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SECOND ORDER APPROXIMATIONS OF THE HOMOCLINIC SOLUTIONS FOR THE FITZHUGH-NAGUMO SYSTEM

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Abstract The FitzHugh -Nagumo system modeling the electrical potential of the heart is considered. The Bogdanov-Takens bifurcation is detected using the normal form method. Then first- and second-order approximations of the curve containing homoclinic bifurcation values are obtained. In addition, we obtain first- and second-order approximations for the homoclinic orbits. The solutions obtained theoretically are compared with those obtained numerically for several values of the parameters.
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1. INTRODUCTION

Consider the FitzHugh-Nagumo system [5] written into the form

$$\begin{cases} \frac{dx}{dt} = c\left(x + y - \frac{x^3}{3}\right), \\ \frac{dy}{dt} = -\frac{1}{c}(x - a + by), \end{cases} \tag{1}$$

where x is the electrical potential of the nodal cell membrane, y is an auxiliary variable depending on the refractory period, the parameters a and b are related to the number of channels of the cell membrane which are opened to the Na^+ and K^+ ions and $c \neq 0$ is the relaxation parameter. For $a = b = 0$, the well-known Van der Pol system is obtained [1].

A complete study concerning the bifurcations of system (1) can be found in [9], [10], where a rich phase dynamics is emphasized as the parameters a and b vary, while c is fixed.

Since we are interested on the homoclinic orbits of system (1) which are present for parameters situated on some curves in the (b, a) parameter plane, emerging from the Bogdanov-Takens bifurcation points, we mention some results concerning this type of bifurcation.

Consider the following family of systems of o.d.e.'s

$$\dot{\mathbf{x}} = X_\mu(\mathbf{x}), \mathbf{x} \in \mathbf{R}^2, \boldsymbol{\mu} \in \mathbf{R}^2, X_\mu \in C^\infty\left(\mathbf{R}^2\right). \tag{2}$$

Let $\mathbf{x} = \mathbf{x}_0$ be a nonhyperbolic equilibrium point of (2), for $\boldsymbol{\mu} = \boldsymbol{\mu}_0$.

The point $(\mathbf{x}_0, \boldsymbol{\mu}_0)$ is a Bogdanov-Takens bifurcation for the family (2) if (2) is topologically equivalent around $(\mathbf{x}_0, \boldsymbol{\mu}_0)$ with the following family of o.d.e.'s [7]:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = \nu_1 + \nu_2 x_1 + x_1^2 + \sigma x_1 x_2, \end{cases} \tag{3}$$

around the origin $(\mathbf{0}, \mathbf{0})$. Here $\sigma = \pm 1$. This is a codimension two bifurcation.

As the Jacobi matrix at the Bogdanov-Takens bifurcation point is not diagonalizable, the normal form for this bifurcation is not unique. Another normal form slightly different of (3), but equivalent to it, is [6], [3]:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = \gamma_1 + \gamma_2 x_2 + x_1^2 + \sigma x_1 x_2, \end{cases} \tag{4}$$

with $\sigma = \pm 1$.

For $\sigma = -1$, the bifurcation diagram for (4) is given in Figure 1. The parametric portrait is divided in 9 different strata by the origin and the following curves:

(i) $SN^+ = \{\gamma \in \mathbf{R}^2, \gamma_1 = 0, \gamma_2 > 0\}$;

(ii) $SN^- = \{\gamma \in \mathbf{R}^2, \gamma_1 = 0, \gamma_2 < 0\}$;

(iii) $H = \{\gamma \in \mathbf{R}^2, \gamma_1 = -\gamma_2^2, \gamma_2 < 0\}$;

(iv) $HL = \{\gamma \in \mathbf{R}^2, \gamma_1 = -\frac{49}{25}\gamma_2^2 + O(\gamma_2^{5/2}), \gamma_2 < 0\}$.

The curve H corresponds to supercritical Hopf bifurcation values, SN^+, SN^- correspond to saddle-node bifurcation values, whilst HL corresponds to homoclinic bifurcation values. The system (4) has a single attractive limit cycle in region 5 and has no limit cycle outside this region.

Note that if $\sigma = 1$ in (4), by the change of coordinates, time and parameters $(x_1, x_2, \gamma_1, \gamma_2, t) \to (x_1, -x_2, \gamma_1, -\gamma_2, -t)$, one can obtain (4) with $\sigma = -1$. Consequently, for $\sigma = 1$ the parametric portrait is symmetric of the one from Figure 1 with respect to the $0\gamma_1$ axis, while in the phase portraits, the time is reversed on the trajectories, thus the limit cycle is repulsive. In this case the curve H corresponds to subcritical Hopf bifurcation values.

This paper is structured as follows. In Section 2 the normal form (4) corresponding to the Bogdanov-Takens bifurcation is derived for the FitzHugh-Nagumo system (1). Using this normal form, the local representation of the curve of homoclinic bifurcation values, emerging from the Bogdanov-Takens bifurcation point is found. In Section 3, using the method from [8], we obtain second-order asymptotic approximations both for the curve of homoclinic bifurcation values and for the homoclinic orbits existing for parameters situated on this curve, in a neighborhood of the Bogdanov-Takens bifurcation point. In the last Section the numerical solutions are compared with the theoretical ones and concluding remarks are presented.

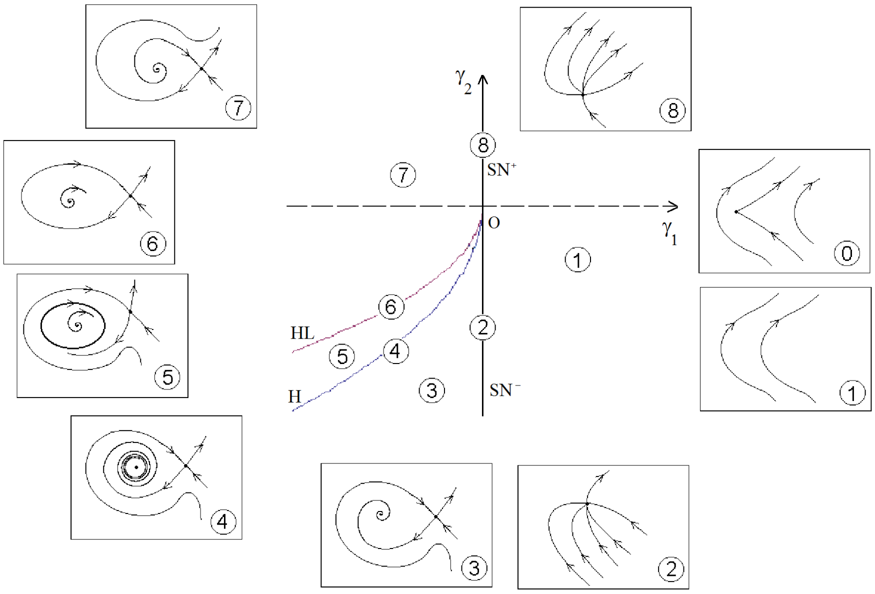

Fig. 1.: The bifurcation diagram for the Bogdanov-Takens bifurcation.

2. BOGDANOV-TAKENS BIFURCATION FOR THE FITZHUGH-NAGUMO SYSTEM

The equilibria and their corresponding eigenvalues for the FitzHugh-Nagumo system (1) were computed in [9]. In the following we emphasize some of the codimension one bifurcations. The saddle-node bifurcation values, corresponding to saddle-node equilibria, are situated on the surfaces

$$S_{1,2} : a = \pm\frac{2b}{3}\left(1-\frac{1}{b}\right)^{3/2}, \ b \in (-\infty, 0) \cup [1, \infty). \tag{5}$$

The Hopf bifurcation values, corresponding to nonhyperbolic Hopf equilibria, are situated on the surfaces

$$H_{1,2} : a = \pm\frac{1}{3c^3}(3c^2 - b^2 - 2bc^2)\sqrt{c^2 - b}, \tag{6}$$

with $b \in (-c, c)$, for $c \geq 1$, or $b \in (-c, c^2]$, for $c < 1$.

For a fixed $c > 1$, the equilibria possessing two zero eigenvalues correspond to the points where $H_{1,2}$ and $S_{1,2}$ intersect tangentially, namely:

$$Q_1(-c, \frac{2c}{3}\left(1+\frac{1}{c}\right)^{3/2}), \quad b < 0, \tag{7}$$

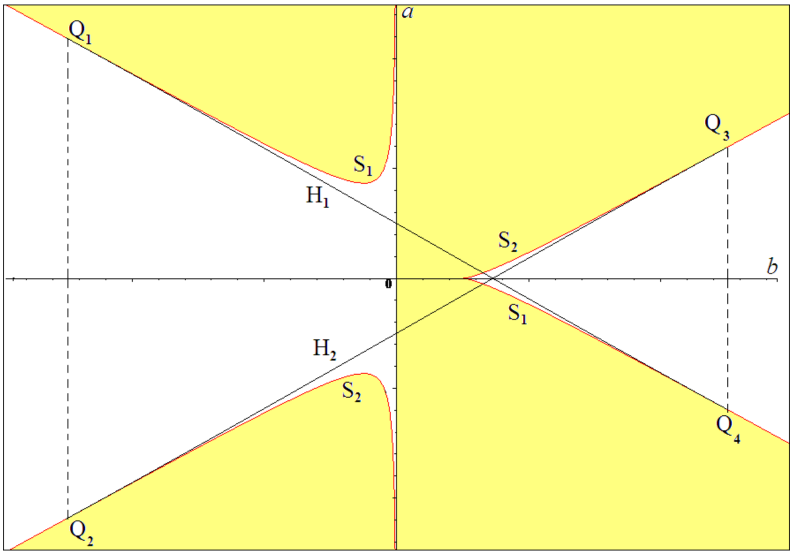

Fig. 2.: The Hopf and saddle-node bifurcation curves.

$$Q_3(c, \frac{2c}{3}\left(1-\frac{1}{c}\right)^{3/2}), \quad b>1. \tag{8}$$

and their symmetrical with respect to the Ob-axis, Q_2 and Q_4 (Figure 2).

In order to derive the normal form around a double equilibrium point (x^*, y^*) existing for the parameters values (b^*, a^*), new variables ψ_1 and ψ_2, and new parameters α_1 and α_2 are introduced, using the translations:

$$\begin{aligned}\psi_1 &= x - x^*, \psi_2 = y - y^*, \\ \alpha_1 &= b - b^*, \alpha_2 = a - a^*.\end{aligned}$$

The equilibrium point (x^*, y^*) of the phase plane (x, y) corresponds to the equilibrium $(0, 0)$ of the (ψ_1, ψ_2) plane, while the point (b^*, a^*) of the (b, a)-plane corresponds to the point $(0, 0)$ in the (α_1, α_2)-plane.

Taking into account that the coordinates x^*, y^* of the equilibrium satisfy $x^* + y^* - \frac{x^{*3}}{3} = 0$, $x^* - a^* + b^* y^* = 0$, system (1) becomes

$$\begin{aligned}\dot{\psi}_1 &= c[\psi_1(1 - x^{*2}) + \psi_2 - x^*\psi_1^2 - \tfrac{1}{3}\psi_1^3], \\ \dot{\psi}_2 &= -\tfrac{1}{c}[\psi_1 + (b^* + \alpha_1)\psi_2 - \alpha_2 + \alpha_1 y^*].\end{aligned} \tag{9}$$

In the following we derive the Bogdanov-Takens bifurcation at the point Q_1.

If the parameters (b^*, a^*) are fixed at the point Q_1, we get $b^* = -c, a^* = \frac{2c}{3}\left(1+\frac{1}{c}\right)^{3/2}$, $x^* = \sqrt{1+\frac{1}{c}}, y^* = \frac{1-2c}{3c}x^*$. Thus, system (9) is written as:

$$\begin{aligned} \dot{\psi}_1 &= c[-\tfrac{1}{c}\psi_1 + \psi_2 - x^*\psi_1^2 - \tfrac{1}{3}\psi_1^3], \\ \dot{\psi}_2 &= -\tfrac{1}{c}[\psi_1 + (\alpha_1 - c)\psi_2 - \alpha_2 + \alpha_1 y^*]. \end{aligned} \tag{10}$$

and has for $(\alpha_1, \alpha_2) = (0,0)$ the double equilibrium point $(\psi_1, \psi_2) = (0,0)$, with the corresponding Jacobi matrix
$\begin{pmatrix} -1 & c \\ -1/c & 1 \end{pmatrix}$. In this case the corresponding eigenvalues are $\lambda_1 = \lambda_2 = 0$. Using the eigenvector $v_0 = (c,1)^T$ and the generalized eigenvector $v_1 = (0,1)^T$ we obtain the transformation

$$\eta_1 = \frac{\psi_1}{c}, \quad \eta_2 = -\frac{\psi_1}{c} + \psi_2, \tag{11}$$

that brings the system (10) with $(\alpha_1, \alpha_2) = (0,0)$ into a system whose Jacobi matrix is in the Jordan canonical form $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. We apply this transformation to system (10) for all parameters α with small $\|\alpha\|$. Neglecting the third-order terms, we obtain:

$$\begin{aligned} \dot{\eta}_1 &= \eta_2 - c^2 x^* \eta_1^2, \\ \dot{\eta}_2 &= -\tfrac{\alpha_1}{c}\eta_1 - \tfrac{\alpha_1}{c}\eta_2 + c^2 x^*\eta_1^2 - \tfrac{1}{c}(\alpha_1 y^* - \alpha_2). \end{aligned} \tag{12}$$

The next transformation reads:

$$\xi_1 = \eta_1, \xi_2 = \eta_2 - c^2 x^* \eta_1^2 \tag{13}$$

so system (12) becomes up to third-order terms:

$$\begin{aligned} \dot{\xi}_1 &= \xi_2, \\ \dot{\xi}_2 &= -\tfrac{\alpha_1}{c}\xi_1 - \tfrac{\alpha_1}{c}\xi_2 + cx^*(c-\alpha_1)\xi_1^2 - 2c^2x^*\xi_1\xi_2 \\ &\quad -\tfrac{1}{c}(\alpha_1 y^* - \alpha_2). \end{aligned} \tag{14}$$

Consider now a transformation of the form:

$$w_1 = \xi_1 - \delta, w_2 = \xi_2,$$

where $\delta = \delta(\alpha_1, \alpha_2)$ will be chosen such as the term containing w_1 in the second equation of the obtained system will be annihilated. In the new variables w_1, w_2 the system reads:

$$\begin{aligned} \dot{w}_1 &= w_2, \\ \dot{w}_2 &= -\tfrac{\alpha_1}{c}\delta + cx^*(c-\alpha_1)\delta^2 - \tfrac{1}{c}(\alpha_1 y^* - \alpha_2) \\ &\quad + w_1[-\tfrac{\alpha_1}{c} + 2cx^*(c-\alpha_1)\delta] + w_2(-\tfrac{\alpha_1}{c} - 2c^2x^*\delta) \\ &\quad + cx^*(c-\alpha_1)w_1^2 - 2c^2x^*w_1w_2. \end{aligned} \tag{15}$$

Consequently, we take $\delta = \frac{\alpha_1}{2c^2 x^*(c-\alpha_1)}$, so system (15) becomes:

$$\begin{aligned} \dot{w}_1 &= w_2, \\ \dot{w}_2 &= \beta_1 + \beta_2 w_2 + cx^*(c-\alpha_1)w_1^2 - 2c^2 x^* w_1 w_2. \end{aligned} \tag{16}$$

where $\beta_1 = -\frac{\alpha_1^2}{4c^3 x^*(c-\alpha_1)} - \frac{1}{c}(\alpha_1 y^* - \alpha_2), \beta_2 = -\frac{\alpha_1(2c-\alpha_1)}{c(c-\alpha_1)}$.

Finally, by a time scaling and by introducing new variables:

$$t = \frac{2c}{c-\alpha_1}\tau,\ x_1 = \frac{4c^3 x^*}{c-\alpha_1}w_1,\ x_2 = \frac{8c^4 x^*}{(c-\alpha_1)^2}w_2 \tag{17}$$

system (16) takes the normal form (4) with $\sigma = -1$, where $\gamma_1 = \frac{16c^5 x^*}{(c-\alpha_1)^3}\beta_1$, $\gamma_2 = \frac{2c}{c-\alpha_1}\beta_2$ and the dot stands for the derivation with respect to the new time τ.

Consequently, the FitzHugh-Nagumo system (1) exhibits at Q_1 a codimension-two Bogdanov-Takens bifurcation. In addition, in every plane $c = const.$, at the point Q_1 emerges the curve HL corresponding to homoclinic bifurcation values. Its equation approximated by $\gamma_1 = -\frac{49}{25}\gamma_2^2$, is written in the initial parameters as:

$$HL_1 : a = \frac{(b+c)^2}{4bc^2 x^*}\left(\frac{49(c-b)^2}{25c^2} - 1\right) + \frac{x^*}{3c}(b + 3c - 2bc), \tag{18}$$

for $b \in (-c, 0)$.

3. SECOND-ORDER APPROXIMATIONS OF THE HOMOCLINIC ORBITS

For a generic Bogdanov-Takens bifurcation, first-order asymptotic approximations for the curve of homoclinic bifurcation values emerging from the Bogdanov-Takens bifurcation point and for the corresponding small homoclinic orbits were first obtained by Beyn [2]. Recently, Kuznetsov [8] obtained a quadratic approximation of the curve of homoclinic bifurcation values and second-order corrections of the homoclinic orbit. Following the lines in [8], we derive in this section second-order approximation for the curve HL emerging at the point Q_1 and for the homoclinic orbits of the phase-plane of the FitzHugh-Nagumo system. First we apply a blowup transformation for system (16):

$$\begin{aligned} u &= \tfrac{cx^*(c-\alpha_1)}{\varepsilon^2}w_1,\ v = \tfrac{cx^*(c-\alpha_1)}{\varepsilon^3}w_2 \\ \beta_1 &= -\tfrac{4}{cx^*(c-\alpha_1)}\varepsilon^4,\ \beta_2 = -\tfrac{2c}{c-\alpha_1}\varepsilon^2\theta, \end{aligned} \tag{18}$$

and consider $s = \varepsilon t$ the new time. Thus, system (16) becomes:

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= -4 + u^2 - \tfrac{2c\varepsilon}{c-\alpha_1}v(u+\theta), \end{aligned} \tag{20}$$

where the dot stands for the derivation with respect to the time s. The branch of homoclinic orbits of system (20) parametrized by ε is

$$\begin{aligned} u &= u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + ... + \varepsilon^k u_k + ... \\ v &= v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + ... + \varepsilon^k v_k + ... \\ \text{with } \theta &= \theta_0 + \varepsilon\theta_1 + \varepsilon^2\theta_2 + ... + \varepsilon^k\theta_k + ... \end{aligned} \tag{21}$$

where k stands for the order of approximation. In addition we require that $\lim_{s\to\infty} u(s) = \lim_{s\to-\infty} u(s)$, $\lim_{s\to\infty} v(s) = \lim_{s\to-\infty} v(s)$ and the above limits must be finite. We also fix that $v(0) = 0$.

In the following we replace (21) into the system (20) and collect the ε^k terms.

For $k = 0$, we get:

$$\begin{aligned} \dot{u}_0 &= v_0, \\ \dot{v}_0 &= u_0^2 - 4. \end{aligned}$$

This system is hamiltonian, with the Hamilton function $H(u_0, v_0) = 4u_0 - \frac{u_0^3}{3} + \frac{v_0^2}{2}$. Its solution is:

$$\begin{aligned} u_0(s) &= 2 - 6\operatorname{sec h}^2(s), \\ v_0(s) &= 12\sec h^2(s)\tanh(s). \end{aligned} \tag{22}$$

For $k = 1$, we get the linear non-homogeneous system:

$$\begin{aligned} \dot{u}_1 &= v_1, \\ \dot{v}_1 &= 2u_0u_1 + \frac{2c}{b}v_0(\theta_0 + u_0). \end{aligned}$$

The homogenous corresponding system is equivalent with the second-order differential equation $\ddot{u}_1 = 2u_0u_1$ which have a solution $\mu_1(s) = v_0$. Another linear independent solution is

$$\mu_2(s) = 2\cosh^2(s) + \frac{15}{\cosh^3(s)}(s\sinh(s) - \cosh(s)) + 5.$$

Thus, using the general solution $c_1\mu_1 + c_2\mu_2$ of the homogenous equation we obtain the desired solution u_1 of the nonhomogenous system by the variation of constants method, while $v_1 = \dot{u}_1$. Thus we have:

$$\begin{aligned} u_1(s) &= \frac{144c}{7(c-\alpha_1)}\frac{\sinh(s)\log(\cosh(s))}{\cosh^3(s)}, \\ v_1(s) &= \frac{144c}{7(c-\alpha_1)}\frac{\sinh^2(s) + (1 - 2\sinh^2(s))\log(\cosh(s))}{\cosh^4(s)}, \end{aligned} \tag{23}$$

and $\theta_0 = \frac{10}{7}$.

For $k = 2$, we get the linear non-homogeneous system:

$$\begin{aligned}\dot{u}_2 &= v_2,\\ \dot{v}_2 &= 2u_0u_2 + \frac{2c}{b}v_0\left(\theta_1 + u_1\right) + \frac{2c}{b}v_1\left(\theta_0 + u_0\right) + u_1^2.\end{aligned}$$

By similar computations, we obtain:

$$\begin{aligned}u_2(s) = &-\frac{\gamma}{\cosh^4(s)}[12\log^2(\cosh(s))(\cosh(2s) - 2)\\ &\qquad + 12\log(\cosh(s))(1 - \cosh(2s))\\ &\qquad + 6s\sinh(2s) - 7\cosh(2s) + 8]\\ v_2(s) = &\frac{12\gamma}{\cosh^5(s)}[s\cosh(s)(2\cosh^2(s) - 3)\\ &- \sinh(s)(12\log^2(\cosh(s) - 14\log(\cosh(s)) - 3)]\\ &+ \frac{16\gamma\sinh(s)}{\cosh^3(s)}[3\log^2(\cosh(s)) - 6\log(\cosh(s)) - 1]\end{aligned}$$

and $\theta_1 = 0$. Here $\gamma = \frac{72c^2}{49(c-\alpha_1)^2}$.

For $k = 3$, we get a linear non-homogeneous system that give $u_3(s), v_3(s)$ and θ_2. As we are interested only in the second-order approximations, we mention only the value of θ_2, namely

$$\theta_2 = \frac{1152c^2}{2401(c - \alpha_1)^2}.$$

Thus the second-order approximations (21) of the homoclinic orbits are completely determined in the (u, v) phase plane.

In the following we transform the approximate homoclinic orbits into the initial phase-plane (x, y). This is possible because all the above transformations of the variables are invertible. Thus, taking into account the inverse transformations

$$\begin{aligned}&x = \psi_1 + x^*,\ y = \psi_2 + y^*\\ &\psi_1 = c\eta_1,\ \psi_2 = \eta_1 + \eta_2\\ &\eta_1 = \xi_1,\ \eta_2 = \xi_2 + c^2x^*\xi_1^2\\ &\xi_1 = w_1 + \delta,\ \xi_2 = w_2\\ &w_1 = \frac{u\varepsilon^2}{cx^*(c - \alpha_1)},\ w_2 = \frac{v\varepsilon^3}{cx^*(c - \alpha_1)}\end{aligned}$$

we get the second-order approximations of the homoclinic orbits in the initial variables:

$$\begin{aligned}x &= -\frac{1}{bx^*}(\varepsilon^2u(\varepsilon t) - \frac{b}{2c} - b + \frac{1}{2}),\\ y &= \frac{\varepsilon^2}{b^2x^*}u\left(\varepsilon t\right) - \frac{\varepsilon^3}{bcx^*}v\left(\varepsilon t\right) + \frac{\varepsilon^4}{b^2x^*}u^2\left(\varepsilon t\right) + \frac{c^2 - b^2}{4b^2c^2x^*} + y^*.\end{aligned} \tag{23}$$

Eliminating ε between the last two equations (19), with $\theta = \theta_0 + \varepsilon\theta_1 + \varepsilon^2\theta_2$, we obtain the second-order approximation of the curve of homoclinic bifurcations values in terms of β_1 and β_2 :

$$HL_2 : \beta_2 = \frac{10c}{7b}\sqrt{bcx^*\beta_1} + \frac{576}{2401}\frac{c^4x^*}{b^2}\beta_1. \tag{25}$$

Taking into account that the parameters β_1 and β_2 can be written using the initial parameters a, b and c as:

$$\beta_1 = \frac{a}{c} + \frac{(b+c)^2}{4bc^3x^*} - \frac{x^*}{3c^2}(b + 3c - 2bc),$$
$$\beta_2 = \frac{c^2 - b^2}{bc}$$

and replacing these expressions into (25) we obtain the implicit equation of the second-order approximation of the curve of homoclinic bifurcation values situated in the parameter-space (b, a) of the FitzHugh-Nagumo system.

Remark that if we eliminate ε between the last two equations (19), with $\theta = \theta_0 + \varepsilon\theta_1$, then we obtain $\beta_2 = \frac{10c}{7b}\sqrt{bcx^*\beta_1}$, which in terms of a, b, c is the curve HL_1 given in (18).

Note that a similar study can be done in the neighborhood of the point Q_3. In this case we have $b^* = c$, $a^* = \frac{2c}{3}\left(1 - \frac{1}{c}\right)^{3/2}$, while the coordinates of the double equilibrium point corresponding to the parameters fixed at Q_3 are $x^* = -\sqrt{1 - \frac{1}{c}}$, $y^* = \frac{a^* - x^*}{b^*}$. Using similar transformations, we get the normal form (4) with $\sigma = 1$. Then, second-order approximations for the homoclinic orbits can be obtained.

4. COMPARISON BETWEEN ANALYTICAL AND NUMERICAL RESULTS

In Figure 3 we represented the second quadrant of the parameter-plane (b, a), the point Q_1 of Bogdanov-Takens bifurcation, the curve H_1 of Hopf bifurcation values, the curve S_1 of saddle-node bifurcation values and the curve HL of homoclinic bifurcation values obtained numerically. The two curves HL_1 and HL_2 of first and second-order approximation of homoclinic bifurcation values are very close to HL in the neighborhood of Q_1, so they can't be distinguished in our representation. This is why in the Table 1, for some values of b, the corresponding values of the parameters a for the curves HL_1, HL and HL_2 are given for $c = 5$. We remark that HL_1 is situated below HL, while HL_2 is above HL and, as expected, is more accurate than HL_1. Far away from Q_1 the curves HL_1 and HL_2 are not good approximations of HL any more. The last column of the table contains the corresponding values of the parameter ε.

In the next three figures, we represented homoclinic orbits situated in the phase-plane (x, y) for three values of the parameter b, namely $b = -4.75$ (Figure 4),

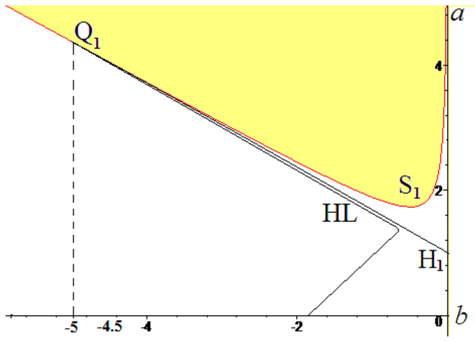

Fig. 3.: Parameter portrait around Q_1, for $c = 5$.

Table 1: Values of homoclinic bifurcation.

b	a_{HL_1}	$a_{numeric}$	a_{HL_2}	ε
-5.0	4.38178	4.38178	4.38178	0
-4.75	4.21669	4.216707	4.21671	0.183
-4.5	4.05007	4.050217	4.05025	0.2544
-4.0	3.7123	3.71363	3.71395	0.3444
-3.5	3.36851	3.37344	3.37478	0.4009

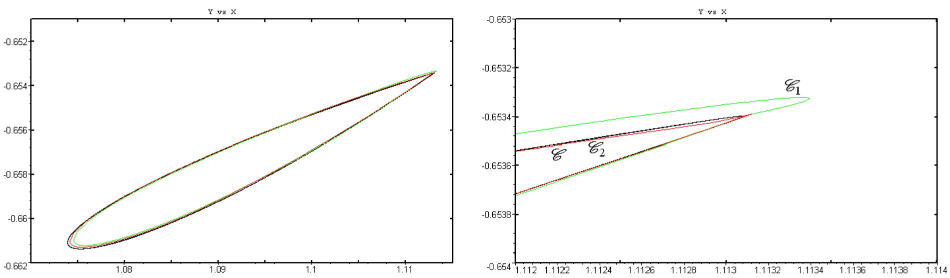

Fig. 4.: Homoclinic orbits for $b = -4.75$, $\varepsilon = 0.183$.

$b = -4.5$ (Figure 5) and $b = -4.0$ (Figure 6). In each of these figures, we represented the first-order approximation of the homoclinic orbit $\mathcal{C}_1$ (colored by green), the second-order approximation $\mathcal{C}_2$ (colored by red) and the homoclinic orbit $\mathcal{C}$ obtained numerically (by black), using [4]. A zoom of these homoclinic orbits in the neighborhood of the saddle equilibrium point is given in each figure. The approxima-

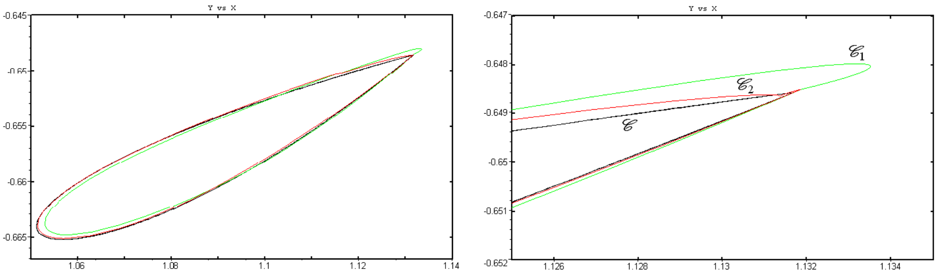

Fig. 5.: Homoclinic orbits for $b = -4.5$, $\varepsilon = 0.2544$.

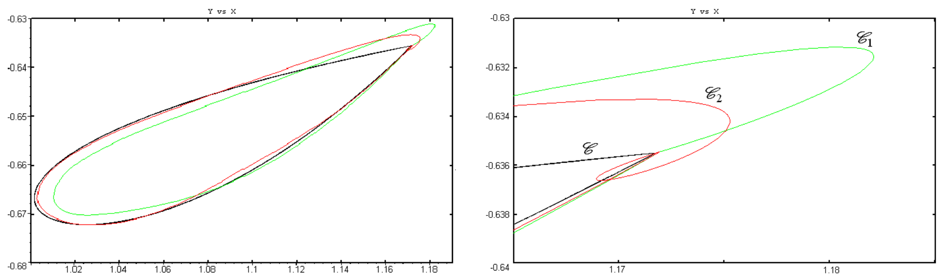

Fig. 6.: Homoclinic orbits for $b = -4.0$, $\varepsilon = 0.3444$.

tions of the homoclinic orbits were represented using the equations (24) with u and v from (21) and $k = 1$ for the first-order approximation, while $k = 2$ for the second-order one. Remark that in Figure 4 the second-order approximation homoclinic orbit almost collides with the numerical orbit, while the first-order approximation is a little different. In addition, in the neighborhood of the saddle equilibrium point, the first-order approximation makes a parasitic turn, approaching the equilibrium along a wrong direction. In Figure 5, corresponding to a greater value of ε, the second-order approximation orbit is better than the first-order one, but both of them are less accurate than in the previous figure. As ε continues to increase, that is far away from the point Q_1, neither the first-order approximation of the homoclinic orbit, nor the second-order one are accurate any more (Figure 6). The parasitic turn of the first-order approximation orbit continues to exist, while the second-order approximation orbit makes in the neighborhood of the saddle equilibrium point a parasitic loop.

As a conclusion, the second-order approximations of the homoclinic orbits are significantly better than the first-order ones, but only for values of ε smaller than 0.3.

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