\dot{B}(t)$, where $B(t)$ is a standard Brownian motion. Optimal control problems involving a random final time are considered for this model, and explicit solutions are obtained by making use of the method of similarity solutions.

Keywords: Brownian motion, dynamic programming, first passage time, method of similarity solutions.

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1. INTRODUCTION

Mixing theory appears in an area with far from being completely solved problems: flow kinematics [6]. Its methods and techniques developed the significant relation between turbulence and chaos. Turbulence is an important feature of dynamic systems with few degrees of freedom, the so-called "far from equilibrium systems". These are widespread among excitable media models. Numeric simulations of mixing flow models are currently being carried out, using different computational appliances. In the mathematical framework, the flow complexity implies the following three stages (see [1]):

- modelling the global swirling streamlines;
- local modelling of the concentrated vorticity structure;
- introducing the elements of chaotic turbulence.

Various computational standpoints for the mixing phenomena were considered, starting with the analysis of the mixing efficiency ([1]) for the three-dimensional model, and continuing with some phase-portrait analysis ([2]). The computational standpoints are very important, in order to achieve more features for unifying the mixing theory. In this sense, we must mention the phase-portrait analysis for the three-dimensional model which enables one to compare the behaviour of the model for different appliances approaches (see [2]).

The issue of repetitive events resulted from the simulations cases (over 60 cases overall), giving rise to the consistency with the interdisciplinary area.

In the present paper, a stochastic optimal control problem for the mixing flow model

$$\begin{aligned} \dot{x}_1(t) &= G x_2(t), \\ \dot{x}_2(t) &= K G x_1(t), \end{aligned} \tag{1}$$

where G (> 0) and $-1 < K < 1$ are constants, is considered. Let

$$\begin{aligned} \dot{x}_1(t) &= G x_2(t), \\ \dot{x}_2(t) &= K G x_1(t) + b_0 x_2(t) u(t) + m_0 x_2(t) + \sigma_0 x_2(t) \dot{B}(t), \end{aligned} \tag{2}$$

where b_0 ($\neq 0$), m_0 and σ_0 (> 0) are constants, and $\{B(t), t \geq 0\}$ is a standard Brownian motion.

Assume that $x_i(0) = x_i$ for $i = 1, 2$ and that the cost function is given by

$$J(x_1, x_2) = \int_0^{T(x_1,x_2)} \left\{ \frac{1}{2} q[x_1(t), x_2(t)] u^2(t) + \lambda \right\} dt, \tag{3}$$

where T is a random variable, q is a positive function and λ is a constant. Our aim is to find the control u^* that minimizes the expected value of J. Problems of this type are called "LQG homing" (see [7]). Depending on the sign of the constant λ, the optimizer is either trying to maximize or minimize the time spent by the controlled process in the continuation region, taking the quadratic control costs into account.

The first author has worked intensively on this topic (see, for example, [3] and [4]). These problems are generally very difficult to solve explicitly in two or more dimensions; one such problem was solved in [5]. Here, we will obtain analytical solutions to particular problems by making use of the method of similarity solutions to solve the appropriate partial differential equations.

In the next section, the differential equation that must be solved to obtain the explicit solution to the problem set up above will be derived. Then, three particular cases will be considered and solved exactly in Section 3. We will illustrate the effect of the optimal control on the system's trajectory in one of these particular cases. An outline of possible future work will be given in Section 4.

2. COMPUTATION OF THE OPTIMAL CONTROL

We define the value function

$$F(x_1, x_2) = \inf_{u(t),\ 0 \leq t \leq T(x_1,x_2)} E[J(x_1, x_2)]. \tag{4}$$

By making use of dynamic programming, we find that F satisfies the partial differential equation (p.d.e.)

$$\frac{1}{2} q u^2 + \lambda + G x_2 F_{x_1} + (K G x_1 + b_0 x_2 u + m_0 x_2) F_{x_2} + \frac{\sigma_0^2}{2} x_2^2 F_{x_2 x_2} = 0, \tag{5}$$

where $q = q(x_1, x_2)$ and $u = u(0)$. That is, we only need to determine the optimal control u^* at the *initial time* $t = 0$.

We deduce from the above p.d.e. that

$$u^* = -\frac{b_0 x_2 F_{x_2}}{q}. \tag{6}$$

Substituting this expression into the p.d.e. we obtain the *non-linear* p.d.e.

$$-\frac{1}{2}\frac{b_0^2}{q} x_2^2 F_{x_2}^2 + \lambda + G x_2 F_{x_1} + (KG x_1 + m_0 x_2)\, F_{x_2} + \frac{\sigma_0^2}{2} x_2^2 F_{x_2 x_2} = 0. \tag{7}$$

Now, assume that

$$q(x_1, x_2) \equiv q_0 \; (> 0) \tag{8}$$

and let us define

$$\Phi(x_1, x_2) = e^{-\alpha F(x_1, x_2)}, \tag{9}$$

where α is a positive constant defined by

$$\alpha = \frac{b_0^2}{q_0 \sigma_0^2}. \tag{10}$$

We find that the function Φ satisfies the linear p.d.e.

$$-\alpha\lambda\Phi + G x_2 \Phi_{x_1} + (KG x_1 + m_0 x_2)\, \Phi_{x_2} + \frac{\sigma_0^2}{2} x_2^2 \Phi_{x_2 x_2} = 0. \tag{11}$$

Thus, to obtain the optimal control u^*, we must solve the above p.d.e., subject to the appropriate boundary conditions.

To simplify our problem, we will make use of the method of similarity solutions. We assume that the function $\Phi(x_1, x_2)$ can actually be written as follows:

$$\Phi(x_1, x_2) = \Psi(z), \tag{12}$$

where

$$z := \frac{x_2}{x_1} \tag{13}$$

is the similarity variable.

The function Ψ satisfies the ordinary differential equation (o.d.e.)

$$-\alpha\lambda\Psi(z) + \left(KG + m_0 z - G z^2\right)\Psi'(z) + \frac{\sigma_0^2}{2} z^2 \Psi''(z) = 0. \tag{14}$$

Remark 2.1. *For the method of similarity solutions to work, we must be able to express (after simplification) all the terms in the original equation as a function of the similarity variable z as well as the boundary conditions.*

In the next section, we will consider various particular cases.

3. PARTICULAR CASES

First, for the sake of simplicity, let us assume that

$$q_0 = b_0 = \sigma_0 = 1, \quad m_0 = 0, \quad K = \frac{1}{2} \quad \text{and} \quad G = 1. \tag{15}$$

We then have $\alpha = 1$, and the o.d.e. is reduced to

$$-\lambda\Psi(z) + \left(\frac{1}{2} - z^2\right)\Psi'(z) + \frac{1}{2}z^2\Psi''(z) = 0. \tag{16}$$

Case I. Assume that the final time T is the random variable

$$T(x_1, x_2) = \inf\left\{t \geq 0 : \frac{x_2(t)}{x_1(t)} = k_1 \text{ or } k_2\right\}, \tag{17}$$

where $0 < k_1 < x_2/x_1 < k_2$.

Let us choose $\lambda = -1$. Then, the aim is to maximize the time that the ratio $x_2(t)/x_1(t)$, starting from x_2/x_1, remains between the constants k_1 and k_2, taking the quadratic control costs into account.

The boundary conditions are

$$\Psi(k_1) = \Psi(k_2) = 1 \tag{18}$$

(because $F(x_1, x_2) = 0$ if $x_2/x_1 = k_1$ or k_2).

The mathematical software *Maple* is able to obtain the explicit solution to the above boundary value problem, which is expressed in terms of the special function HeunD. From this explicit solution, we can deduce the value of the optimal control u^*.

Case II. Assume next that the constant K is equal to 0 (rather than 1/2). Then, the o.d.e. becomes

$$-\alpha\lambda\Psi(z) + \left(m_0 z - Gz^2\right)\Psi'(z) + \frac{\sigma_0^2}{2}z^2\Psi''(z) = 0. \tag{19}$$

Again, *Maple* gives the general solution of this equation, in terms of the special functions WhittakerM and WhittakerW. Moreover, if we choose the same constants as in Case I, then the solution is given in terms of the Bessel functions I and K.

Case III. Finally, assume that the parameter λ is equal to $+1$. Then, we want the ratio $x_2(t)/x_1(t)$ to reach k_1 or k_2 as soon as possible (taking the control costs into account). Using the same constants as in Case II, we must solve the o.d.e.

$$-\Psi(z) - z^2\Psi'(z) + \frac{1}{2}z^2\Psi''(z) = 0. \tag{20}$$

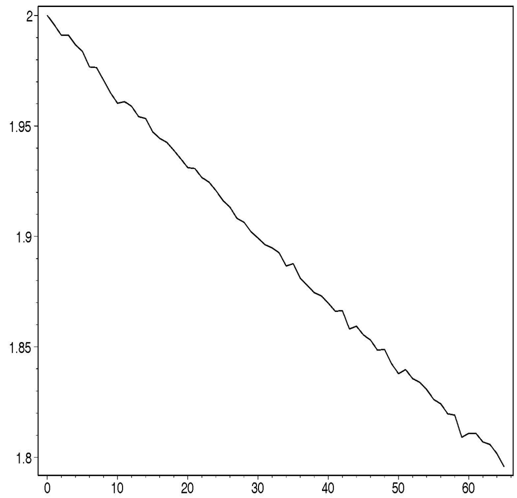

Fig. 1.: An example of a trajectory in the case of the uncontrolled process.

Its general solution is simply

$$\Psi(z) = c_1 \frac{1+z}{z} + c_2 e^{2z} \frac{z-1}{z}, \tag{21}$$

where the constants c_1 and c_2 are uniquely determined from the boundary conditions $\Psi(k_1) = \Psi(k_2) = 1$.

To illustrate the effect of the optimal control on the system's trajectory, we used *Maple* to simulate both the uncontrolled and the optimally controlled system in this particular case. We chose the following values: $x_1 = 1$, $x_2 = 2$, $k_1 = 1{,}8$ and $k_2 = = 2{,}2$. With a discretization step size of 0,001, it took the uncontrolled process $z(t) := x_2(t)/x_1(t)$ 65 steps in order to leave the interval $(1{,}8; 2{,}2)$ for the first time; see Figure 1.

Now, with the choices that we made above, it is a simple matter to compute $\Psi(z)$ explicitly and hence the optimal control u^*. We simulated the optimally controlled system. As can be seen in Figure 2, it took only three steps for the controlled process to leave the interval $(1{,}8; 2{,}2)$. Moreover, the value of the cost function was equal to approximately 0,0037, whereas it was 0,065 in the case of the uncontrolled process. We see the huge difference in the two cases. Hence, it is much more efficient to optimally control the system than to let the process eventually leave the interval without using any control.

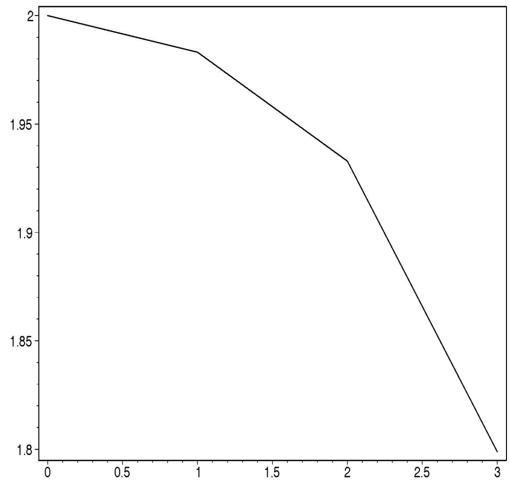

Fig. 2.: An example of a trajectory in the case of the controlled process.

4. CONCLUSIONS

By making use of the method of similarity solutions, we were able to obtain explicit solutions to LQG homing problems in two dimensions, which is generally a difficult task. Next, we could

- try to do the same in three dimensions;
- try to solve other problems in two (or three) dimensions, with different choices for the various functions that appear in the model;
- consider discrete versions of this problem, or the case when there are jumps in the value of $x_1(t)$ and/or $x_2(t)$.

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