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QUALITATIVE AND QUANTITATIVE ANALYSIS FOR A NONLINEAR REACTION-DIFFUSION EQUATION

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Abstract The paper concerns with the existence, uniqueness, regularity and the approximation of solutions to the reaction-diffusion equation endowed with a cubic nonlinearity and Neumann boundary conditions, relevant in a wide class of physical phenomena, including phase separation and transition. The convergence and error analysis for an iterative scheme of fractional steps type, are also established. We prove L^∞ stability by maximum principle arguments and derive error estimates using energy methods for two implicit-explicit approaches, a linearized scheme and a fractional steps type scheme. A numerical experiment validates the theoretical results, comparing the accuracy of the classical Newton method with the linearized method as well as with two different schemes of fractional steps type.

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1. INTRODUCTION

Consider a one dimensional nonlinear reaction-diffusion equation with respect to the unknown function $v(t, x)$:

$$\begin{cases} p_1 \dfrac{\partial}{\partial t} v - p_2 \Delta v + p_3 (v^3 - v) = f(t, x) & \text{in } Q = [0, T] \times \Omega \\ p_2 \dfrac{\partial}{\partial \mathbf{n}} v = 0 & \text{on } \Sigma = [0, T] \times \partial\Omega \\ v(0, x) = v_0(x) & \text{on } \Omega, \end{cases} \tag{1}$$

where:

- Ω is a bounded domain in $\boldsymbol{R}$ with smooth boundary $\partial\Omega = \Gamma$ and $T > 0$ stands for some final time;

- $v(t,x)$ is the unknown function; in particular, $v(t,x)$ is the *phase function* (used to distinguish between the states (phases) of a material which occupies the region Ω at every time $t \in (0,T]$);
- p_1, p_2, p_3 are positive values;
- $f(t,x) \in L^p(Q)$ is a given function and $p \geq 2$ (see (2));
- $v_0 \in W_\infty^{2-\frac{2}{p}}(\Omega)$ verifying the compatibility condition $p_2 \frac{\partial}{\partial \mathbf{n}} v_0 = 0$;
- $\mathbf{n}$=$\mathbf{n}$(x) is a vector of the outward (from Ω) unit normal to the surface Σ; $\frac{\partial}{\partial \mathbf{n}}$ denotes differentiation along $\mathbf{n}$.

Here we have used the standard notation for Sobolev spaces, namely, given a positive integer k and $1 \leq p \leq \infty$, we denote by $W_p^{k,2k}(Q)$ the usual Sobolev space on Q:

$$W_p^{k,2k}(Q) = \left\{ y \in L^p(Q) : \ \frac{\partial^r}{\partial t^r} \frac{\partial^s}{\partial x^s} y \in L^p(Q), \text{ for } 2r + s \leq k \right\},$$

i.e., the space of functions whose t-derivatives and x-derivatives up to the order k and $2k$, respectively, belong to $L^p(Q)$. Also, we have used the Sobolev spaces $W_p^l(\Omega)$ with nonintegral l for the initial and boundary conditions, respectively (see [28] - Chapter 1 and references therein).

The nonlinear parabolic equation (1) occurs in the phase-field transition system [9], where the phase function $v(t,x)$ describes the transition between the solid and liquid phases in the solidification process of a material occupying a region Ω. (1) it is a particular instance of the Allen-Cahn equation [1, 2, 3], which was introduced to describe the motion of anti-phase boundaries in crystalline solids, and it has been widely applied to many [16] complex moving interface problems, e.g., the *mixture of two incompressible fluids*, the *nucleation of solids*, the *vesicle membranes*.

A great deal of work has be done on reaction-diffusion problems and Allen-Cahn equations [22], [24], [44], [49]-[51]. For more general assumptions and with various types of boundary conditions, equation (1) has been numerically investigated in e.g., [4], [8], [10]-[13], [15], [18], [21], [25]-[27], [29], [32]-[35], [38], [40], [43], [45].

Under an assumption on the ω-m-accretivity of a more general nonlinear operator, error estimates for several first-order approximations are presented in [6, 27].

The error analysis for the implicit backward Euler and finite elements approximation is presented in [19, 23, 40], while a discontinuous-Galerkin in time method is analyzed in [17]. Computations with several different higher-order time-stepping schemes, such as BDF2-AB2 and Crank-Nicolson, are used for the sharp interface limit in [52]. For finite element analysis and adapting meshes we refer to [41], [42], [47] and [48], while for the existence, uniqueness and a maximum principle in Hilbert Sobolev spaces we refer to [39, 46].

The outline of the paper is as follows. In Section 2 we prove the existence, regularity, stability and uniqueness of the solution to the nonlinear problem (1) in the presence of homogeneous Neumann boundary conditions, while, in Section 3 we are concerned with the convergence of fractional steps type scheme associated to the nonlinear reaction-diffusion equation $(1)_1$. In Section 4 we introduce the two semi-discrete in time approximations and prove L^∞ stability by maximum principle. The convergence of the numerical methods is derived in Section 5 by energy estimates arguments, proving consistency and stability results for error equations. The numerical experiment in Section 6 confirms the theoretical rates of convergence. The concluding remarks are formulated in Section 7.

2. WELL-POSEDNESS OF SOLUTIONS TO THE NONLINEAR EQUATION (1)

In the present Section we will investigate the solvability of the first boundary value problems of the form (1) in the class $W_p^{1,2}(Q)$. One proves the existence, the regularity and the uniqueness of solutions (Theorem 2.1 below) to the nonlinear parabolic problem (1) considering the cubic nonlinearity $p_3(v - v^3)$ which verifies for $N = 1$ and $r = 3$ the general assumptions H_0 and H_2 formulated in [39], that is:

H_0 : $(v - v^3)|v|^{3p-4}v \leq 1 + |v|^{3p-1} - |v|^{3p}$.

H_2 : There exist a function $\bar{F} : \boldsymbol{R}^2 \to \boldsymbol{R}$ and a constant $b_0 > 0$ verifying the relations:

$((v_1 - v_1^3) - (v_2 - v_2^3))^2 \leq \bar{F}(v_1, v_2)(v_1 - v_2)^2$; $\bar{F}(v_1, v_2) \leq b_0(1 + |v_1|^4 + |v_2|^4)$, $\forall v_1, v_2 \in \boldsymbol{R}$.

The basic tools in the analysis of the problem (1) are the Leray-Schauder degree theory [20], the L^p-theory of linear and quasi-linear parabolic equations [28], as well as the Lions and Peetre embedding Theorem [30], p. 24, which ensures the existence of a continuous embedding $W_p^{1,2}(Q) \subset L^\mu(Q)$, where the number μ is defined as follows

$$\mu = \begin{cases} \infty & if \quad p > \dfrac{3}{2}, \\ \\ any\ positive\ number\ \geq 3p & if \quad p = \dfrac{3}{2}. \end{cases} \tag{2}$$

The main result of this section establishes the dependence of the solution $v(t, x)$ in the nonlinear parabolic equation (1) on the term $f(t, x)$ in the right-hand side.

Theorem 2.1 *There exists a unique solution $v \in W_p^{1,2}(Q)$ for* (1) *and v satisfies*

$$\|v\|_{W_p^{1,2}(Q)} \leq C \left\{1 + \|v_0\|^{3-\frac{2}{p}}_{W_\infty^{2-\frac{2}{p}}(\Omega)} + \|f\|_{L^p(Q)}\right\}, \tag{3}$$

where the constant C depends on $|\Omega|$, T, M, p, p_1, p_2 *and* p_3, *but is independent of* v *and* f.

If v_1, v_2 *are two solutions of* (1) *corresponding to the data* $f^1, f^2 \in L^p(Q)$, *respectively, for the same initial conditions, such that*

$$\|v_1\|_{W_p^{1,2}(Q)} \leq M, \quad \|v_2\|_{W_p^{1,2}(Q)} \leq M, \tag{4}$$

then

$$\|v_1 - v_2\|_{W_p^{1,2}(Q)} \leq C\, \|f^1 - f^2\|_{L^p(Q)}, \tag{5}$$

where the constant C depends on $|\Omega|$, T, M, p, p_1, p_2, p_3 *and* b_0 *but is independent of* v_1, v_2, f^1 *and* f^2.

Proof. The proof of Theorem 2.1 was given in [7] and [36] noting that there formulation differs from this by certain physical parameters. Moreover, corresponding to different boundary conditions (including nonlinear and nonhomogeneous boundary conditions), similar results we proved in [14], [31], [32], [37] and [39]. For reader's convenience, we present here an outline of the proof.

In order to use the Leray-Schauder degree theory, we choose as suitable Banach space $B = L^{3p}(Q)$, endowed with the norm $\|v\|_B = \|v\|_{L^{3p}(Q)}$, $\forall v \in B$.

Let us define the nonlinear operator $T : L^{3p}(Q) \times [0,1] \to L^{3p}(Q)$ as

$$T(\bar{v}, \lambda) = v = v(\bar{v}, \lambda) \quad \forall\, \bar{v} \in L^{3p}(Q),\ \forall\, \lambda \in [0,1], \tag{6}$$

where v is the solution of the linear problem

$$\begin{cases} p_1 \frac{\partial}{\partial t} v - p_2 \Delta v = \lambda \Big[p_3(\bar{v} - \bar{v}^3) + f(t,x) \Big] & \text{on } Q \\ \frac{\partial}{\partial \mathbf{n}} v = 0 & \text{on } \Sigma \\ v(0,x) = \lambda v_0(x) & \text{on } \Omega. \end{cases} \tag{7}$$

Applying the L^p-theory of parabolic equations to problem (7) and making use of (2), which guarantees the continuous embedding $W_p^{1,2}(Q) \subset L^\mu(Q) \subset L^{3p}(Q)$, we easily deduce that problem (1) has a solution, which means that the operator T is *well defined* and it maps $L^{3p}(Q) \times [0,1]$ into $L^{3p}(Q)$.

L^p-theory, combined with the continuity of Nemytski operator (see [20]) and relation (2), allow us to conclude that the mapping T defined in (6) is *continuous and compact* (see also [7] and [36]).

The regularity of the solution. We show now that there exists $\delta > 0$ such that (see (6))

$$(v, \lambda) \in L^{3p}(Q) \times [0,1],\ v = T(v, \lambda) \implies \|v\|_{L^{3p}(Q)} < \delta. \tag{8}$$

Let $v \in L^{3p}(Q)$ solving the problem

$$\begin{cases} p_1 \frac{\partial}{\partial t} v - p_2 \Delta v = \lambda \left\{ p_3 (v - v^3) + f(t,x) \right\} & \text{on } Q \\ \frac{\partial}{\partial \mathbf{n}} v = 0 & \text{on } \Sigma \\ v(0,x) = \lambda v_0(x) & \text{on } \Omega. \end{cases} \tag{9}$$

Following exactly the same reasoning as in [7], corresponding to solution v of problem (9) we then get the following estimate

$$\|v\|_{W_p^{1,2}(Q)} \leq C(|\Omega|, T, M, p, p_1, p_2, p_3) \left(1 + \|v_0\|_{W_\infty^{2-\frac{2}{p}}(\Omega)} + \|v_0\|_{L^{3p-2}(\Omega)}^{3-\frac{2}{p}} + \|f\|_{L^p(Q)} \right). \tag{10}$$

The continuous embedding $W_p^{1,2}(Q) \subset L^\mu(Q) \subset L^{3p}(Q)$ ensures that

$$\|v\|_{L^{3p}(Q)} \leq C \|v\|_{W_p^{1,2}(Q)}. \tag{11}$$

Combining (10) and (11) we see that the claim in (8) holds true.
Denoting

$$B_\delta := \left\{ v \in L^{3p}(Q) : \ \|v\|_{L^{3p}(Q)} < \delta \right\}, \tag{12}$$

relation (8) ensures that

$$T(v, \lambda) \neq v \qquad \forall v \in \partial B_\delta, \ \ \forall \lambda \in [0,1],$$

provided that $\delta > 0$ is sufficiently large.

Further, we conclude that problem (1) has a solution $v \in W_p^{1,2}(Q)$. Finally, making use of the embedding $W_\infty^{2-\frac{2}{p}}(\Omega) \subset L^{3p-2}(\Omega)$ and estimate (10), we conclude that (3) is true.

The uniqueness of the solution. Now we will establish the stability result (5) and so, as a consequence, the uniqueness of the solution of (1). By hypothesis, $v_1, v_2 \in W_p^{1,2}(Q)$ solve problem (1) corresponding to f^1 and f^2, respectively, for the same initial conditions v_0. Thus $v_1 - v_2 \in W_p^{1,2}(Q)$ and it satisfies

$$\begin{cases} p_1 \frac{\partial}{\partial t}(v_1 - v_2) - p_2 \Delta (v_1 - v_2) = \left\{ p_3 [(v_1 - v_2) - (v_1^3 - v_2^3)] + (f^1 - f^2) \right\} & \text{on } Q \\ \frac{\partial}{\partial \mathbf{n}}(v_1 - v_2) = 0 & \text{on } \Sigma \\ (v_1 - v_2)(0,x) = 0 & \text{on } \Omega. \end{cases} \tag{13}$$

Multiplying $(13)_1$ by $|v_1 - v_2|^{p-2}(v_1 - v_2)$, integrating over Q_t, $t \in (0, T]$, and using Green's formula and Cauchy-Schwarz inequality, we obtain

$$\frac{p_1}{p}\int_\Omega |v_1 - v_2|^p\, dx + (p-1)p_2 \int_{Q_t} |\nabla(v_1 - v_2)|^2 |v_1 - v_2|^{p-2}\, dsdx$$

$$\leq \frac{2^p}{p}\|f^1 - f^2\|^p_{L^p(Q)} + \frac{p-1}{p}\frac{1}{2^{\frac{p}{p-1}}}\int_{Q_t} |v_1 - v_2|^p\, dsdx$$

$$+p_3 \int_{Q_t} \Big[(v_1 - v_2) - (v_1^3 - v_2^3)\Big]|v_1 - v_2|^{p-2}(v_1 - v_2)\, dsdx, \quad \forall t \in (0, T].$$

Due to the inequality $\Big[(v_1 - v_1^3) - (v_2 - v_2^3)\Big](v_1 - v_2) \leq (v_1 - v_2)^2, \quad \forall v_1, v_2 \in \boldsymbol{R}$, and by means of Gronwall's inequality, it results from foregoing inequality

$$\|v_1 - v_2\|^p_{L^p(\Omega)} \leq C(T, p, p_1, p_2, p_3)\|f^1 - f^2\|^p_{L^p(Q)}. \tag{14}$$

We have $v_1, v_2 \in W^{1,2}_p(Q) \subset L^\mu(Q) \subset L^{3p}(Q)$, which yields that $p_3[(v_1 - v_2) - (v_1^3 - v_2^3)] \in L^p(Q)$. So we may apply the L^p-theory to the linear problem (13) which gives the estimate

$$\|v_1 - v_2\|^p_{W^{1,2}_p(Q)} \leq C(|\Omega|, T, p)\Big(\|(v_1 - v_2) - (v_1^3 - v_2^3)\|^p_{L^p(Q)} + \|f^1 - f^2\|^p_{L^p(Q)}\Big). \tag{15}$$

The inequality $\mu > 3p$ allows us to fix a number m such that (see [39])

$$2 \leq p \leq \frac{\mu p}{\mu + p - 3p} \leq 3p < m \leq \mu. \tag{16}$$

Consequently, the next sequence of embeddings holds

$$W^{2,1}_p(Q) \subset L^\mu(Q) \subset L^m(Q) \subset L^{3p}(Q) \subset L^p(Q) \subset L^2(Q). \tag{17}$$

From (H_2), (17) and Hölder's inequality it is seen that

$$\|(v_1 - v_2) - (v_1^3 - v_2^3)\|_{L^p(Q)} \leq \|\bar{F}(v_1, v_2)^{\frac{1}{2}}|v_1 - v_2|\|_{L^p(Q)} \tag{18}$$

$$= \Big(\int_Q \bar{F}(v_1, v_2)^{\frac{p}{2}}|v_1 - v_2|^p\, dtdx\Big)^{\frac{1}{p}} \leq \Big(\int_Q \bar{F}(v_1, v_2)^{\frac{n_0}{2}}\, dtdx\Big)^{\frac{1}{n_0}}\|v_1 - v_2\|_{L^m(Q)},$$

where we denoted $n_0 := \frac{mp}{m-p}$. The computation above makes sense because $\bar{F}(v_1, v_2)^{\frac{n_0}{2}} \in L^1(Q)$. Indeed (see [39]), taking into account the growth condition in (H_2), $\bar{F}(v_1, v_2) \in L^{\frac{1}{2}\frac{\mu}{2}}(Q)$ whenever $v_1, v_2 \in L^\mu(Q)$, and by (17) it is true that

$$\frac{\mu}{2} > n_0 = \frac{mp}{m-p} > 2. \tag{19}$$

Then, the inequalities in (19) lead to the claim above.

Combining (15) and (18), we arrive at

$$\|v_1 - v_2\|_{W_p^{1,2}(Q)} \leq C(|\Omega|, T, p)\left(\|\bar{F}(v_1, v_2)\|^{\frac{1}{2}}_{L^{\frac{n_0}{2}}(Q)} \|v_1 - v_2\|_{L^m(Q)} + \|f^1 - f^2\|_{L^p(Q)}\right). \tag{20}$$

In addition, we have (see (H_2))

$$\left(1 + |v_1|^4 + |v_2|^4\right)^{\frac{n_0}{2}} \leq C(p, b_0)\left(1 + |v_1)|^{2n_0} + |v_2|^{2n_0}\right).$$

The relation above, inequality (19) and estimate (20) imply

$$\|v_1 - v_2\|_{W_p^{1,2}(Q)} \leq C(|\Omega|, T, p)\Big[\|f^1 - f^2\|_{L^p(Q)} \tag{21}$$

$$+C(|\Omega|, T, p, b_0)\left(1 + \|v_1\|^2_{L^{2n_0}(Q)} + \|v_2\|^2_{L^{2n_0}(Q)}\right)\|v_1 - v_2\|_{L^m(Q)}\Big]$$

$$\leq C(|\Omega|, T, p, b_0)\left[\|f^1 - f^2\|_{L^p(Q)} + \left(1 + \|v_1\|^2_{W_p^{1,2}(Q)} + \|v_2\|^2_{W_p^{1,2}(Q)}\right)\|v_1 - v_2\|_{L^m(Q)}\right]$$

$$\leq C(|\Omega|, T, p, b_0)(1 + 2M^2)\left(\|v_1 - v_2\|_{L^m(Q)} + \|f^1 - f^2\|_{L^p(Q)}\right).$$

By the embeddings in (17), the interpolation inequality (see [30], pp. 58) yields that $\forall \varepsilon > 0$, $\exists C(\varepsilon) > 0$ such that

$$\|v\|_{L^m(Q)} \leq \varepsilon\|v\|_{W_p^{1,2}(Q)} + C(\varepsilon)\|v\|_{L^p(Q)}, \quad \forall v \in W_p^{1,2}(Q). \tag{22}$$

From (21), (22) and (14), we derive that

$$\begin{aligned}(1 - \varepsilon C(|\Omega|, \quad & T, M, p, b_0))\|v_1 - v_2\|_{W_p^{1,2}(Q)} \leq \\ & \leq C(|\Omega|, T, M, p, b_0)\Big(\|f^1 - f^2\|_{L^p(Q)} + \\ & + C(\varepsilon)C(T, p, p_1, p_2, p_3)C(|\Omega|, T, M, p, b_0)\|f^1 - f^2\|_{L^p(Q)}\Big).\end{aligned} \tag{23}$$

For $\varepsilon > 0$ with $1 - \varepsilon C(|\Omega|, T, M, p, b_0) > 0$, (23) implies estimate (5) and thus the proof of Theorem 2.1 is finished. ∎

As a consequence, the uniqueness of solution to problem (1) is valid.

Corollary 2.2 *Under hypotheses* H_0 *and* H_2 *the problem* (1) *possesses a unique solution* $v \in W_p^{1,2}(Q)$.

Proof. Let $f^1 = f^2 = f$ in the Theorem 2.1. Then (2.4) shows that the conclusion of the corollary is true. ∎

3. APPROXIMATING SCHEME

The aim of this Section is to use the *fractional steps method* in order to approximate the solution of the nonlinear problem $(1)_1$, whose uniqueness is guaranteed by Corollary 2.2. To do that, let us associate to the time-interval $[0, T]$ the equidistant grid of length $\varepsilon_M = \frac{T}{M}$, for any integer $M \geq 1$, and corresponding to it, the approximate problem (24), (25) written below:

$$\begin{cases} \dfrac{\partial}{\partial t} v_M(t,\cdot) - \dfrac{p_2}{p_1}\,\Delta v_M(t,\cdot) = \dfrac{1}{p_1} f(t,\cdot) & t \in [i\varepsilon_M, (i+1)\varepsilon_M], \\[2ex] \dfrac{p_2}{p_1}\dfrac{\partial}{\partial \mathbf{n}} v_M(t,\cdot) = 0 \\[2ex] v_M(i\varepsilon_M,\cdot) = z_M(\varepsilon_M,\cdot), \quad i = 0, ..., M-1, \end{cases} \tag{24}$$

where $z_M(\varepsilon_M, \cdot)$ is the solution of Cauchy problem

$$\begin{cases} z_M'(\tau,\cdot) + \dfrac{p_3}{p_1}\left(z_M^3(\tau,\cdot) - z_M(\tau,\cdot)\right) = 0, & \tau \in [0, \varepsilon_M], \\[2ex] z_M(0,\cdot) = v_M^-(i\varepsilon_M,\cdot), \quad v_M^-(0,\cdot) = v_0(x) & i = 0, 1, ..., M-1, \end{cases} \tag{25}$$

while v_M^- stands for the left-hand limit of v_M at $i\varepsilon_M$, that is: $v_M^-(i\varepsilon_M,\cdot) = \lim\limits_{t \uparrow i\varepsilon_M} v_M(t,\cdot)$.

We point out that the sequence of approximating problems (24)-(25) supplies a decoupling method for the original problem (1) into a linear parabolic boundary value problem (24) and a nonlinear evolution equation (25). The advantage of this approach consists in simplifying the numerical computation of the approximations to (1), due to that the fractional steps method avoids the iterative process in passing from a time level to the next one.

Remark. *As a novelty regarding the structure of the fractional steps type scheme (24)-(25), we mention that the linear term $p_3 v$ from the original problem* (1) *appears now in Cauchy problem* (25) *and not in the linear problem* (24)*, as was done in previous works:* [4], [8], [21], [25], [32]-[35], [37]-[38], [40], [43].

The main question is the convergence of the sequence $\{v_M\}$ of solutions to the approximate problems (24)-(25) to the unique solution $v \in W_p^{1,2}(Q)$ of problem (1) as $M \longrightarrow \infty$. We will treat the convergence of this numerical scheme on the basis of an abstract approximation result (see [5], [32], [34], [37], [43] for a detailed discussion).

We have

Theorem 3.1 *Assume that the function $y \mapsto p_3(y^3 - y) + \omega y$ is strictly increasing on $\mathbf{R}$ for some $\omega > 0$. Then for all $v_0 \in W_\infty^{2-\frac{2}{p}}(\Omega) \subset L^p(\Omega)$ and $f \in L^p(Q)$, the sequence $\{v_M\}$ solving* (24)-(25) *converges to the unique solution v of problem* (1) *in the following sense*

$$\lim_{M\to\infty} v_M(t) = v(t) \quad in \quad L^p\Omega)$$

uniformly with respect to $[0, T]$.

Proof. We will apply Theorem 3.1 (see [37]) with $X = L^p(\Omega)$. For this, let us put (1) in an abstract framework. To do that, we define the operator

$$A : D(A) = \{v \in W_p^2(\Omega) : \frac{p_2}{p_1}\frac{\partial}{\partial \mathbf{n}} v = 0\} \subset X \to X, \quad \text{by } A(v) = -\frac{p_2}{p_1}\Delta v$$

and, the operator

$$B : D(B) = L^{3p}(\Omega) \subset X \to X, \quad \text{by } B(\varphi) = \frac{p_3}{p_1}(v^3 - v).$$

Thus the system (1) becomes

$$\begin{cases} \dfrac{d}{dt}v(t) + Av(t) + Bv(t) = \dfrac{1}{p_1}f(t) & t \in (0, T], \\ \\ v(0) = v_0. \end{cases} \tag{26}$$

Lemma 3.1. *The operator A is m-accretive and surjective in $X = L^p(\Omega)$.*

Proof. For beginning, let us show that A is accretive. Thus, if $\bar{v}, v \in D(A) \subset L^p(\Omega)$ and $y_1 = Av$, $y_2 = A\bar{v}$, by Green's formula we get

$$\|v - \bar{v}\|_{L^p(\Omega)}^{p-2}(y_1 - y_2, J(v - \bar{v}))_{L^p(\Omega)\times(L^p(\Omega))^*} = \frac{p_2}{p_1}\int_\Omega -\Delta(v - \bar{v})|v - \bar{v}|^{p-2}(v - \bar{v})dx$$

$$= \frac{p_2}{p_1}\int_\Omega |\nabla(v - \bar{v})|^2|v - \bar{v}|^{p-2}dx \geq 0,$$

which ensure that A is accretive. In the above writing, $J(y)(x) = |y(x)|^{p-2}y(x)\|y\|_{L^p(\Omega)}^{2-p}$ is the duality mapping of the space $L^p(\Omega)$.

It is well-known that for every $f \in L^p(\Omega) = \overline{D(A)}$ the problem

$$v - \lambda\frac{p_2}{p_1}\Delta v = f \in L^p(\Omega),$$

has a unique solution $v \in D(A)$ for every $\lambda > 0$ (see [5]), so the operator A is m-accretive. ∎

Lemma 3.2. *The operator B is m-ω-accretive.*

Proof. Let us choose

$$\omega \geq \frac{p_3}{p_1}. \tag{27}$$

If $v_1, v_2 \in D(B) = L^{3p}(\Omega) \subset X = L^p(\Omega) \subset L^2(\Omega)$ (see (17)), we have

$$\begin{aligned}
&\langle (B+\omega I)(v_1) - (B+\omega I)(v_2), v_1 - v_2\rangle_{L^2(\Omega)} \\
&= \frac{p_3}{p_1}\left\langle (v_1^3 - v_2^3) - (v_1 - v_2), v_1 - v_2\right\rangle_{L^2(\Omega)} + \omega\|v_1 - v_2\|^2_{L^2(\Omega)} \\
&\geq \frac{p_3}{p_1}\left\langle |v_1|^2 v_1 - |v_2|^2 v_2, v_1 - v_2\right\rangle_{L^2(\Omega)} - \frac{p_3}{p_1}\|v_1 - v_2\|^2_{L^2(\Omega)} + \omega\|v_1 - v_2\|^2_{L^2(\Omega)} \\
&= \frac{p_3}{p_1}\left\langle |v_1|^2 v_1 - |v_2|^2 v_2, v_1 - v_2\right\rangle_{L^2(\Omega)} + (\omega - \frac{p_3}{p_1})\|v_1 - v_2\|^2_{L^2(\Omega)} \geq 0,
\end{aligned}$$

where (27), as well as that the function $|v|^2 v$ is increasing, has been used.

It remains to check that $R(B + (\lambda + \omega)I) = X$, $\forall \lambda > 0$. Given $\beta \in L^p(\Omega)$, the problem

$$\frac{p_3}{p_1}(v^3 - v) + (\lambda + \omega)\varphi = \beta,$$

has a unique solution $v \in D(B)$ for every $\lambda > 0$. This expresses that the operator $B : D(B) \subset X \to X$ is m-ω-accretive. ∎

Lemmas 3.1 and 3.2 show that the conditions of Theorem 3.1 in [37] are satisfied and applying it we obtain the convergence stated by the present Theorem 3.1. ∎

4. TWO METHODS AND STABILITY

Let the time step $\varepsilon_M = \frac{T}{M}$, $M \geq 1$, be fixed, arbitrary, $t_n = n\varepsilon_M$, $n = 0, 1, \ldots, M$ (see the previous section), and assume that the initial data v_0 as well as the forcing term $f^n := f(t_n)$ are given (superscripts denote the time level of approximation). We consider the following two first-order methods for semi-discretization in time:

- the first method we analyze is an implicit-explicit (IMEX) scheme, which treats the nonlinearity by partial lagging (see also [40])

$$p_1 \frac{v^{n+1} - v^n}{\varepsilon_M} - p_2 \Delta v^{n+1} + p_3\left((v^n)^2 v^{n+1} - v^{n+1}\right) = f^{n+1}; \tag{28}$$

- the second method we consider is also an IMEX scheme, similar to the fractional steps type method considered in [4], [8], [21], [25], [32]-[35], [37], [40],

[43] and, of course, adapted to our new structure which is described in (24)-(25)

$$\begin{cases} p_1 \dfrac{v^{n+1} - \phi^n}{\varepsilon_M} - p_2 \Delta v^{n+1} = f^{n+1} \\ \\ \phi^n = v^n \left[1 \Big/ \left(z_M^3(0,\cdot) e^{-2\frac{p_3}{p_1}\varepsilon_M} + z_M^2(0,\cdot) e^{2\frac{p_3}{p_1}\varepsilon_M} \right)^{\frac{1}{2}} \right]. \end{cases} \tag{29}$$

We note that that ϕ_n is the value at ε_M of the exact solution of the ordinary differential equation (see (25))

$$z' = \frac{p_3}{p_1}\left(z - z^3\right)$$

on $[0, \varepsilon_M]$, with the initial condition $z_M(0,\cdot) = v^n = v_M^-(n\varepsilon_M, \cdot)$, $n = 0, 1, \ldots, M-1$, $v^0 = v_M^-(0,\cdot) = v_0(x)$. Moreover, the fractional step method (29) can be equivalently written as

$$p_1 \frac{v^{n+1} - v^n}{\varepsilon_M} - p_2 \Delta v^{n+1} +$$

$$+ \frac{p_1}{\varepsilon_M}(v^n)^3 \frac{v^n e^{-2\frac{p_3}{p_1}\varepsilon_M} + e^{2\frac{p_3}{p_1}\varepsilon_M}}{\left((v^n)^3 e^{-2\frac{p_3}{p_1}\varepsilon_M} + (v^n)^2 e^{2\frac{p_3}{p_1}\varepsilon_M}\right)^{\frac{1}{2}} \left(\left((v^n)^3 e^{-2\frac{p_3}{p_1}\varepsilon_M} + (v^n)^2 e^{2\frac{p_3}{p_1}\varepsilon_M}\right)^{\frac{1}{2}} + 1\right)} = f^{n+1}.$$

4.1. STABILITY OF THE IMEX METHOD

In this section we establish the numerical stability of the approximations in (28) by using a maximum principle (see also [40]). First we recall that (see, e.g., [5, Proposition 3.7.1]) if $u \in H^1(\Omega)$ then the function $u^+ := \max\{u, 0\}$ belongs to $H^1(\Omega)$ and moreover

$$\frac{\partial}{\partial x_i} u^+(x) = \begin{cases} \frac{\partial u}{\partial x_i}(x) & \text{a.e. in } \{x \in \Omega;\ u > 0\} \\ \\ 0 & \text{a.e. in } \{x \in \Omega;\ u \le 0\}. \end{cases} \tag{30}$$

In the remainder we assume that the initial data is smooth, and satisfies the following bound

$$v_0 \in W_p^1(\Omega), \qquad v_0(x) \in [-1, 1] \qquad \text{for a.e. } x \in \Omega. \tag{31}$$

In particular this implies that $v_0 \in W_p^1(\Omega) \subset L^\infty(\Omega)$ for all $p > 1$. Moreover, in this section we also assume that the forcing term f is zero.

Theorem 4.1. *Suppose that the initial data v_0 satisfies* (31) *and the time step is sufficiently small, i.e., $\varepsilon_M \leq \frac{p_1}{p_3}$. Then for all $n \geq 0$, the weak solutions v^n of* (28) *starting from the initial condition v_0 satisfy*

$$v^n(x) \in [-1, 1] \quad \textit{for a.e.} \;\; x \in \Omega. \tag{32}$$

Proof. The proof is based on the maximum principle for elliptic equations in Sobolev spaces, using homogeneous Neumann boundary conditions (see [5, Theorem 3.7.2]). We provide it here for the reader's convenience. We consider first the upper bound inequality, that is $v^n(x) \leq 1$ a.e. $x \in \Omega$. Let us proceed by induction on the index time level n. For time level zero, the result follows immediately. Assume that (32) holds for the nth time level and consider the $(n+1)$st time level.

From the weak formulation of (28), that is

$$\int_\Omega \left(\frac{p_1}{\varepsilon_M} - p_3 + p_3(v^n)^2\right) v^{n+1}\varphi dx + p_2 \int_\Omega \nabla v^{n+1} \cdot \nabla\varphi dx = \frac{p_1}{\varepsilon_M} \int_\Omega v^n \varphi dx \, \forall \varphi \in H^1(\Omega),$$

we obtain after some manipulation

$$\frac{p_1}{\varepsilon_M} \int_\Omega (v^{n+1} - 1)\varphi \, dx + + \int_\Omega p_3((v^n)^2 - 1)v^{n+1}\varphi \, dx \quad + p_2 \int_\Omega \nabla v^{n+1} \cdot \nabla\varphi \, dx = \\ = \frac{p_1}{\varepsilon_M} \int_\Omega (v^n - 1)\varphi \, dx,$$

$$\frac{p_1}{\varepsilon_M} \int_\Omega (v^{n+1} - 1)\varphi \, dx + \int_\Omega p_3((v^n)^2 - 1)(v^{n+1} - 1)\varphi \, dx + \quad \int_\Omega p_3((v^n)^2 - 1)\varphi \, dx + \\ + p_2 \int_\Omega (\nabla v^{n+1} - 1) \cdot \nabla\varphi dx \quad = \frac{p_1}{\varepsilon_M} \int_\Omega (v^n - 1)\varphi \, dx,$$

or equivalently

$$\int_\Omega \Big[\frac{p_1}{\varepsilon_M} + p_3((v^n)^2 - 1)\Big](v^{n+1} - 1)\varphi \, dx + \int_\Omega p_3((v^n)^2 - 1)\varphi \, dx + \\ + p_2 \int_\Omega (\nabla v^{n+1} - 1) \cdot \nabla\varphi \, dx = \frac{p_1}{\varepsilon_M} \int_\Omega (v^n - 1)\varphi \, dx,$$

$$\left(\frac{p_1}{\varepsilon_M} - p_3\right) \int_\Omega (v^{n+1} - 1)\varphi \, dx + p_3 \int_\Omega (v^n)^2 (v^{n+1} - 1)\varphi \, dx + \\ + p_2 \int_\Omega (\nabla v^{n+1} - 1) \cdot \nabla\varphi \, dx = \int_\Omega \Big[\frac{p_1}{\varepsilon_M} - p_3(v^n + 1)\Big](v^n - 1)\varphi \, dx.$$

Now we take as test function $\varphi = (v^{n+1} - 1)^+$ and use (30) to get

$$\left(\frac{p_1}{\varepsilon_M} - p_3\right)\int_\Omega |(v^{n+1} - 1)^+|^2\,dx + p_3\int_\Omega (v^n)^2|(v^{n+1} - 1)^+|^2\,dx +$$
$$+p_2\int_\Omega |\nabla(v^{n+1} - 1)^+|^2\,dx = \int_\Omega \left[\frac{p_1}{\varepsilon_M} - p_3(v^n + 1)\right](v^n - 1)(v^{n+1} - 1)^+ dx.$$

Finally, using the induction assumption we have

$$\left(\frac{p_1}{\varepsilon_M} - p_3\right)\int_\Omega |(v^{n+1} - 1)^+|^2\,dx + p_3\int_\Omega (v^n)^2|(v^{n+1} - 1)^+|^2\,dx +$$
$$+p_2\int_\Omega |\nabla(v^{n+1} - 1)^+|^2\,dx \le 0,$$

and, under the time step restriction $\varepsilon_M \le \frac{p_1}{p_3}$, we obtain that $(v^{n+1} - 1)^+ = 0$ a.e. in Ω, i.e., $v^{n+1}(x) \le 1$ a.e. $x \in \Omega$, completing thus the induction argument.

The lower bound inequality follows in a similar manner. ∎

Remark 3. *In* [6]*, a more restrictive conditional stability was reported for a slightly different linearized method, for the sharp interface problem, namely* $\varepsilon_M \le 2\varepsilon^2 \equiv 2\frac{p_1}{p_3}$.

5. ERROR ANALYSIS

Error estimates for the numerical schemes (28)-(29) will be presented in this Section. Let us introduce the notation

$$\mathcal{E}(\alpha,\beta) = \alpha e^{-2\frac{p_3}{p_1}\varepsilon_M} + \beta e^{2\frac{p_3}{p_1}\varepsilon_M}.$$

We begin by defining the *point-wise truncation errors* with respect to the time discretization, that is:

$$e^n := v(t_n) - v^n, \qquad \forall\ n = 0, 1, \dots M,$$

and the *local truncation errors*, respectively:

$$E^{n+1}_{imex} := p_1\frac{v(t_{n+1}) - v(t_n)}{\varepsilon_M} - p_2\Delta v(t_{n+1}) - p_3\left(v(t_{n+1}) - v(t_{n+1})v^2(t_n)\right) - f(t_{n+1}), \quad (33)$$

$$E^{n+1}_{os} := p_1\frac{v(t_{n+1}) - v(t_n)}{\varepsilon_M} - p_2\Delta v(t_{n+1}) \quad (34)$$
$$- \frac{p_1}{\varepsilon_M}v^3(t_n)\frac{\mathcal{E}(v(t_n), 1)}{\mathcal{E}^{1/2}\left(z_M^3(0,\cdot), z_M^2(0,\cdot)\right)\left[\mathcal{E}^{1/2}\left(z_M^3(0,\cdot), z_M^2(0,\cdot)\right) + 1\right]} - f(t_{n+1})$$

$\forall n = 0, 1, \dots, M$, with $z_M(0,\cdot) = v^-(n\varepsilon_M, \cdot) = \lim_{t\uparrow n\varepsilon_M} v(t,\cdot)$.

By subtracting (28)-(29) respectively from (51)-(52), we obtain the following equations in errors

$$\frac{p_1}{\varepsilon_M}\left(e^{n+1}-e^n\right)-p_2\Delta e^{n+1}-p_3e^{n+1}+p_3\mathcal{D}_{imex}^{n+1}=E_{imex}^{n+1}, \tag{35}$$

$$\frac{p_1}{\varepsilon_M}\left(e^{n+1}-e^n\right)-p_2\Delta e^{n+1}+p_3\mathcal{D}_{os}^{n+1}=E_{os}^{n+1}, \tag{36}$$

where

$$\mathcal{D}_{imex}^{n+1}=v^2(t_n)v(t_{n+1})-(v^n)^2v^{n+1}, \tag{37}$$

$$\begin{aligned}\mathcal{D}_{os}^{n+1}=&\frac{p_1}{\varepsilon_M}v^3(t_n)\frac{\mathcal{E}(v(t_n),1)}{\mathcal{E}^{1/2}\left(z_M^3(0,\cdot),z_M^2(0,\cdot)\right)\left[\mathcal{E}^{1/2}\left(z_M^3(0,\cdot),z_M^2(0,\cdot)\right)+1\right]}-\\&-\frac{p_1}{\varepsilon_M}(v^n)^3\frac{\mathcal{E}(v^n,1)}{\mathcal{E}^{1/2}\left((v^n)^3,(v^n)^2\right)\left[\mathcal{E}^{1/2}\left((v^n)^3,(v^n)^2\right)+1\right]}\end{aligned} \tag{38}$$

5.1. CONSISTENCY RESULT

In what follow we will assume that there exists a maximum principle result for the fractional steps method (29). We have the following result regarding consistency.

Lemma 5.1. *Assuming that the weak solution of* (1) *also satisfies* $v\in W_p^{1,2}(Q)$. *Then the local truncation errors satisfy*

$$\varepsilon_M\sum_{n=0}^{M-1}\|E_{imex}^{n+1}\|_{L^p(\Omega)}^p\le\varepsilon_M^p2^{p-1}\left[p_1^p\int_{t_0}^{t_M}\|v''(\tau)\|_{L^p(\Omega)}^pd\tau+2^pp_3^p\int_{t_0}^{t_M}\|v'(\tau)\|_{L^p(\Omega)}^pd\tau\right], \tag{39}$$

$$\begin{aligned}&\varepsilon_M\sum_{n=0}^{M-1}\|E_{os}^{n+1}\|_{L^p(\Omega)}^p\le\\&\le\varepsilon_M^p2^{2(p-1)}\int_{t_0}^{t_M}\left(p_1^p\|v''(\tau)\|_{L^p(\Omega)}^p+4^pp_3^p\|v'(\tau)\|_{L^p(\Omega)}^p\right)d\tau+2^{p-1}p_1^p\varepsilon_M^{1-p}.\end{aligned} \tag{40}$$

Proof. From the Taylor expansion we have

$$\frac{v(t_{n+1})-v(t_n)}{\varepsilon_M}=v'(t_{n+1})+\int_{t_n}^{t_{n+1}}v''(\tau)\frac{t_n-\tau}{\varepsilon_M}d\tau,$$

which substituting into (51) and using the original equation (1) evaluated at t_{n+1}, i.e.,

$$p_1v'(t_{n+1})-p_2\Delta v(t_{n+1})+p_3v^3(t_{n+1})-p_3v(t_{n+1})=f(t_{n+1}),$$

we obtain (see [40])

$$E_{imex}^{n+1} = p_1 \int_{t_n}^{t_{n+1}} v''(\tau)\frac{t_n - \tau}{\varepsilon_M} d\tau - p_3 v(t_{n+1})\big(v(t_n) + v(t_{n+1})\big) \int_{t_n}^{t_{n+1}} v'(\tau) d\tau.$$

Now, using the maximum principle for the exact solution yields

$$\|E_{imex}^{n+1}\|_{L^p(\Omega)} \le p_1 \left\| \int_{t_n}^{t_{n+1}} v''(\tau) d\tau \right\|_{L^p(\Omega)} + 2p_3 \left\| \int_{t_n}^{t_{n+1}} |v'(\tau)| d\tau \right\|_{L^p(\Omega)}.$$

Moreover, using $(a+b)^m \le 2^{m-1}(a^m + b^m)$ and the Hölder inequality we get

$$\|E_{imex}^{n+1}\|_{L^p(\Omega)}^p \le 2^{p-1}\varepsilon_M^{p-1} \int_{t_n}^{t_{n+1}} \Big(p_1^p \|v''(\tau)\|_{L^p(\Omega)}^p + 2^p p_3^p \|v'(\tau)\|_{L^p(\Omega)}^p\Big) d\tau,$$

then sum for $n = 0, \dots, M-1$ and multiply with ε_M to obtain (39).

Further, we will focus our attention on the inequality (40) expressing the consistency of the fractional steps method (29). Thus, making use of the following computation (see the original equation (1) evaluated at t_{n+1})

$$\begin{aligned}
&-p_3 v^3(t_{n+1}) + p_3 v(t_{n+1}) \\
&= -p_3\big(v^3(t_{n+1}) - v^3(t_n)\big) + p_3\big(v(t_{n+1}) - v(t_n)\big) - p_3\big(v^3(t_n) - v(t_n)\big) \\
&= -p_3\big(v(t_{n+1}) - v(t_n)\big)\big[v^2(t_{n+1}) + v(t_{n+1})v(t_n) + v^2(t_n) + 1\big] - p_3 v(t_n)\big(v^2(t_n) - 1\big),
\end{aligned}$$

then the local truncation error (52) writes

$$\begin{aligned}
E_{os}^{n+1} &= p_1 \int_{t_n}^{t_{n+1}} v''(\tau)\frac{t_n - \tau}{\varepsilon_M} d\tau - p_3 v^3(t_{n+1}) + p_3 v(t_{n+1}) \\
&\quad - \frac{p_1}{\varepsilon_M} v^3(t_n) \frac{\mathcal{E}(v(t_n), 1)}{\mathcal{E}^{1/2}\left(z_M^3(0,\cdot), z_M^2(0,\cdot)\right)\left[\mathcal{E}^{1/2}\left(z_M^3(0,\cdot), z_M^2(0,\cdot)\right) + 1\right]} \\
&= p_1 \int_{t_n}^{t_{n+1}} v''(\tau)\frac{t_n - \tau}{\varepsilon_M} d\tau - p_3\big[v^2(t_{n+1}) + v(t_{n+1})v(t_n) + v^2(t_n) + 1\big] \int_{t_n}^{t_{n+1}} v'(\tau)\, d\tau - \\
&\quad - \frac{p_1}{\varepsilon_M} v^3(t_n) \frac{\mathcal{E}(v(t_n), 1)}{\mathcal{E}^{1/2}\left(z_M^3(0,\cdot), z_M^2(0,\cdot)\right)\left[\mathcal{E}^{1/2}\left(z_M^3(0,\cdot), z_M^2(0,\cdot)\right) + 1\right]} - p_3 v(t_n)\big(v^2(t_n) - 1\big)
\end{aligned}$$

Let's observe that the square brackets from the last term in the relationship above is smaller or equal to 1. Consequently, in accordance with our work assumption regarding the existence of a maximum principle result for the exact solution in the fractional steps method (29), we have

$$\|E_{os}^{n+1}\|_{L^p(\Omega)} \leq p_1 \left\| \int_{t_n}^{t_{n+1}} |v''(\tau)| d\tau \right\|_{L^p(\Omega)} + 4p_3 \left\| \int_{t_n}^{t_{n+1}} |v'(\tau)|\, d\tau \right\|_{L^p(\Omega)} + \frac{p_1}{\varepsilon_M},$$

Now, using $(a+b)^m \leq 2^{m-1}(a^m + b^m)$ as well as the Hölder's inequality, we obtain

$$\|E_{os}^{n+1}\|_{L^p(\Omega)}^p \leq 2^{p-1} \left[\left\| \int_{t_n}^{t_{n+1}} (p_1|v''(\tau)| + 4p_3|v'(\tau)|)\, d\tau \right\|_{L^p(\Omega)}^p + \left(\frac{p_1}{\varepsilon_M}\right)^p \right]$$

$$\leq 2^{p-1} \left[2^{p-1}\varepsilon_M^{p-1} \int_{t_n}^{t_{n+1}} \left(p_1^p \|v''(\tau)\|_{L^p(\Omega)}^p + 4^p p_3^p \|v'(\tau)\|_{L^p(\Omega)}^p \right) d\tau + \left(\frac{p_1}{\varepsilon_M}\right)^p \right].$$

Finally, summing the previous inequality for $n = 0, ..., M-1$ and then multiplying the result with ε_M, we get (40) which conclude the proof of the Lemma 5.1. ∎

5.2. STABILITY RESULT

For future reference, we recall here the following form of the discrete Grönwall lemma. Assume $w_n, \alpha_n, q_n \geq 0, \beta \in [0,1)$ satisfy $w_n + q_n \leq \alpha_n + \beta \sum_{\kappa=1}^{n} w_\kappa \quad \forall \kappa \geq 0$, where $\{\alpha_n\}$ is non-decreasing. Then

$$w_n + \frac{q_n}{1-\beta} \leq \frac{\alpha_n - \beta w_0}{1-\beta} \exp\left(\frac{n\beta}{1-\beta}\right). \tag{41}$$

To obtain convergence results for the numerical methods (28) and (29), we will prove a stability result using energy estimates. In this respect, we begin by testing (53) and (55) with $e^{n+1}|e^{n+1}|^{p-2}$. We get

$$\underbrace{p_1 \int_\Omega \frac{e^{n+1} - e^n}{\varepsilon_M} e^{n+1}|e^{n+1}|^{p-2} dx}_{\mathbf{T1}} \underbrace{- p_2 \int_\Omega \Delta e^{n+1} e^{n+1}|e^{n+1}|^{p-2} dx}_{\mathbf{T2}} \tag{42}$$

$$- p_3 \int_\Omega |e^{n+1}|^p dx + \underbrace{p_3 \int_\Omega \mathcal{D}_{imex}^{n+1} e^{n+1}|e^{n+1}|^{p-2} dx}_{\mathbf{T3}} = \underbrace{\int_\Omega E_{imex}^{n+1} e^{n+1}|e^{n+1}|^{p-2} dx}_{\mathbf{T4}},$$

$$\underbrace{p_1 \int_\Omega \frac{e^{n+1}-e^n}{\varepsilon_M} e^{n+1}|e^{n+1}|^{p-2}dx}_{\mathbf{T1}} - \underbrace{p_2 \int_\Omega \Delta e^{n+1} e^{n+1}|e^{n+1}|^{p-2}dx}_{\mathbf{T2}} \tag{43}$$

$$+ \underbrace{p_3 \int_\Omega \mathcal{D}_{os}^{n+1} e^{n+1}|e^{n+1}|^{p-2}dx}_{\mathbf{T5}} = \underbrace{\int_\Omega E_{os}^{n+1} e^{n+1}|e^{n+1}|^{p-2}dx}_{\mathbf{T6}},$$

for any $p \geq 2$.

In the following, we will analyze each term in (56)-(43) separately.

T1. The first term, involving the finite difference approximation of the time derivative, is evaluated using Young's inequality

$$ab \leq \frac{1}{m}a^m + \frac{1}{n}b^n, \qquad \frac{1}{m} + \frac{1}{n} = 1, \qquad a, b \in \mathbb{R}^+, \tag{44}$$

with

$$a = (e^{n+1})^{p-1}, \quad b = e^n, \quad m = \frac{p}{p-1}, \quad n = p.$$

We have

$$\int_\Omega (e^{n+1}-e^n)e^{n+1}|e^{n+1}|^{p-2}\,dx = \int_\Omega (|e^{n+1}|^p - \underbrace{e^{n+1}|e^{n+1}|^{p-2}}_{\frac{p}{p-1}} \underbrace{e^n}_{p})dx$$

$$\geq \int_\Omega (|e^{n+1}|^p - \frac{p-1}{p}|e^{n+1}|^p - \frac{1}{p}|e^n|^p)dx = \frac{1}{p}\int_\Omega (|e^{n+1}|^p - |e^n|^p)dx,$$

which enables us to obtain the following inequality

$$\frac{1}{p}(\|e^{n+1}\|^p_{L^p(\Omega)} - \|e^n\|^p_{L^p(\Omega)}) \leq \int_\Omega (e^{n+1}-e^n)e^{n+1}|e^{n+1}|^{p-2}\,dx. \tag{45}$$

T2. Using (30) and the homogeneous Neumann boundary conditions, the diffusion term gives by integration by parts:

$$\begin{aligned}
&(-\Delta e^{n+1}, e^{n+1}|e^{n+1}|^{p-2}) \qquad (46)\\
&= -\int_{\partial\Omega} \frac{\partial e^{n+1}}{\partial \nu} e^{n+1}|e^{n+1}|^{p-2}d\sigma + (p-1)\int_\Omega |\nabla e^{n+1}|^2|e^{n+1}|^{p-2}dx\\
&= (p-1)\int_\Omega |\nabla e^{n+1}|^2|e^{n+1}|^{p-2}dx.
\end{aligned}$$

T3 & T5. Next we analyze the nonlinear terms $\mathcal{D}_{imex}^{n+1}$ and $\mathcal{D}_{os}^{n+1}$ defined in (37)-(38).

Lemma 5.2. *Assume that hypotheses of Theorem 4.1 hold, and also there exists a maximum principle result in the case of method* (29)*. Then*

$$\int_\Omega \mathcal{D}_{imex}^{n+1}\, e^{n+1}|e^{n+1}|^{p-2}\,dx \geq \int_\Omega |e^{n+1}|^p v^2(t_n)\,dx - \frac{p-1}{p}\int_\Omega |e^{n+1}|^p\,dx - \frac{1}{p}\int_\Omega |e^n|^p\,dx, \tag{47}$$

$$\int_\Omega \mathcal{D}_{os}^{n+1} e^{n+1}|e^{n+1}|^{p-2}\,dx \geq \frac{p_1\varepsilon_M}{4}\left[\frac{1}{e^{8\frac{p_3}{p_1}\varepsilon_M}} + e^{4\frac{p_3}{p_1}\varepsilon_M}\right]\int_\Omega \left(\frac{p-1}{p}|e^{n+1}|^p + \frac{1}{p}|e^n|^p\right)dx. \tag{48}$$

Proof. For the term $\mathcal{D}_{imex}^{n+1}$ corresponding to the linearized scheme (28), it follows that

$$\begin{aligned}
\int_\Omega \mathcal{D}_{imex}^{n+1}\, e^{n+1}|e^{n+1}|^{p-2}\,dx &= \int_\Omega \Big(v(t_{n+1})v^2(t_n) - v^{n+1}(v^n)^2\Big)e^{n+1}|e^{n+1}|^{p-2}\,dx \\
&= \int_\Omega \Big((v(t_{n+1}) - v^{n+1})v^2(t_n) + v^{n+1}(v^2(t_n) - (v^n)^2)\Big)e^{n+1}|e^{n+1}|^{p-2}\,dx \\
&= \int_\Omega \Big(e^{n+1}v^2(t_n) + v^{n+1}e^n(v(t_n) + v^n)\Big)e^{n+1}|e^{n+1}|^{p-2}\,dx \\
&= \int_\Omega |e^{n+1}|^p v^2(t_n)\,dx + \int_\Omega e^{n+1}|e^{n+1}|^{p-2}\,e^n\;v^{n+1}(v(t_n) + v^n)\,dx.
\end{aligned}$$

Since the time-step $\varepsilon_M \leq \frac{p_1}{p_3}$ (see Theorem 4.1), by the maximum principle for the exact solution as well as the relation (32) and Young's inequality, we derive

$$\begin{aligned}
&\int_\Omega \mathcal{D}_{imex}^{n+1}\, e^{n+1}|e^{n+1}|^{p-2}\,dx \\
&\quad= \int_\Omega |e^{n+1}|^p v^2(t_n)\,dx + \int_\Omega e^{n+1}|e^{n+1}|^{p-2}\,e^n\;v^{n+1}(v(t_n) + v^n)\,dx \\
&\quad\geq \int_\Omega |e^{n+1}|^p v^2(t_n)\,dx - 2\int_\Omega |e^{n+1}|^{p-1}|e^n|\,dx \\
&\quad\geq \int_\Omega |e^{n+1}|^p v^2(t_n)\,dx - \frac{p-1}{p}\int_\Omega |e^{n+1}|^p\,dx - \frac{1}{p}\int_\Omega |e^n|^p\,dx,
\end{aligned}$$

from where we easily obtain the relation (47).

Now, we pass to verify the estimate (48). For beginning, let's make the following notations:

$A = |v(t_n)|\left(v(t_n)e^{-2\frac{p_3}{p_1}\varepsilon_M} + e^{2\frac{p_3}{p_1}\varepsilon_M}\right)^{\frac{1}{2}}$, with $v(t_n) = v_M(t_n, \cdot)$ (see $(24)_3$)

and

$B = |v^n|\left(v^n e^{-2\frac{p_3}{p_1}\varepsilon_M} + e^{2\frac{p_3}{p_1}\varepsilon_M}\right)^{\frac{1}{2}}$.

Then, in the case of fractional steps method (29), assuming also that there is a maximum principle, after some manipulations we obtain from (38)

$$\int_\Omega \mathcal{D}_{os}^{n+1} e^{n+1}|e^{n+1}|^{p-2}\,dx =$$

$$= \frac{p_1}{\varepsilon_M}\int_\Omega \left[v^3(t_n)\frac{\mathcal{E}\,(v(t_n),1)}{A(A+1)} - (v^n)^3\frac{\mathcal{E}\,(v^n,1)}{B(B+1)}\right]e^{n+1}|e^{n+1}|^{p-2}\,dx$$

$$= \frac{p_1}{\varepsilon_M}\int_\Omega \frac{v^2(t_n)(v^n)^2\mathcal{E}\,(v(t_n),1)\,\mathcal{E}\,(v^n,1)}{A(A+1)B(B+1)}e^n|e^{n+1}|^{p-1}\,dx$$

$$+ \frac{p_1}{\varepsilon_M}\int_\Omega \frac{v^3(t_n)\mathcal{E}\,(v(t_n),1)\,B - (v^n)^3\mathcal{E}\,(v^n,1)\,A}{A(A+1)B(B+1)}|e^{n+1}|^{p-1}\,dx$$

$$\geq \frac{p_1}{\varepsilon_M}\int_\Omega \frac{e^{-2\frac{p_3}{p_1}\varepsilon_M}\,(v(t_n)+1)\,e^{-2\frac{p_3}{p_1}\varepsilon_M}\,(v^n+1)}{A(A+1)B(B+1)}e^n|e^{n+1}|^{p-1}\,dx$$

$$+ \frac{p_1}{\varepsilon_M}\frac{1}{(A+1)(B+1)}\int_\Omega \left[\frac{\mathcal{E}^{1/2}\,(v^n,1)}{|v^n|} - \frac{\mathcal{E}^{1/2}\,(v(t_n),1)}{|v(t_n)|}\right]|e^{n+1}|^{p-1}\,dx$$

$$\geq \frac{p_1}{\varepsilon_M}\frac{(\varepsilon_M e^{-2\frac{p_3}{p_1}\varepsilon_M})^2}{4\left(e^{2\frac{p_3}{p_1}\varepsilon_M}\right)^2}\int_\Omega e^n|e^{n+1}|^{p-1}\,dx + \frac{p_1}{\varepsilon_M}\frac{\left(\varepsilon_M e^{2\frac{p_3}{p_1}\varepsilon_M}\right)^2}{4}\int_\Omega e^n|e^{n+1}|^{p-1}\,dx$$

$$= \frac{p_1\varepsilon_M}{4}\left[\frac{1}{e^{8\frac{p_3}{p_1}\varepsilon_M}} + e^{4\frac{p_3}{p_1}\varepsilon_M}\right]\int_\Omega e^n|e^{n+1}|^{p-1}\,dx$$

$$\geq \frac{p_1\varepsilon_M}{4}\left[\frac{1}{e^{8\frac{p_3}{p_1}\varepsilon_M}} + e^{4\frac{p_3}{p_1}\varepsilon_M}\right]\int_\Omega \Big(\frac{1}{p}|e^n|^p + \frac{p-1}{p}|e^{n+1}|^p\Big)\,dx.$$

which means that the claim in (48) holds true. This achieves the proof of Lemma 5.2. ∎

T4 & T6. Using the Young inequality (44) with $a = |E^{n+1}|$, $b = |e^{n+1}|^{p-1}$, $m = p$ and $n = \frac{p}{p-1}$, the local truncation error term gives,

$$(E^{n+1}, e^{n+1}|e^{n+1}|^{p-2}) \leq \frac{1}{p}\|E^{n+1}\|^p_{L^p(\Omega)} + \frac{p-1}{p}\|e^{n+1}\|^p_{L^p(\Omega)}, \tag{49}$$

where $E^{n+1} = E^{n+1}_{imex}$ or $E^{n+1} = E^{n+1}_{os}$.

As we have proposed previously, at this point we have the estimations (45)-(49), corresponding to the terms **T1 - T6**. Now we will continue with the evaluation of relations (56)-(43). We first substitute in (56) the relations (45)-(46) and (49) corresponding to $E^{n+1} = E^{n+1}_{imex}$ to obtain

$$\begin{aligned}
&\frac{p_1}{p\varepsilon_M}\int_\Omega (|e^{n+1}|^p - |e^n|^p)\,dx \qquad (50)\\
&\quad + p_2(p-1)\int_\Omega |\nabla e^{n+1}|^2|e^{n+1}|^{p-2}\,dx + p_3\int_\Omega \mathcal{D}^{n+1}_{imex}e^{n+1}|e^{n+1}|^{p-2}\,dx\\
&\quad \leq p_3\|e^{n+1}\|^p_{L^p(\Omega)} + \frac{1}{p}\|E^{n+1}_{imex}\|^p_{L^p(\Omega)} + \frac{p-1}{p}\|e^{n+1}\|^p_{L^p(\Omega)}\\
&\quad = \frac{1}{p}\|E^{n+1}_{imex}\|^p_{L^p(\Omega)} + \Big(p_3 + \frac{p-1}{p}\Big)\|e^{n+1}\|^p_{L^p(\Omega)}.
\end{aligned}$$

Sum for $n = 0, \dots, M-1$

$$\frac{p_1}{p\varepsilon_M}\|e^M\|^p_{L^p(\Omega)} + p_2(p-1)\sum_{n=1}^{M}\int_\Omega |\nabla e^n|^2|e^n|^{p-2}\,dx + p_3\sum_{n=1}^{M}\int_\Omega \mathcal{D}^n_{imex}\,e^n|e^n|^{p-2}\,dx$$
$$\le \frac{p_1}{p\varepsilon_M}\|e^0\|^p_{L^p(\Omega)} + \frac{1}{p}\sum_{n=1}^{M}\|E^n_{imex}\|^p_{L^p(\Omega)} + \Big(p_3 + \frac{p-1}{p}\Big)\sum_{n=1}^{M}\|e^n\|^p_{L^p(\Omega)},$$

and multiplying by $\varepsilon_M \frac{p}{p_1}$ yields

$$\begin{aligned}&\|e^M\|^p_{L^p(\Omega)} \\ &+ p(p-1)\frac{p_2}{p_1}\varepsilon_M\sum_{n=1}^{M}\int_\Omega |\nabla e^n|^2|e^n|^{p-2}\,dx + p\frac{p_3}{p_1}\varepsilon_M\sum_{n=1}^{M}\int_\Omega \mathcal{D}^n_{imex}e^n|e^n|^{p-2}\,dx \\ &\le \|e^0\|^p_{L^p(\Omega)} + \frac{1}{p_1}\varepsilon_M\sum_{n=1}^{M}\|E^n_{imex}\|^p_{L^p(\Omega)} + \Big(p_3 + \frac{p-1}{p}\Big)\frac{p}{p_1}\varepsilon_M\sum_{n=1}^{M}\|e^n\|^p_{L^p(\Omega)}.\end{aligned} \tag{51}$$

In the IMEX case we obtain from (47) and (51)

$$\begin{aligned}&\|e^M\|^p_{L^p(\Omega)} \\ &+ p(p-1)\frac{p_2}{p_1}\varepsilon_M\sum_{n=1}^{M}\int_\Omega |\nabla e^n|^2|e^n|^{p-2}\,dx + p\frac{p_3}{p_1}\varepsilon_M\sum_{n=1}^{M}\int_\Omega |e^n|^p v^2(t_n)\,dx \\ &\le (1 + \frac{p_3}{p_1}\varepsilon_M)\|e^0\|^p_{L^p(\Omega)} + \frac{1}{p_1}\varepsilon_M\sum_{n=1}^{M}\|E^n_{imex}\|^p_{L^p(\Omega)} \\ &\qquad + \Big(2p_3 + \frac{p-1}{p}\Big)\frac{p}{p_1}\varepsilon_M\sum_{n=1}^{M}\|e^n\|^p_{L^p(\Omega)}.\end{aligned}$$

Assume now that the time step of the IMEX method satisfies

$$\varepsilon_M \le \varepsilon_{imex} := \frac{p_1}{2pp_3 + p - 1},$$

then again by Grönwall inequality (41) we obtain

$$\|e^M\|^p_{L^p(\Omega)} + p(p-1)\frac{p_2}{p_1}\frac{\varepsilon_M}{1-\varepsilon_M/\varepsilon_{imex}}\sum_{n=1}^{M}\int_\Omega |\nabla e^n|^2|e^n|^{p-2}\,dx$$
$$+\,p\frac{p_3}{p_1}\frac{\varepsilon_M}{1-\varepsilon_M/\varepsilon_{imex}}\sum_{n=1}^{M}\int_\Omega |e^n|^p v^2(t_n)\,dx$$
$$\le \frac{\left(1-\frac{\varepsilon_M}{\varepsilon_{imex}}+\frac{p_3}{p_1}\varepsilon_M\right)\|e^0\|^p_{L^p(\Omega)}+\frac{1}{p_1}\varepsilon_M\sum_{n=1}^{M}\|E^n_{imex}\|^p_{L^p(\Omega)}}{1-\varepsilon_M/\varepsilon_{imex}}\times\exp\left(\frac{M\varepsilon_M}{\varepsilon_{imex}-\varepsilon_M}\right).$$

In the fractional steps method (29) we obtain from (45)-(46) and (49) corresponding to $E^{n+1} = E^{n+1}_{os}$

$$\|e^M\|^p_{L^p(\Omega)} + p_2(p-1)\frac{p}{p_1}\varepsilon_M\sum_{n=1}^{M}\int_\Omega |\nabla e^n|^2|e^n|^{p-2}\,dx+$$
$$+\,\frac{p}{p_1}\varepsilon_M\frac{p_1\varepsilon_M}{4}\left[\frac{1}{e^{8\frac{p_3}{p_1}\varepsilon_M}}+e^{4\frac{p_3}{p_1}\varepsilon_M}\right]\sum_{n=1}^{M}\int_\Omega |e^n|^p\,dx$$
$$\le \|e^0\|^p_{L^p(\Omega)}+\frac{1}{p_1}\varepsilon_M\sum_{n=1}^{M}\|E^n_{os}\|^p_{L^p(\Omega)}+\frac{p-1}{p}\frac{p}{p_1}\varepsilon_M\sum_{n=1}^{M}\|e^n\|^p_{L^p(\Omega)}.$$

Therefore, assuming that the time-step is small enough

$$\varepsilon_M \le \varepsilon_{os} := \frac{p_1}{p-1},$$

we obtain from Grönwall's inequality (41) the following stability estimate

$$\|e^M\|^p_{L^p(\Omega)} + p(p-1)\frac{p_2}{p_1}\frac{\varepsilon_M}{1-\varepsilon_M/\varepsilon_{os}}\sum_{n=1}^{M}\int_\Omega |\nabla e^n|^2|e^n|^{p-2}\,dx+$$
$$+\,\frac{p}{4}\varepsilon_M^2\left[\frac{1}{e^{8\frac{p_3}{p_1}\varepsilon_M}}+e^{4\frac{p_3}{p_1}\varepsilon_M}\right]\sum_{n=1}^{M}\int_\Omega |e^n|^p\,dx$$
$$\le \frac{\left(1-\frac{\varepsilon_M}{\varepsilon_{os}}\right)\|e^0\|^p_{L^p(\Omega)}+\frac{1}{p_1}\varepsilon_M\sum_{n=1}^{M}\|E^n_{os}\|^p_{L^p(\Omega)}}{1-\varepsilon_M/\varepsilon_{os}}\times\exp\left(\frac{M\varepsilon_M}{\varepsilon_{os}-\varepsilon_M}\right).$$

Now we collect the stability estimates for the error equations we proven so far in the following result.

Lemma 5.3. *Assuming the time-step ε_M for methods* (28) *and* (29) *satisfy respectively*

$$\begin{aligned} \varepsilon_M &\le \varepsilon_{imex} := \tfrac{p_1}{2pp_3+p-1}, \\ \varepsilon_M &\le \varepsilon_{os} := \tfrac{p_1}{p-1}, \end{aligned} \tag{52}$$

then the errors satisfy the following estimates

$$\begin{aligned} &\|e^M\|^p_{L^p(\Omega)} + p(p-1)\frac{p_2}{p_1}\frac{\varepsilon_M}{1-\varepsilon_M/\varepsilon_{imex}}\sum_{n=1}^{M}\int_\Omega |\nabla e^n|^2|e^n|^{p-2}\,dx + \\ &+p\frac{p_3}{p_1}\frac{\varepsilon_M}{1-\varepsilon_M/\varepsilon_{imex}}\sum_{n=1}^{M}\int_\Omega |e^n|^p v^2(t_n)\,dx \le \exp\left(\frac{M\varepsilon_M}{\varepsilon_{imex}-\varepsilon_M}\right)\cdot \\ &\quad\cdot\left[\left(1+\frac{p_3}{p_1}\frac{\varepsilon_M}{1-\varepsilon_M/\varepsilon_{imex}}\right)\|e^0\|^p_{L^p(\Omega)} + \frac{1}{p_1}\frac{\varepsilon_M}{1-\varepsilon_M/\varepsilon_{imex}}\sum_{n=1}^{M}\|E^n_{imex}\|^p_{L^p(\Omega)}\right], \end{aligned} \tag{53}$$

$$\begin{aligned} &\|e^M\|^p_{L^p(\Omega)} + p(p-1)\frac{p_2}{p_1}\frac{\varepsilon_M}{1-\varepsilon_M/\varepsilon_{os}}\sum_{n=1}^{M}\int_\Omega |\nabla e^n|^2|e^n|^{p-2}\,dx + \\ &\quad+\frac{p}{4}\varepsilon_M^2\left[\frac{1}{e^{8\frac{p_3}{p_1}\varepsilon_M}} + e^{4\frac{p_3}{p_1}\varepsilon_M}\right]\sum_{n=1}^{M}\int_\Omega |e^n|^p\,dx \le \\ &\quad\le \exp\left(\frac{M\varepsilon_M}{\varepsilon_{os}-\varepsilon_M}\right)\left(\|e^0\|^p_{L^p(\Omega)} + \frac{1}{p_1}\frac{\varepsilon_M}{1-\varepsilon_M/\varepsilon_{os}}\sum_{n=1}^{M}\|E^n_{os}\|^p_{L^p(\Omega)}\right). \end{aligned} \tag{54}$$

Finally we can prove that methods (28)-(29) are **first order** accurate in time.

Theorem 5.1. *Assume that the time steps ε_M satisfies the assumption formulated in* (52) *and that the exact solution v to problem* (1) *is also $W^{1,2}_p(Q)$ regular. Then*

methods (28)-(29) *satisfy the following error estimates*

$$\|e^M\|^p_{L^p(\Omega)} + p(p-1)\frac{p_2}{p_1}\frac{\varepsilon_M}{1-\varepsilon_M/\varepsilon_{imex}}\sum_{n=1}^{M}\int_\Omega |\nabla e^n|^2|e^n|^{p-2}\,dx+$$
$$+p\frac{p_3}{p_1}\frac{\varepsilon_M}{1-\varepsilon_M/\varepsilon_{imex}}\sum_{n=1}^{M}\int_\Omega |e^n|^p v^2(t_n)\,dx$$
$$\leq \exp\Big(\frac{M\varepsilon_M}{\varepsilon_{imex}-\varepsilon_M}\Big)\Bigg[\Big(1+\frac{p_3}{p_1}\frac{\varepsilon_M}{1-\varepsilon_M/\varepsilon_{imex}}\Big)\|e^0\|^p_{L^p(\Omega)}$$
$$+\frac{1}{p_1}\frac{\varepsilon_M^p 2^{p-1}}{1-\varepsilon_M/\varepsilon_{imex}}\left(p_1^p\int_{t_0}^{t_M}\|v''(\tau)\|^p_{L^p(\Omega)}d\tau + 2^p p_3^p\int_{t_0}^{t_M}\|v'(\tau)\|^p_{L^p(\Omega)}d\tau\right)\Bigg], \tag{55}$$

$$\|e^M\|^p_{L^p(\Omega)} + p(p-1)\frac{p_2}{p_1}\frac{\varepsilon_M}{1-\varepsilon_M/\varepsilon_{os}}\sum_{n=1}^{M}\int_\Omega |\nabla e^n|^2|e^n|^{p-2}\,dx+$$
$$+\frac{p}{4}\varepsilon_M^2\left[\frac{1}{e^{8\frac{p_3}{p_1}\varepsilon_M}}+e^{4\frac{p_3}{p_1}\varepsilon_M}\right]\sum_{n=1}^{M}\int_\Omega |e^n|^p\,dx$$
$$\leq \exp\Big(\frac{M\varepsilon_M}{\varepsilon_{os}-\varepsilon_M}\Big)\Big[\|e^0\|^p_{L^p(\Omega)}+$$
$$+\frac{1}{p_1}\frac{\varepsilon_M^p 2^{p-1}}{1-\varepsilon_M/\varepsilon_{os}}\left(\int_{t_0}^{t_M}\big\|p_1|v''(\tau)|+3p_3|v'(\tau)|\big\|^p_{L^p(\Omega)}\,d\tau + T(\frac{3}{2}\frac{p_3^2}{p_1})^p|\Omega|\right)\Bigg]. \tag{56}$$

Proof. Using Lemmas 5.1 and 5.3, we can easy deduce that relations (55) and (56) are true. ∎

6. NUMERICAL EXAMPLES

We will compare the two numerical solutions obtained by (28) and (29) with the following exact solution to (1)

$$v_e(t,x) := \exp(-2\omega^2 t)\cos\left(\pi\frac{x}{b}\right), \quad t\in[0,T], \quad x\in[0,b],$$

with the forcing term

$$f_{v_e}(t,x) = e^{-2\omega^2 t}\cos\left(\pi\frac{x}{b}\right)\left[-2\omega^2 p_1 - p_2\left(\frac{\pi}{b}\right)^2 - p_3\left(1-\exp(-4\omega^2 t)\cos^2\left(\pi\frac{x}{b}\right)\right)\right].$$

In numerical tests we will consider a particular case of the nonlinear reaction-diffusion equation (1), namely, the Allen-Cahn equation ([1]), which means $p_1 = \alpha * \xi$, $p_2 = \xi$ and $p_3 = \frac{1}{2\xi}$. For beginning, taking $T = 1$, $\omega = 0.5$, $b = 1$, $\alpha = 1.0e + 2$, $\xi = .5$, $N = 31$, $\varepsilon_M = 0.1$, $M = T/\varepsilon_M$, the errors $\|v_e - v_j^N\|_\infty$ produced by three methods analyzed in [40] (the Newton method, the linearized method and the old fractional steps method), as well as the new fractional steps method (29), are shown in Figure 1. The numerical experiment was performed further with $T = 2, b = 1$ (see Figure 2) and $T = 2, b = 2$ (see Figure 3). The approximate solution v_j^M, $j = 1.2, ..., N$ was computed iteratively for $\varepsilon_M = \varepsilon_M/k$, $k = 1, 2, ..., 5$.

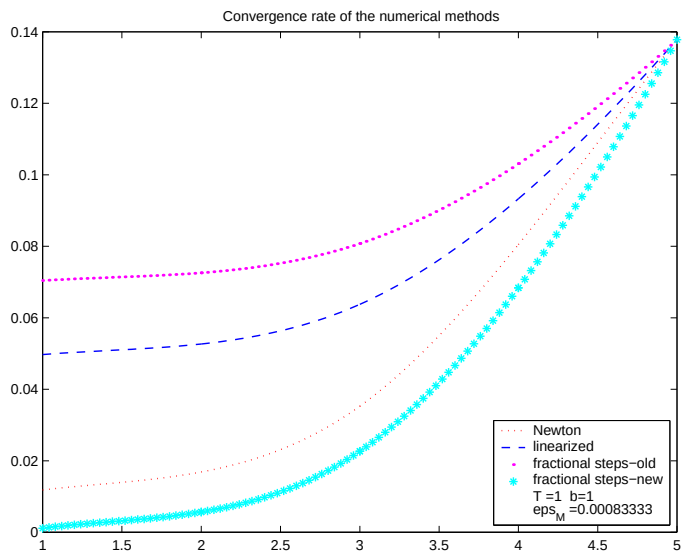

Fig. 1.: *Errors* $\|v_e - v_j^M\|_\infty$ *of the Newton, the linearized and the fractional steps methods*

7. CONCLUSIONS

As a novelty of this work we refer firstly to the new scheme of fractional steps type, introduced by (29) in order to approximate the solution to the nonlinear reaction-diffusion problem (1). Corresponding, we have considered an IMEX scheme and we have proved stability result for the error equation, which shows us that the new fractional steps method depend linearly on the small parameter, like as in the methods studied in [40].

Secondly, in the numerical experiments we focus our attention on a particular case of (1) - the Allen-Cahn equation, which serves as a mathematical model for many complex moving interface problems and in which the challenge in terms of numerical analysis is due to the thickness of the interface separating different phases.

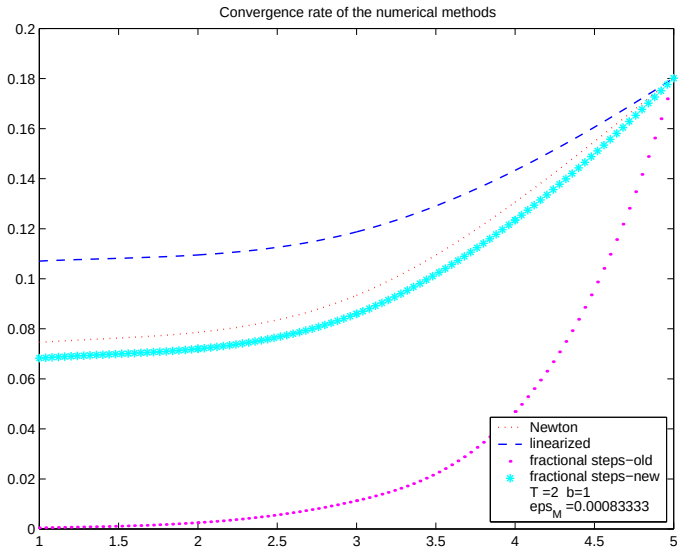

Fig. 2.: *Errors* $\|v_e - v_j^M\|_\infty$ *of the Newton, the linearized and the fractional steps methods*

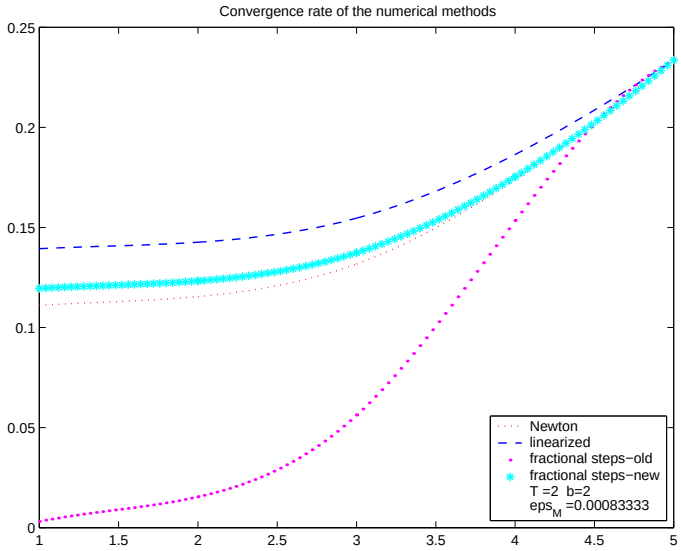

Fig. 3.: *Errors* $\|v_e - v_j^M\|_\infty$ *of the Newton, the linearized and the fractional steps methods*

Not least, let's remark from the graphical representation of the errors (see Figures 1-3), produced by those four methods analyzed, that conditions of consistency

and stability are sustained by both theory and numerical experiment and that are significantly influenced by the parameters of time and space (see [38] for detailed discussions regarding the dependence on the physical parameters).

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