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A NUMERICAL SOLUTION OF FREDHOLM FUZZY INTEGRAL EQUATIONS OF THE SECOND KIND BY RADIAL BASIS FUNCTIONS

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Abstract In this paper, we present a numerical method to solve linear Fredholm fuzzy integral equations of the second kind based on Radial Basis Functions. The error estimate of the proposed method is given and compared with another approach. This approach is based on the meshless method as well as some definitions of the fuzzy solution and Radial Basis Functions method. Several examples at the end of this paper are employed to demonstrate the effectiveness of the proposed method.

Keywords: Fuzzy integral equation, Fuzzy number, kansa method, Radial basis function, Fuzzy set, α -level set.

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1. INTRODUCTION

1.1. FUZZY INTEGRAL EQUATION

At the first time Dubois and Prade introduced the idea of integration of fuzzy function [1] and later several methods were presented by Goetschel and Voxman [2], Kaleva [3], Nanda [4], Ralescu and Adams [5], Wan [6], Bede and Gal [7], and others. One of the first applications of fuzzy integration was given by Wu and Ma [8] who examined the fuzzy Fredholm integral equation of the second kind FFIE-2. In 2006, Deb et al [9] introduced a new set of orthogonal function; HBT approach for solving linear Fredholm integral equations has been presented by Maleknejad and Mahmodi [10] and then has been extended by Marzban and Razzaghi to multi-delay systems [11]. Recently, Mirzaee and Hoseini have used the HBT method to find approximate solutions of nonlinear Volterra and Fredholm integral equations [12]. The formation of the paper is prepared as follows. In section 2, some useful definitions are in fuzzy integral equation and the RBFs interpolation based on Kansa approach. In section 3 the new solution is introduced and then several examples are solved with this method in section 4.

1.2. RADIAL BASIS FUNCTION

Several meshless methods have been proposed over the last decade. Although these methods usually all bear the generic label meshless, not all are truly meshless but the Kansa method actually do not require an auxiliary mesh or cell structure. Since they are radially symmetric functions which are shifted by point in multidimensional Euclidean space and then linearity combined, they form data dependent approximation spaces. This data dependence makes the spaces formed suitable for providing approximations to large classes of given functions. It also opens the door to existence and uniqueness results for interpolating scattered data by radial basis functions in very general settings. They are usually applied to approximate functions. Advanced numerical methods for computing the Radial basis function approximations are required when the data set is large while more standard software is required for instance for finite elements. Most often RBF approximate are used in combination with interpolation. The concept of using RBFs for solving DEs was first introduced by Kansa [14] who directly collocated the Radial basis functions for the approximate solution of differential equations. However, in recent years RBFs have been extensively researched and applied in a wider range of analysis. Partial differential equations (PDEs) and ordinary differential equations (ODEs) have been solved using RBFs with recent work [14, 15, 16, 17, 18, 19, 20, 21, 22]. The flexibility of the approach is also based on the radial symmetry of each term. Clearly, a good choice of the φ is important for the quality of the approximation and for the existence of the interpolates, and for more details:

1) Linear radial basis function $\varphi(r) = 1$.

2) Multi quadrate radial basis function $\varphi(r) = \sqrt{(r^2 + c^2)}$, which contains another scalar parameter (c) which may be adjusted to improve the approximation, where the choice $c = 0$ gives the previous example.

3) Gussian kernel $\varphi(r) = (e^{-r^2 c^2})$, which also contains another scalar parameter $c \neq 0$ which may be adjusted to adapt the approximation.

4) Invers multi quadrate radial basis function $\varphi(r) = \dfrac{1}{\sqrt{(r^2 + c^2)}}$, which contains another scalar parameter $c \neq 0$ and provides further flexibility.

where r $= |x - x_i|$ and c is a free positive parameter, often referred to as the shape parameter, to be specified by the user. The shape parameter c within the Gaussian (GA) and multiquadratic RBFs requires fine tuning and can dramatically alter the quality of the interpolation. Too large or too small shape

parameter c make the GA too flat or too peaked. Despite many studies conducted to find algorithms for selecting the optimum values of c, the optimal choice of shape parameter is an open problem which is still under intensive investigation, One should consider this point from this article which is we use GA function and we initial c to 1 in implement our methods. The standard radial basis functions are categorized into two major classes[23]:
Class 1. Infinitely smooth RBFs: These basis functions are infinitely differentiable and heavily depend on the shape parameter c e.g. Hardy multiquadratic (MQ), Gaussian (GA), inverse multiquadratic (IMQ), and inverse quadratic (IQ).
Class 2. Infinitely smooth (except at centers) RBFs: The basis functions of this category are not infinitely differentiable. These basis functions are shape parameter free and have comparatively less accuracy than the basis functions discussed in the Class 1. For example, thin plate spline.

2. PRELIMINARIES

Definition 2.1. *[24] A fuzzy number u is a fuzzy set $u : R \to [0,1]$ Which satisfies the following conditions:*

- *u is normal, that is exists an $x_0 \in R$ such that $u(x_0) = 1$.*
- *u is a convex fuzzy set, $\forall \eta \in [0,1]$, $x, y \in R$ it holds that $(\eta.x + (1-\eta).y) \geq min\,[u(x), u(y)]$.*
- *u is upper semi-continues.*
- *$[u]^\alpha$=$\{x \in R | u(x) \geq \alpha\}$ $\quad 0 \leq \alpha \leq 1$, is compact subset of R.*

Definition 2.2. *[25] We determined the α-level set fuzzy as follow:*

$$[u]^\alpha = \{x \in R | u(x) \geq \alpha\}, 0 \leq \alpha \leq 1,$$

and we determined $[u]^\alpha = [\underline{u}^\alpha, \overline{u}^\alpha]$, if we have the following conditions:

- *$\overline{u}^\alpha$ is bounded left continues non decreasing function over[0,1], with respect to any α.*
- *$\underline{u}^\alpha$ is bounded right continues non decreasing function over[0,1], with respect to any α.*
- *$\underline{u}^\alpha \leq \overline{u}^\alpha$ for all $\alpha \in$[0,1].*
  where
  $\overline{u}^\alpha = max\,\{x | x \in [u]^\alpha\}$.
  $\underline{u}^\alpha = min\,\{x | x \in [u]^\alpha\}$.

Definition 2.3. *[25] Two fuzzy set u and v is* $u = v$ *if and only if* $[u]^\alpha = [v]^\alpha$ *for all* $\alpha \in$ *[0,1]. The following arithmetic operations on fuzzy numbers are well known and frequently used below:*

$$[v+u]^\alpha = [\underline{u}^\alpha + \underline{v}^\alpha, \overline{u}^\alpha + \overline{v}^\alpha],$$

$$[v-u]^\alpha = [\underline{u}^\alpha - \underline{v}^\alpha, \overline{u}^\alpha - \overline{v}^\alpha],$$

$$[\eta.u]^\alpha = \eta.[u]^\alpha = \begin{cases} [\eta.\underline{u}^\alpha, \eta.\overline{u}^\alpha] & \eta \geq 0 \\ [\eta.\overline{u}^\alpha, \eta.\underline{u}^\alpha] & \eta < 0 \end{cases}$$

Definition 2.4. *[25] A fuzzy real number valued function* f *is said to be continuous in* $x_0 \in [a,b]$*, if for each* $\epsilon > 0$ *there is* $\delta > 0$ *such that* $D(f(x), f(x_0)) < \epsilon$*, whenever* $x \in [a,b]$ *and* $|x - x_0| < \delta$*. we say that* f *is fuzzy continuous on* $[a,b]$ *if* f *is continuous at each* $x_0 \in [a,b]$ *and denote the space of all such functions by* $C_F([a,b])$*.*

Theorem 2.1. *[24] Let* $E = \{U : R \longrightarrow [0,1]\}$*,* $T \in R$ *and* $f : T \longrightarrow E$ *the integral of* f *over* T*, denoted by* $\int f(t)dt$*, is defined level wise by the:*

$[\int f(t)dt]^\alpha = \int f^\alpha(t)dt = \{\int f(t)dt | f : T \longrightarrow R^n$ *is a measurable selection for* $f^\alpha\}$*,*

for all $0 < \alpha < 1$*, a strongly measurable and integrally bounded mapping* $f : T \longrightarrow E^n$ *is said to be integral over* T *if* $\int f(t)dt \in E$*.*

Theorem 2.2. *[25] If* $f : T \longrightarrow E$ *is strongly measurable and integrable bounded, then* f *is integrable.*

Theorem 2.3. *[24] Let* $f : T \longrightarrow E$ *be integrable and* $c \in T$ *then*

$\int_{t_0}^{t_0+p} f(t)dt = \int_{t_0}^{c} f(t)dt + \int_{c}^{t_0+p} f(t)dt.$

Theorem 2.4. *[24] Let* $f, g : T \longrightarrow E$ *be integrable and* $\lambda \in R$ *then*

1)$\int (f(t) + g(t))dt = \int f(t)dt + \int g(t)dt.$

2)$\int \lambda.f(t)dt = \lambda \int f(t)dt.$

3)$D(f,g)$*is integrable.*

4)$D(\int f(t)dt, \int g(t)dt) < \int D(f,g).$

Theorem 2.5. *[24] Let* $f : T \longrightarrow E$ *the integral of* f *over* T *denoted by* $\int f(t)dt$ *is defined level wise by the equation.*

Theorem 2.6. *[24] If the fuzzy function $f(t)$ is continuous in the metric D, its definite integral exists, and also,*

$$\underline{(\int_a^b f(t,\alpha)dt)} = \int_a^b \underline{f}(t,\alpha),$$

$$\overline{(\int_a^b f(t,\alpha)dt)} = \int_a^b \overline{f}(t,\alpha).$$

2.1 Properties of Radial Basis Functions

Let $R^+ = \{x \in R, x \geqslant 0\}$ be the non-negative half-line and let $\phi(0) \geq 0$. A Radial basis functions on R^d is a function as below:

$$\phi(\|X - X_i\|),$$

where $X, X_i \in R^d$, and $\|.\|$ denotes the Euclidean distance between X and X_i. If one chooses N points $\{X_i\}_{i=1}^N$ is R^d then by custom

$$\xi(X) = \sum_{i=1}^{N} \lambda_i \phi(\|X - X_i\|), \lambda_i \in R.$$

is called Radial basis function as well [22].

2.2 RBFs Interpolation

The one dimensional function $\xi(x)$ to be interpolated or approximated can be represented by RBFs as

$$y(x) \approx y_N(x) = \sum_{i=1}^{N} \lambda_i \phi_i(x) = \mathbf{\Phi}^T(x)\mathbf{\Lambda}, \tag{1}$$

where

$$\phi_i(x) = \varphi(\|x - x_i\|),$$

$$\mathbf{\Phi}^T(x) = [\phi_1(x), \phi_2(x), ...\phi_N(x)],$$

$$\mathbf{\Lambda} = [\lambda_1, \lambda_2, ...\lambda_N]^T.$$

x is the input and $\{\lambda_i\}_{i=1}^N$ are the set of coefficients to be determined. By choosing N interpolate nodes $\{x_i\}_{i=1}^N$, we can approximate the function $y(x)$

$$y_j = \sum_{i=1}^{N} \lambda_i \phi_i(x_j), \quad j = 1, 2, ...N, \tag{2}$$

to summarize discussion on coefficient matrix, we define

$$\mathbf{\Psi}\mathbf{\Lambda} = \mathbf{\Xi},$$

where

$$\mathbf{\Xi} = [y_1, y_2, ..., y_N]^T,$$

$$\mathbf{\Psi} = [\Phi^T(x_1), \Phi^T(x_2), ..., \Phi^T(x_N)]^T =$$

$$\begin{pmatrix} \phi_1(x)_1 & ... & \phi_1(x)_N \\ \vdots & :: & \vdots \\ \phi_N(x)_1 & ... & \phi_N(x)_N \end{pmatrix}$$

Note that $\phi_i(x_j) = \varphi(\|x_i - x_j\|)$ therefore we have $\phi_i(x_j) = \phi_j(x_i)$ and consequently $\mathbf{\Psi} = \mathbf{\Psi}^T$.

All the infinitely smoth RBFs choices, will give the coefficient matrices $\mathbf{\Psi}$ in the matrix which are symmetric and nonsingular [17]. There is unique interpolant of the form (1), no matter how the distinct data points are scattered in any number of space dimensions. In the cases of inverse quadratic, inverse multiquadric (IMQ), hyperbolic secant (sech), and Gaussian (GA) the matrix $\mathbf{\Psi}$ is positive and, for multiquadric (MQ), it has one positive eigenvalue and the remaining ones are all negative [17].

Algorithm.

The algorithm works in the following manner:

1. Choose$N$ center points $\{x_j\}_{j=1}^{N}$form the domain set [a,b].

2. Approximate $\xi(x)$ as $\xi_N(x) = \mathbf{\Phi}^T(x)\mathbf{\Lambda}$.

3. Substitute $\xi_N(x)$ into the main problem and creat residual function Res(x).

4. Substitute collocation point $\{x_j\}_{j=1}^{N}$ into the Res(x) and creat the N equations.

5. Solve the N equations with N unknown coefficients of members of $\mathbf{\Lambda}$ and find the numerical solution.

We have the following theorem about the convergence of RBFs interpolation:

Theorem 2.7. *Assume* $x_i, (i = 1, 2, ...N)$, *are* N *nodes in convex* Ω,

$$h = \max_{x\in\Omega} \min_{1\le i\le N} \|x - x_i\|_2, \tag{3}$$

when $\hat{\phi}(\eta) < c(1 + |\eta|)^{-(2l+d)}$ *for any* $u(x)$ *satisfies* $\int(\hat{u}(\eta))^2/\hat{u}(\eta)d\eta < \infty$, *we have*

$$\|u^{N(\alpha)} - u^{\alpha}\|_{\infty} \le ch^{(l-\alpha)},$$

where $\hat{\phi}(x)$ *is an RBF and constant* c *depends on the RBFs,* d *is the space dimension,* l *and* α *are non-negative integer. It can be seen that not only the RBF itself but also its any order derivative has a good novergence [17].*

In this problem, we use Gaussian, inverse quadric, and secant hyperbolic RBFs which are positive definite functions and can get high accurate solutions.

3. SOLUTION OF FUZZY FREDHOLM INTEGRAL EQUATION VIA RBF

Throughout this paper, we consider fuzzy Fredholm integral equation with $a = 0$, $b = 1$, $\gamma > 0$ where $u(x)$ and $f(x)$ are in fuzzy function see prove [22].

The linear ordinary Fredholm integral equation with fuzzy coefficient as below:

$$u_k(x) = f(x) + \gamma \int_a^b k(x,t)u_{k-1}(t)dt. \tag{4}$$

In fuzzy case at the first step consider the spacial system then try to solve it, in other words we solve two linear fuzzy Fredholm integral equation with separate boundary conditions [1]

$$\begin{cases} \underline{u}_k(x) = \underline{f}(x) + \gamma \int_a^b k(x,t)\underline{u}_{k-1}(t)dt, \\ \overline{u}_k(x) = \overline{f}(x) + \gamma \int_a^b k(x,t)\overline{u}_{k-1}(t)dt. \end{cases}$$

Consider $\underline{y}_n(x)$ be the approximate equation of $\underline{u}_k(x)$:

$$\underline{y}_n(x) = \sum_{i=1}^{n} \lambda_i \varphi_i(x), \tag{5}$$

and then substitute equation (5) in equation (4) now the problems is find $\lambda_i : i = 1...N$

$$\underline{y}_{n+1}(x) = f(x) + \gamma \int_a^b k(x,t).\ ((y_n(x) = \sum_{i=1}^{n} \lambda_i \varphi_i(x))dt, \tag{6}$$

therefore $\underline{y_n\ (\varphi,\ \text{x}\ )}$ is now a function of the unknown $\lambda_1, \lambda_2, ...\lambda_n$ which may be rewritten as $\underline{y(\lambda_1\lambda_2..\lambda_n, x\ )}$ and therefore $\underline{y(\lambda_1\lambda_2..\lambda_n, x\ )} \backsimeq 0$ (Res function) for all x$\in$ [a,b], so to simplify the problem we write:

$$\underline{y_n(\lambda_1, \lambda_2, ..\lambda_n, x)} = f(x) + \gamma \int_a^b (k(x,t)).(y_n(x) = \sum_{i=1}^{n} \lambda_i \varphi_i(x))dt, \tag{7}$$

to find the coefficient $\lambda : i = 1...n$; solve equation (7) at n-distinic collocations points $x_1, x_2, ...x_n \in [a,b]$ and now try to solve the linear systems below :

$$\begin{aligned}
&\underline{y_n(\lambda_1, \lambda_2, ..\lambda_n, x_1)} = 0, \\
&\underline{y_n(\lambda_1, \lambda_2, ..\lambda_n, x_2)} = 0, \\
&\vdots \\
&\underline{y_n(\lambda_1, \lambda_2, ..\lambda_n, x_n)} = 0.
\end{aligned}$$

Then solve the upper bound equation $(\overline{u}(x))$ non fuzzy linear system. Finally there are two approximate function for lower and upper equations $(\underline{u}(x), \overline{u}(x))$ that leads to a one fuzzy system, this theory has been proved [1].

4. NUMERICAL RESULTS

Here we consider three example to illustrate the presented method for FFIE.
Example 4.1.
Consider the following FFIE :

$$k(x,t) = (2t-1)^2(1-2x) \qquad 0 < x, t < 1, \lambda = 1,$$

$$\begin{aligned}
\overline{f}^{\alpha}(x) &= \frac{1}{3}x^2\alpha - \frac{1}{4}\alpha + \frac{5}{3}x^2 + \frac{5}{12}, \\
\underline{f}^{\alpha}(x) &= -\frac{1}{3}x^2 + x^2\alpha + \frac{1}{3}x + \frac{1}{4}\alpha - \frac{1}{12}.
\end{aligned}$$

The exact solution is as follows :

$$\underline{y}^{\alpha}(x) = \alpha.x,$$

$$\overline{y}^{\alpha}(x) = (2-\alpha).x.$$

Table (1) shows the comparison of the approximate solutions at x = 0.5 for any $\alpha \in [0,1]$ with another approach.

Example 4.2.
Consider the following FFIE :

$$k(x,t) = x^2(1-2t) \quad 0 < x, t < 1, \lambda = 1,$$

$$\underline{f}^{\alpha}(x) = \alpha x - x^2\left[\frac{2}{3}\alpha x^3 - \frac{4}{3}x^3 - \frac{1}{2}\alpha x^2 + x^2 + \frac{1}{12}\alpha - \frac{1}{12}\right],$$

$$\overline{f}^{\alpha}(x) = (2-\alpha)x + x^2\left[\frac{2}{3}\alpha x^3 - \frac{1}{3}\alpha x^2 + \frac{1}{12}\alpha - \frac{1}{12}\right].$$

The exact solution is as follows:

$$\underline{y}^{\alpha}(x) = \alpha x,$$

$$\overline{y}^{\alpha}(x) = (2-\alpha)x.$$

Table (2) shows the comparison of the approximate solutions at x=0.1 for any $\alpha \in [0,1]$ with another approach.

Table 1 x=0.5

α_i	$Error.\underline{y}[26]$	$Error.\underline{y}(RBF)$	$Error.\overline{y}[26]$	$Error.\overline{y}(RBF)$
0	$0.7e-3$	0	$-2e-2$	0
0.1	$6e-3$	0	$-2e-2$	$1e-16$
0.2	$4e-3$	$2e-18$	$2e-2$	0
0.3	$3e-3$	$3e-17$	$2e-2$	0
0.4	$1e-3$	$2e-17$	$3e-2$	$2e-16$
0.5	$1e-3$	$1e-17$	$1e-2$	$3e-16$
0.6	$1e-2$	$4e-17$	$1e-2$	$4e-16$
0.7	$1e-2$	$2e-17$	$1e-2$	0
0.8	$1e-2$	$1e-16$	$3e-2$	$2e-16$
0.9	$1e-2$	$2e-16$	$2e-2$	0

Example 4.3.
Consider the following FFIE :

$$k(x,t) = 0.1sin(x)sin\left(\frac{t}{2}\right) \qquad 0 < x, t < 1, \lambda = 1,$$

$$\underline{f}^{\alpha}(x) = (\alpha^2 + \alpha)\left(sin(\frac{x}{2}) - 0.05sin(x)(1 - sin(1))\right),$$

$$\overline{f}^{\alpha}(x) = (4 - \alpha^3 - \alpha)\left(sin(\frac{x}{2}) - 0.05sin(x)(1 - sin(1))\right).$$

The exact solution is as follows:

$$\underline{y}^{\alpha}(x) = (\alpha^2 + \alpha)sin(\frac{x}{2}),$$

$$\overline{y}^{\alpha}(x) = (4 - \alpha^3 - \alpha)sin(\frac{x}{2}).$$

Table (3) shows the comparison of the approximate solutions at $x = 0.5$ for any $\alpha \in$ [0,1] with another approach.
Convergence of the RBF has been proved for order linear fuzzy integral equation and showed this point in the table (4). In other words we solve example (3) with different number of n and we see the error will be led to acceptable results in comparison with its.

5. CONCLUSIONS

The RBF has been successfully applied to finding solutions of linear Fredholm fuzzy integral equations of the second kind. The results showed that the

Table 2 x=0.1

α_i	$Error.\underline{y}[27]$	$Error.\underline{y}(RBF)$	$Error.\overline{y}[27]$	$Error.\overline{y}(RBF)$
0	$0.19e-4$	$0.1e-1$	$-0.1e-4$	$0.21e-2$
0.1	$2e-4$	$1e-5$	$1e-2$	$1e-3$
0.2	$1e-4$	$2e-4$	$1e-2$	$1e-3$
0.3	$1e-2$	$3e-4$	$1e-2$	$3e-3$
0.4	$1e-2$	$2e-4$	$1e-2$	$1e-3$
0.5	$1e-2$	$2e-4$	$1e-2$	$2e-3$
0.6	$2e-2$	$2e-4$	$1e-2$	$1e-3$
0.7	$1e-2$	$1e-3$	$1e-2$	$3e-3$
0.8	$1e-2$	$1e-3$	$1e-2$	$1e-3$
0.9	$2e-2$	$1e-3$	$1e-2$	$1e-3$

Table 3 x=0.5

α_i	$Error.\underline{y}[27]$	$Error.\underline{y}(RBF)$	$Error.\overline{y}[27]$	$Error.\overline{y}(RBF)$
0	$0e-7$	$e-10$	$1.2e-5$	$0.21e-8$
0.1	$6.4e-7$	$2e-10$	$1.2e-5$	$5e-8$
0.2	$1.2e-6$	$4e-10$	$1.1e-5$	$3e-8$
0.3	$1.9e-6$	$3e-9$	$1.09e-5$	$2e-8$
0.4	$2.5e-6$	$5e-9$	$1.02e-5$	$1e-8$
0.5	$3.2e-6$	$1e-9$	$9.6e-5$	$4e-8$
0.6	$3.8e-6$	$2e-9$	$8.9e-6$	$2e-8$
0.7	$4.4e-6$	$4e-9$	$8.3e-6$	$1e-9$
0.8	$5.1e-6$	$2e-9$	$7.7e-6$	$3e-9$
0.9	$5.7e-6$	$2e-9$	$7.0e-6$	$4e-9$

Table 4 Example (3), x=0.5

α_i	$n=15$	$n=20$	$n=25$	$n=30$
0	$0e-10$	$e-11$	$1.2e-12$	$0.21e-13$
0.1	$2e-10$	$2e-11$	$1.2e-12$	$5e-13$
0.2	$4e-10$	$4e-11$	$1.1e-12$	$3e-13$
0.3	$3e-10$	$3e-11$	$1.09e-12$	$2e-13$

RBF is a powerful mathematical tool to solving n-th linear Fredholm fuzzy integral equations of the second kind. For linear Fredholm fuzzy integral equations of the second kind we have compared our approach with the algorithms given in [26, 27] and the results presented that the RBF method is more precise than those method. Convergence of the RBF has been proved for linear Fredholm fuzzy integral equations of the second kind.

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