EMBEDDING THEOREMS FOR WEIGHTED ANISOTROPIC SPACES OF HOLOMORPHIC FUNCTIONS IN STRONGLY PSEUDOCONVEX DOMAINS

Romi F. Shamoyan, Seraphim P. Maksakov

Bryansk University, Bryansk, Russia
rshamoyan@gmail.com, msp222@mail.ru

Abstract
We introduce mixed norm analytic spaces in products of strongly pseudoconvex domains and provide new sharp embedding theorems for them extending previously known assertions.

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Keywords: analytic functions, strongly pseudoconvex domains, Bergman type holomorphic mixed norm spaces.


1. INTRODUCTION

In recent paper [25] characterization of the measures $\mu$ in the unit polydisc for which the differentiation operator maps anisotropic weighted space of holomorphic function with mixed norm into Lebesque space $L^q$ was obtained. We need some definitions.

Let

$$U^n = \{ z = (z_1, z_2, \ldots, z_n) : |z_j| < 1, 1 \leq j \leq n \}$$

be the unit polydisc of $n$-dimensional complex space $\mathbb{C}^n$, $T^n$ be the Shilov boundary of $U^n$, $\vec{p} = (p_1, p_2, \ldots, p_n)$, $0 < p_j < +\infty$, $j = 1, n$; $\vec{\omega}(t) = (\omega_1(t), \omega_2(t), \ldots, \omega_n(t))$, $t \in (0, 1)$, where $\omega_j(t)$ is positive integrable functions on $(0, 1)$, and $\omega$ belongs to certain class (see [25]). We denote by $A^\vec{p}_\vec{\omega}$ the set of all holomorphic functions in $U^n$ for which (see [9])

$$\|f\|_{A^\vec{p}_\vec{\omega}} = \left( \int_U \left( \int_U \left( \int_U \cdots \int_U \right)^{\frac{p_1}{p_1}} \omega_1(1 - |z_1|)dm_2(z_1) \right)^{\frac{p_2}{p_1}} \times 
\cdots \int_U \left( \int_U \cdots \int_U \right)^{\frac{p_n}{p_1}} \omega_n(1 - |z_n|)dm_2(z_n) \right)^{\frac{1}{p_n}} < +\infty,$$

where $m_2$ is planar Lebesgue measure on $U := U^1$. 

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Assume further \( \vec{\mu} = (\mu_1, \mu_2, \ldots, \mu_n) \), where \( \mu_j \) is the Borel nonnegative finite measure on \( U \), \( L^\vec{\mu} \) is corresponding space with mixed norm (see [5], [6]) that is, the space of all measurable functions on \( U^n \) for which

\[
\|f\|_{L^\vec{\mu}} = \left( \int_U \left( \prod_{j=1}^n |f(z_1, z_2, \ldots, z_n)|^{p_j} d\mu \right)^{\frac{\frac{1}{p_j}}{n-1}} d\mu \right)^{\frac{1}{n}} < \infty.
\]

In addition, the authors obtained a complete characterization of the measures \( \vec{\mu} \) for which the operator \( \vec{D}^m f(z_1, z_2, \ldots, z_n) = \frac{\partial^m f(z_1, z_2, \ldots, z_n)}{\partial z_1^{m_1} \partial z_2^{m_2} \cdots \partial z_n^{m_n}} \), \( z = (z_1, z_2, \ldots, z_n) \in U^n \), maps \( A_{\vec{\omega}}^q \) in \( L^\vec{\rho}_\vec{q} \), where \( \vec{\rho} = (p_1, p_2, \ldots, p_n) \), \( \vec{q} = (q_1, q_2, \ldots, q_n) \), \( 0 < p_j \leq q_j, j = 1, n \). The authors also obtained a description of the measures \( \nu \) on \( U^n \) for which the operator \( \vec{D}^m \) maps \( A_{\vec{\omega}}^p \) into \( L^\vec{\rho}_\vec{q} \), where \( 0 < p_j \leq q, j = 1, n \). In connection with these results, we recall that for \( n = 1, m = 0 \) into Hardy classes \( H^p(U) \) corresponding description was obtained in the classical work of L. Carleson [7] and in the case of the Hardy space \( H^p(B_k) \) in the ball was obtained by L. Hörmander in [12]. We also note the work of F.A. Shamoyan [24], where he studied the Hardy space \( H^p(U^n) \), there supposed \( \vec{m} = (m_1, m_2, \ldots, m_n) \), \( m_j \neq 0, j = 1, n \). The case of weighted Bergman spaces was investigated in [17]. Various related assertions (sharp embedding theorems in analytic function spaces) can be seen in [15]-[1].

The theory of analytic spaces on pseudoconvex domains is well-developed by various authors during last decades (see [9 - 11] and the references therein). One of the goals of this paper among other things is to define for the first time in literature new mixed norm analytic spaces in products strictly pseudoconvex domains and to establish some basic properties of these spaces. We believe this new interesting objects can serve as a base for further generalizations and investigations in this active research area. Spaces we mentioned above are closely connected also with so-called multifunctional analytic function spaces on products of strictly pseudoconvex domains \( D \times \cdots \times D \).

Various such connections in analytic and harmonic function spaces were found and mentioned in [29], [27], [33]. Note that some basic properties of the last spaces on product domains are closely connected with the so-called trace operator [24 - 25]. In main part of paper we will turn to study of certain embedding theorems for some new mixed norm analytic classes in strictly pseudoconvex domains in \( \mathbb{C}^n \). We note that in this paper we extend some theorems from [27] and [30], where they can be seen in context of unit disk. Proving estimates and embedding theorems in pseudoconvex domains we heavily use the technique which was developed recently in [19], [20]. In our embedding theorem and inequalities for analytic function spaces in pseudoconvex domains with smooth boundary the so-called Carleson-type measures.
constantly appear. We add some historical remarks on this important topic now. Carleson measures were introduced by Carleson [7] in his solution of the corona problem in the unit disk of the complex plane; and, since then, they have become an important tool in analysis, and an interesting object of study per se. Let \( A \) be a Banach space of holomorphic functions in a domain \( D \subset \mathbb{C}^n \); given \( p \geq 1 \), a finite positive Borel measure \( \mu \) on \( D \) is a Carleson measure of \( A \) (for \( p \)) if there is a continuous inclusion \( A \to L^p_{\mu} \), that is, there exists a constant \( C > 0 \) such that
\[
\forall f \in A, \quad \int_D |f|^p d\mu \leq C \|f\|^p_A.
\]

Carleson studied this property [7] taking as Banach space \( A \) the Hardy spaces in unit disk \( U H^p(U) \), and proved that a finite positive Borel measure \( \mu \) is a Carleson measure of \( H^p(U) \) for \( p \) if and only if there exists a constant \( C > 0 \) such that \( \mu(S_{\theta_0,h}) \leq Ch \) for all sets
\[
S_{\theta_0,h} = \{ r e^{i\theta} \in U : 1 - h \leq r < 1, \ |\theta - \theta_0| < h \}
\]
(see also [8], [17]); in particular the set of Carleson measures of \( H^p(U) \) does not depend on \( p \).

In 1975, W. Hastings [11] (see also V. Oleinik and B. Pavlov [17] and L. Oleinik [16]) proved a similar characterization for the Carleson measure of the Bergman spaces \( A^p(U) \), still expressed in terms of the sets \( S_{\theta_0,h} \). Later J. Cima and W. Wogen [39] characterized Carleson measures for Bergman spaces in the unit ball \( B^k \subset \mathbb{C}^k \), and J. Cima and P. Mercer [15] characterized Carleson measures of Bergman spaces in strongly pseudoconvex domains, showing in particular that the set of Carleson measures of \( A^p(D) \) is independent of \( p \geq 1 \).

J. Cima and P. Mercer’s characterization of Carleson measures of Bergman spaces in expressed using interesting generalizations of \( S_{\theta_0,h} \). Given \( z_0 \in D \) and \( 0 < r < 1 \), let \( B_D(z_0,r) \) denote the ball of center \( z_0 \) and radius \( \frac{1}{2} \log \frac{1 + r}{1 - r} \) for the Kobayashi distance \( k_D \) of \( D \) (that is, of radius \( r \) with respect to the pseudohyperbolic distance \( \rho = \tanh(k_D) \)). Then it is possible to prove (see D. Luecking [13] for \( D = U \), P. Duren and R. Weir [38] and H. Kaptanoğlu [14] for \( D = B^k \), and [19], [20] for \( D \) strongly pseudoconvex) that a finite positive measure \( \mu \) is a Carleson measure of \( A^p(D) \) for \( p \) if and only if for some (and hence all) \( 0 < r < 1 \) there is a constant \( C_r > 0 \) such that
\[
\mu(B_D(z_0,r)) \leq C_r \nu(B_D(z_0,r))
\]
for all \( z_0 \in D \). (The proof of this equivalence in [19] relied on J. Cima and P. Mercer’s characterization [15]).
Thus we will have a new geometrical characterization of Carleson measures of Bergman spaces, and it turns out that this geometrical characterization is very important for the study of the various properties of Toeplitz operators; but first it is necessary to widen the class of Carleson measures under consideration. Given \( \theta > 0 \), we say that a finite positive Borel measure \( \mu \) is a (geometric) \( \theta \)-Carleson measure if for some (and hence all) \( 0 < r < 1 \) there is a constant \( c_r > 0 \) such that

\[
\mu(B_D(z_0, r)) \leq c_r \nu(B_D(z_0, r))^\theta
\]

for all \( z_0 \in D \). Note that 1-Carleson measures are usual Carleson measures of \( A^p(D) \), and we know ([19], [20]) that \( \theta \)-Carleson measures are exactly the Carleson measures of suitably weighted Bergman spaces. Note also that when \( D = B^k \) a \( q \)-Carleson measure in the sense of [14], [40] is a \( (1+ \frac{q}{k+1}) \)- Carleson measure in our sense .

In this paper we are however more interested in Carleson type measures for some new Bergman-type mixed norm spaces in product domains \( D \times \ldots \times D \).

2. PRELIMINARIES ON GEOMETRY OF STRONGLY PSEUDOCONVEX DOMAINS WITH SMOOTH BOUNDARY

In this section we provide a chain of facts, properties on the geometry of strongly convex domains which we will use heavily in all our proofs below. In this section we also introduce in detail all basic lemmas in context of pseudoconvex domains which are needed for formulations and proofs of our results taken from papers of Abate and coauthors [19], [20]. In particular, we following these papers provide several results on the boundary behavior of Kobayashi balls, and we formulate a vital submean property for nonnegative plurisubharmonic function in Kobayashi balls.

We now recall first the standard definition and the main properties of the Kobayashi distance which can be seen in various books and papers; we refer for example to [20], [28] and [21] for details. Let \( k_U \) denote the Poincare distance on the unit disk \( U \subset \mathbb{C}^n \). If \( X \) is a complex manifold, the Lempert function \( \delta_X : X \times X \rightarrow \mathbb{R}^+ \) of \( X \) is defined by

\[
U_X(z, \omega) = \inf \{ k_U(\zeta, \eta) \mid \text{there exists a holomorphic } \phi : U \rightarrow X \text{ with } \phi(\zeta) = z \text{ and } \phi(\eta) = \omega \}
\]

for all \( z, \omega \in X \). The Kobayashi pseudodistance \( k_X : X \times X \rightarrow \mathbb{R}^+ \) of \( X \) is the smallest pseudodistance on \( X \) bounded below by \( \delta_X \). We say that \( X \) (Kobayashi) hyperbolic if \( k_X \) is a true distance, and in that case it is known that the metric topology induced by \( k_X \) coincides with the manifold topology.
of $X$ (see, e.g., Theorem 2.3.10 in [20]). For instance, all bounded domains are hyperbolic (see, e.g., Theorem 2.3.14 in [20]). The following properties are well known in literature. The Kobayashi (pseudo) distance is contracted by holomorphic maps: if $f : X \rightarrow Y$ is a holomorphic map then
\[ k_Y(f(z), f(\omega)) \leq k_X(z, \omega). \]

Next the Kobayashi distance is invariant under biholomorphisms, and decreases under inclusions: if $D_1 \subset D_2 \subset \mathbb{C}^n$ are two bounded domains we have $k_{D_2}(z, \omega) \leq k_{D_1}(z, \omega)$ for all $z, \omega \in D_1$. Further the Kobayashi distance of the unit disk coincides with the Poincaré distance. Also, the Kobayashi distance of the unit ball $B^n \subset \mathbb{C}^n$ coincides with the well known in many applications so-called Bergman distance (see, e.g., Corollary 2.3.6. in [20], see also [40], [30]).

If $X$ is a hyperbolic manifold, $z_0 \in X$ and $r \in (0; 1)$ we shall denote by $B_X(z_0, r)$ the Kobayashi ball of center $z_0$ and radius $\frac{1}{2} \log \frac{1 + r}{1 - r}$:
\[ B_X(z_0, r) = \{ z \in X \mid \tanh k_X(z_0, z) < r \}. \]

We can see that $\rho_X = \tanh k_X$ is still a distance on $X$, because $	anh$ is a strictly convex function on $\mathbb{R}^+$. In particular, $\rho_{B^n}$ is the pseudohyperbolic distance of $B^n$.

The Kobayashi distance of bounded strongly pseudoconvex domains with smooth boundary has several important properties. First of all, it complete (see, e.g., Corollary 2.3.53 in [20]), and hence closed Kobayashi balls are compact. It is vital that we can describe the boundary behavior of the Kobayashi distance: if $D \subset \mathbb{C}^n$ is a strongly pseudoconvex bounded domain and $z_0 \in D$, there exist $c_0, C_0 > 0$ such that
\[ \forall z \in D \quad c_0 - \frac{1}{2} \log d(z, \partial D) \leq k_D(z_0, z) \leq C_0 - \frac{1}{2} \log d(z, \partial D), \]
where $d(\cdot, \partial D)$ denotes the Euclidean distance from the boundary of $D$ (see Theorems 2.3.51 and 2.3.52 in [20]).

Let $\nu$ denote the Lebesgue volume measure of $R^{2n}$, normalized so that $\nu(B^n) = 1$. Then the volume of a Kobayashi ball $B_{B^n}(z_0, r)$ is given by (see [38])
\[ \nu(B_{B^n}(z_0, r)) = r^{2n} \left( \frac{1 - \|z_0\|^2}{1 - r^2 \|z_0\|^2} \right)^{n+1}. \]

A similar estimate is valid for the volume of Kobayashi balls in strongly pseudoconvex bounded domains:

**Lemma 2.1.** (see [19], [20]) Let $D \subset \mathbb{C}^n$ be a strongly pseudoconvex bounded domain. Then there exist $c_1 > 0$ and, for each $r \in (0; 1)$, a $C_{1,r} > 0$...
depending on $r$ such that

$$c_1 r^{2n} d(z_0, \partial D)^{n+1} \leq \nu (B_D(z_0, r)) \leq C_1 r d(z_0, \partial D)^{n+1}$$

for every $z_0 \in D$ and $r \in (0; 1)$.

- given two non-negative functions $f, g : D \to \mathbb{R}^+$ we shall write $f \leq g$ to say that there is $C > 0$ such that $f(z) \leq C g(z)$ for all $z \in D$. The constant $C$ is independent of $z \in D$, but it might depend on other parameters ($r, \theta, \text{etc}$);

- given two strictly positive functions $f, g : D \to \mathbb{R}^+$ we shall write $f \approx g$ if $f \leq g$ and $g \leq f$, that is it there is $C > 0$ such that $C^{-1} g(z) \leq f(z) \leq C g(z)$ for all $z \in D$;

- $dv$ (or $du$) will be the Lebesque measure.

Let $D = \{ z : \rho(z) < 0 \}$ be a bounded strictly pseudoconvex domain of $\mathbb{C}^n$ with $C^\infty$ boundary. We assume that the strictly plurisubharmonic function $\rho$ is of class $C^\infty$ in a neighborhood of $\overline{D}$, that, $-1 \leq \rho(z) < 0, z \in D$ and $|\partial \rho| \geq c_0 > 0$ for $|\rho| \leq r_0$.

- We let $g(z, \zeta)$ be the associated Levi polynomial

$$g(z, \zeta) = 2 \sum_{j=1}^n \frac{\partial \rho}{\partial \zeta_j}(\zeta_j - z_j) - \sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial \zeta_j \partial \zeta_k}(\zeta_j - z_j)(\zeta_k - z_k).$$

It follows from Taylor’s formula and the strict plurisubharmonicity of $\rho$ that there are positive constants $C_1$ and $r$ and a neighborhood $D'$ of $D$ such that

$$\Re g(z, \zeta) \geq \rho(\zeta) - \rho(z) + C_1 |z - \zeta|^2$$

for $z, \zeta \in D'$ and $|z - \zeta| \leq r$. Setting $\tilde{g}(z, \zeta) = g(z, \zeta) - 2\rho(\zeta)$, it follows that

$$\Re \tilde{g}(z, \zeta) = \Re g(z, \zeta) - 2\rho(\zeta) \geq -\rho(\zeta) - \rho(z) + C_1 |z - \zeta|^2$$

(1)

for $z, \zeta \in D'$ and $|z - \zeta| \leq r$ and $\tilde{g}(z, \zeta) = g(z, \zeta)$ for $\zeta \in \partial D$. Let further $N$ be the complex normal vector field of type $(0,1)$ defined by

$$N = \sum_{j=1}^n \frac{\partial \rho}{\partial \zeta_j} \frac{\partial}{\partial \zeta_j}.$$

Also we have

$$N \tilde{g}(z, \zeta) = \mathcal{O}(|z - \zeta|^2).$$
Lemma 2.2. (see [1]). Let $\tilde{g}$, $D'$, $r$ and $C_1$ be as above. There is a neighborhood $\tilde{D}$ of $D$ with $\tilde{D} \supset D'$, a $C^\infty$ function $\tilde{\Phi}$ (Henkin-Ramires function) on $\tilde{D} \times \tilde{D}$, and a positive constant $C_2$ such that

(i) for any $\zeta \in \tilde{D}$ the function $\tilde{\Phi}(\cdot, \zeta)$ is holomorphic on $\tilde{D}$;
(ii) $\tilde{\Phi}(\zeta, \zeta) = -2\rho(\zeta)$ for $\zeta \in \tilde{D}$, and $|\tilde{\Phi}(z, \zeta)| \geq C_2$ for $z, \zeta \in \tilde{D}$ with $|z - \zeta| \geq \frac{r}{2}$;
(iii) there is a non-vanishing $C^\infty$ function $Q(z, \zeta)$ on $\Delta_1 = \{ (z, \zeta) \in \tilde{D} \times \tilde{D} : |z - \zeta| \leq \frac{r}{2} \}$

such that $\tilde{\Phi}(z, \zeta) = \tilde{g}(z, \zeta)Q(z, \zeta)$ on $\Delta_1$.

Lemma 2.3. (see [1]). For each $s > -1$, there is a smooth form $\eta_s \in C^\infty(\tilde{D} \times \tilde{D})$ such that

(i) $\eta_s(z, \zeta)$ is holomorphic in $z$ on $\tilde{D}$ for any fixed $\zeta \in \tilde{D}$, and
(ii) for $f \in A_1^s(D)$ and $z \in D$ we have

$$f(z) = \int_D f(\zeta) \frac{\eta_s(z, \zeta)}{\tilde{\Phi}(z, \zeta)} (-\rho(\zeta))^s d\nu(\zeta).$$

We shall use the following notations:

- $\delta : D \to \mathbb{R}^+$ will denote the Euclidean distance from the boundary, that is $\delta(z) = d(z, \partial D)$.
- Let $d\nu_t(z) = (\delta(z))^t d\nu(z)$, $t > -1$.

We define Bergman kernels now via Henkin-Ramirez function.

Definition 2.1. (see [1]). We say that $K$ is a Bergman kernel of type $t$, $t > 0$, and we will write $K_t$, if $|K(z, \zeta)| \leq c|\Phi(z, \zeta)|^{-t}$.

- $H(D)$ will denote the space of holomorphic on $D$, endowed with the topology of uniform convergence on compact subsets;
- given $1 \leq p \leq +\infty$, the Bergman space $A^p(D)$ is the Banach space $L^p(D) \cap H(D)$, endowed with the $L^p$-norm;
- more generally, given $\beta \in \mathbb{R}$ we introduce the weighted Bergman space $A^p(D, \beta) = L^p(\delta^\beta \nu) \cap H(D)$ endowed with the norm

$$||f||_{p, \beta} = \left( \int_D |f(\zeta)|^p \delta^\beta(\zeta) d\nu(\zeta) \right)^{\frac{1}{p}}.$$
if $1 \leq p \leq +\infty$, and with the norm
\[ \|f\|_{\infty, \beta} = \|f \delta^\beta\|_{\infty} \]
if $p = \infty$;

- $K : D \times D \to \mathbb{C}$ will be the Bergman kernel of $D$. The $K_t$ is a kernel of type $t$ defined with the help of Henkin-Ramirez function $\Phi$. Note if $K$ is kernel of type $t$, $t \in \mathbb{N}$. Then $K^s$ is kernel of type $st$, $s \in \mathbb{N}, t \in \mathbb{N}$. This follows directly from definition (see [1]). Note $K = K_{n+1}$ (see [20], [1]);

- for each $z_0 \in D$ we shall denote by $z_0$ the normalized Bergman kernel defined by
\[ k_{z_0}(z) = \frac{K(z, z_0)}{\sqrt{K(z_0, z_0)}} = \frac{K(z, z_0)}{\|K(\cdot, z_0)\|_2}; \]

- given $r \in (0; 1)$ and $z_0 \in D$, we shall denote by $B_D(z_0, r)$ the Kobayashi ball of center $z_0$ and radius $\frac{1}{2} \log \frac{1 + r}{1 - r}$.

See, e.g., [20], [29], [28], [21] for definitions, basic properties and applications to geometric function theory of the Kobayashi distance; and [30]-[33] for definitions and basic properties of the Bergman kernel. Let us now recall a number of results proved in [19]. We will often use them in proofs of our theorems. The first two give information about the shape of Kobayashi balls:

**Lemma 2.4. (Lemma 2.1 in [19])** Let $D \subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain, and $r \in (0; 1)$. Then
\[ \nu(B_D(\cdot, r)) \approx \delta^{n+1}, \]
(where the constant depends on $r$).

**Lemma 2.5. (Lemma 2.2 in [19])** Let $D \subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain. Then there is $C > 0$ such that
\[ \frac{C}{1 - r} \delta(z_0) \leq \delta(z) \leq \frac{1 - r}{C} \delta(z_0) \]
for all $r \in (0; 1)$, $z_0 \in D$ and $z \in B_D(z_0, r)$.

**Definition 2.2.** Let $D \subset \mathbb{C}^n$ be a bounded domain, and $r > 0$. An $r$-lattice in $D$ is a sequence $\{a_k\} \subset D$ such that $D = \bigcup_k B_D(a_k, r)$ and there exist $m > 0$ such that any point in $D$ belongs to an most $m$ balls of the form $B_D(a_k, R)$, where $R = \frac{1}{2}(1 + r)$. 
Note by Lemma 2.3
\[ \nu_\alpha(B_D(a_k, R)) = \int_{B_D(a_k, R)} \delta^\alpha(z) d\nu(z) \asymp (\delta^\alpha(a_k)) \nu(B_D(a_k, R)), \alpha > -1. \]

The existence of \( r \)-lattice in bounded strongly pseudoconvex domains is ensured by the following

**Lemma 2.6. (Lemma 2.5 in [19])** Let \( D \subset \mathbb{C}^n \) be a bounded strongly pseudoconvex domain. Then for every \( r \in (0; 1) \) there exists an \( r \)-lattice in \( D \), that is there exist \( m \in \mathbb{N} \) and a sequence \( \{a_k\} \subset D \) of points such that \( D = \bigcup_{k=0}^\infty B_D(a_k, r) \) and no point of \( D \) belongs to more than \( m \) of the balls \( B_D(a_k, R) \), where \( R = \frac{1}{2}(1 + r) \).

We will call \( r \)-lattice sometimes the family \( B_D(a_k, r) \). Dealing with \( K \) Bergman kernel we always assume \( |K(z, a_k)| \asymp |K(a_k, a_k)| \) for any \( z \in B_D(a_k, r) \), \( r \in (0; 1) \) (see [19], [20]). Let \( m = (n + 1)l, l \in \mathbb{N} \). Then \( |K_m(z, a_k)| \asymp |K_m(a_k, a_k)|, z \in B_D(a_k, r), r \in (0; 1) \). This fact is crucial for embedding theorems in pseudoconvex domains (see also [15], [35]).

We shall use a submean estimate for nonnegative plurisubharmonic function on Kobayashi balls:

**Lemma 2.7. (Corollary 2.8 in [19])** Let \( D \subset \mathbb{C}^n \) be a bounded strongly pseudoconvex domain. Given \( r \in (0; 1) \), set \( R = \frac{1}{2}(1 + r) \in (0; 1) \). Then there exists a \( C_r > 0 \) depending on \( r \) such that

\[
\forall z_0 \in D, \forall z \in B_D(z_0, r), \quad \chi(z) \leq \frac{C_r}{\nu(B_D(z_0, r))} \int_{B_D(z_0, R)} \chi d\nu
\]

for every nonnegative plurisubharmonic function \( \chi : D \to \mathbb{R}^+ \).

We will use this lemma for \( \chi = |f(z)|^q, f \in H(D), q \in (0, +\infty) \).

Obviously using properties of \( \{B_D(a_k, R)\} \) Kobayashi balls we will have the following estimates for Bergman space \( A^p_D \)

\[
\|f\|_{A^p_D}^p = \int_D |f(\omega)|^p \delta^\alpha(\omega) d\nu(\omega) \asymp \sum_{k=1}^\infty \left[ \max_{z \in B_D(a_k, R)} |f(z)|^p \right] \nu_a B_D(a_k, R) \asymp \sum_{k=1}^\infty \int_{B_D(a_k, R)} |f(z)|^p \delta^\alpha(z) d\nu(z), \quad 0 < p < \infty, \quad \alpha > -1.
\]

Let now

\[
A(p, q, \alpha) = \left\{ f \in H(D) : \sum_{k=1}^\infty \left( \int_{B_D(a_k, R)} |f(z)|^p \delta^\alpha(z) d\nu(z) \right)^\frac{q}{p} < \infty \right\}.
\]
where $0 < p, q < \infty, \alpha > -1$. These are Banach spaces if $\min(p, q) \geq 1$.

These $A(p, q, \alpha)$ spaces (or their multifunctional generalizations) can be considered as natural extensions of classical Bergman spaces in strictly $D$ pseudoconvex domains with smooth boundary for which \( \{ B_D(a_k, R) \} \) family exists related to $r$ - lattice \( \{(a_k)\} \) (see [19], [20]). It is natural to consider the problem of extension of classical results on $A_p^\alpha(D)$ spaces to there $A(p, q, \alpha)$ spaces. Some our results are motivated with this problem.

We now collect a few facts on the (possibly weighted) $L^p$-norms of the Bergman kernel and the normalized Bergman kernel. The first result is classical (see, e.g., [19], [20]):

**Lemma 2.8. (see [20])** Let $D \subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain. Then

\[
\| K(\cdot, z_0) \|_2 = \sqrt{K(z_0, z_0)} \approx \delta^{-\frac{n+1}{2}}(z_0) \quad \text{and} \quad \| k_{z_0} \|_2 \equiv 1
\]

for all $z_0 \in D$.

The next result is the main result of this section, and contains the weighted $L^p$-estimates we shall need:

**Theorem A (see [20])** Let $D \subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain, and let $z_0 \in D$ and $1 < p < \infty$. Then

\[
\int_D |K(\zeta, z_0)|^p \delta^\beta(\zeta) d\nu(\zeta) \approx \begin{cases} 
\delta^{\beta-(n+1)(p-1)}(z_0), & -1 < \beta < (n+1)(p-1); \\
|\log \delta(z_0)|, & \beta = (n+1)(p-1); \\
1, & \beta > (n+1)(p-1).
\end{cases}
\]

A complete analogue of this theorem is valid also for $K_t$ kernel $t > 0$ (see [1]).

We just replace $n+1$ by $t$. We consider in this paper analytic spaces on products of pseudoconvex domains $D \times \cdots \times D = D^m$.

We denote by $H(D \times \cdots \times D) = H(D^m), m \in \mathbb{N}$, the space of analytic functions by each variable on $D^m$.

Let

\[
L_{a}^p(D^m) = \{ f \in L_{loc}^1(D^m) : \left( \int_D \cdots \left( \int_D |f(z_1, \ldots, z_m)|^{p_1} \delta^{\alpha_1}(z_1) d\nu(z_1) \right)^{\frac{p_2}{p_1}} \cdots \delta^{\alpha_m}(z_m) d\nu(z_m) \right)^{\frac{1}{p_m}} < \infty \},
\]

$0 < p_i < \infty, \alpha_i > -1, i = 1, \ldots, m, A_{a}^p(D^m) = L_{a}^p(D^m) \cap H(D^m), m \in \mathbb{N}$.

These are Banach spaces if $\min(p_j) \geq 1$ and complete metric spaces for other values of $p$. Such spaces in $\mathbb{R}^n$ was studied in [6], in polydisk in [26].
3. MAIN RESULTS

In this section we extend main results of [25]. The main idea is to replace \( r \)-lattices of the unit disk heavily used in [25] by \( r \)-lattices in bounded pseudoconvex domains from [19], [20] (see previous section) keeping main steps in old proof of less general case.

**Theorem 3.1.** Let \( \tilde{\mu} = (\mu_1, \ldots, \mu_n), \mu_j \) be Borel nonnegative measures on \( D \subset \mathbb{C}^m, m \in \mathbb{N}, \alpha_j > -1, j = 1, \ldots, n, \tilde{p} = (p_1, \ldots, p_n), \tilde{q} = (q_1, \ldots, q_n) \in \mathbb{R}_+^n \) with \( 0 < p_j \leq q_j, j = 1, \ldots, n \). Then the following assertions are equivalent:

1. \( \|f\|_{L^{\tilde{p}}_{\tilde{\mu}}} \leq C_0 (\tilde{\mu}) \|f\|_{A^\tilde{q}_{\tilde{g}}} \);

2. \( \prod_{j=1}^n \mu_j (B_D (a_{kj}, r)) \leq C_1 \prod_{j=1}^n (1 - |a_{kj}|)^{(\alpha_j + 1)q_j} \), \( j = 1, \ldots, n, k = 0, 1, 2, \ldots \) where \( \{a_k\} \) is \( r \)-lattice of \( D \).

In the case of measures \( \mu \) defined on \( D^n \) we have following result:

**Theorem 3.2.** Let \( \{a_k\} \) be \( r \)-lattice of \( D \subset \mathbb{C}^k \). Let \( p_j \leq q < +\infty, \alpha_j > -1, 1 \leq j \leq n, \mu \) be the Borel nonnegative measure on \( D^n \). Let \( \tilde{z} = (z_1, \ldots, z_n) \), then the following assertions are equivalent:

1. \( \left( \int_{D^n} |f(\tilde{z})|^q d\mu(\tilde{z}) \right)^{1/q} \leq C_2 (\mu) \|f\|_{A^\tilde{q}_{\tilde{g}}} \);

2. \( \mu (B_D (a_{k_1}, r) \times \cdots \times B_D (a_{k_n}, r)) \leq C_3 \prod_{j=1}^n \left( 1 - |a_{kj}| \right)^{(\alpha_j + 1)q_j} \), \( j = 1, \ldots, n, k_j = 0, 1, 2, \ldots \).

To prove Theorems 3.1 and 3.2 we need some auxiliary results. All results below are known for unit polydisk and ball (see [25], [24], [17], [40], [18]). Proof are similar in all these cases and in case of pseudoconvex domains based on properties of \( r \)-lattice in particular (see [25]) for unit disk case.

These lemmas at the same time provide basic properties of new holomorphic mixed norm \( A^p_{\tilde{g}} \) spaces in products of bounded strongly pseudoconvex domains with smooth boundary and may have various applications.

**Lemma 3.1.** Let \( f \in A^p_{\tilde{g}}, 0 < p_j < +\infty, j = 1, \ldots, n \). Then the following estimate holds

\[
\sum_{k_n=0}^{+\infty} \left( \max_{\zeta_n \in B_D (a_{nk_n}, r)} \left( \cdots \sum_{k_2=0}^{+\infty} \left( \max_{\zeta_2 \in B_D (a_{k_2}, r)} \left( \sum_{k_1=0}^{+\infty} \left( \max_{\zeta_1 \in B_D (a_{k_1}, r)} |f(\zeta_1, \ldots, \zeta_n)|^{p_1} \right)^{1/p_1} \right) \right) \right) \right) \leq C_4 (\tilde{\mu}) \|f\|_{A^p_{\tilde{g}}}
\]
\[ \times |B_D(a_{k_1}, r)|^{\frac{\alpha_1}{\pi r^2}} |B_D(a_{k_1}, r)|^{\frac{\alpha_2}{\pi r^4}} |B_D(a_{k_2}, r)|^{\frac{\alpha_2}{\pi r^4}} \cdots \times |B_D(a_{k_n}, r)|^{\frac{\alpha_n}{\pi r^2}} \]

\[ \leq C_4(\vec{\omega}, \vec{p}) \|f\|_{A^p_n}, \]

where \( |B_D(a_{k_j}, r)| \) Lebesgue measure of Kobayashi ball \( B_D(a_{k_j}, r), j = 1, \ldots, n \).

The proof follows from lemma 2.7 the fact that

\[ \Phi(z_2) = \left( \int_D |f(z_1, z_2)|^{p_1} (\delta(z_1))^{\alpha_1} d\nu(z_1) \right)^{\beta}, \]

\[ \beta > 0, \alpha_1 > -1, 0 < p_1 < \infty, z_2 \in D, \]

is plurisubharmonic in \( D \) (see [35], [39]).

The proof of the following theorem is based again on lemma 2.7 and some standard arguments known in the unit disk, in the unit ball (see [26] for same proof in the polydisk case).

**Lemma 3.2.** In the context of the previous lemma we have the estimate

\[ |f(z_1, \ldots, z_m)| \leq C_5 \frac{\|f\|_{A^p_n}}{\prod_{j=1}^m (\delta(z_j))^{\frac{k_{j-1}}{p_j}} (\delta(z_j))^{\frac{\alpha_j}{p_j}}}, \]

\( z = (z_1, \ldots, z_m) \in D^m. \)

We introduce the product kernel \( D_{\vec{\alpha}}(\zeta, z) : \)

\[ D_{\vec{\alpha}}(\zeta, z) = \prod_{j=1}^n K_{\alpha_j}(\zeta_j, z_j), \]

\( \zeta = (\zeta_1, \ldots, \zeta_n), \ z = (z_1, \ldots, z_n) \in D^n, \ \vec{\alpha} = (\alpha_1, \ldots, \alpha_n), \ \alpha_j \in \mathbb{N}, \ j = 1, \ldots, n. \)

**Lemma 3.3.** Let

\[ f \in A^p_{\vec{\alpha}}, \ \vec{\alpha} = (\alpha_1, \ldots, \alpha_m), \ \beta_j > (\vec{\beta}_j), \ \vec{\beta}_j = (\vec{\beta}_j(k, \vec{p}), j = 1, \ldots, m. \tag{3} \]

Then the following representation holds

\[ f(z) = \int_{D^m} D_{\vec{\beta}}(\zeta, z) f(\zeta) d\nu_m(\zeta), \ z \in D^m, \tag{4} \]
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where \( dv_{m}(\zeta) = \prod_{j=1}^{m} dv_{j}(\zeta_{j}), \zeta_{j} \in D, j = 1, \ldots, m. \)

**Proof.** First we note that using Lemma 2.8, we obtain

\[
|f(z_{1}, \ldots, z_{m})| \leq C_{6} \prod_{j=1}^{m} \left[ \delta^{\frac{k+1}{p_{j}^{\ast}}} (z_{j}) \right] \left[ \delta (z_{j}) \right]^{\frac{\alpha_{j}}{p_{j}^{\ast}}},
\]

\( z = (z_{1}, \ldots, z_{m}) \in D^{m}. \)

Hence, in view of (2), we obtain that the space \( A_{p}^{\vec{\alpha}} \) is embedded in \( A_{1}^{\vec{p}} \), where \( A_{1}^{\vec{p}} \) coincides with the class \( A_{\vec{p}}^{\vec{\alpha}} ; \vec{p} = (p_{1}, \ldots, p_{m}) \), for \( p_{j} = 1, j = 1, \ldots, k. \)

Using the result of [1], [19]-[20] we obtain that \( f \) admits a representation (4). The lemma is proved. \( \square \)

The following lemma is proved in [26] for the unit disk and polydisk. The same proof based on parallel properties of \( r \)-lattices is valid in context of bounded strongly pseudoconvex domains. It is interesting enough and may have various applications.

The complete proof will be presented elsewhere.

This lemma is valid under condition \( |K_{t}(z, w)| \asymp |K_{t}(v, w)|, w \in D, z \in B_{D}(v, r), t > 0, v \in D. \) This is valid in the ball and milder version is valid in pseudoconvex domains (see [40] - [20]).

**Lemma 3.4.** Let \( \vec{p} = (p_{1}, \ldots, p_{m}) \), \( 0 < p_{j} < +\infty, \vec{\alpha}, \vec{\alpha} = (\alpha_{1}, \ldots, \alpha_{m}) \), \( j = 1, \ldots, m. \) Then the operator

\[
T_{\vec{p}}(f) (z) = \int_{D^{m}} D_{\vec{p}} (\zeta, z) |f(\zeta)| dv(\zeta_{1}) \ldots dv(\zeta_{m}), z \in D^{m},
\]

maps the space \( A_{p}^{\vec{\alpha}} \) in \( L_{\vec{p}}^{\vec{\alpha}} \), where \( L_{\vec{p}}^{\vec{\alpha}} \) means the class \( L_{\vec{p}}^{\vec{\alpha}} \) with \( d_{\mu_{j}} = \delta^{\gamma_{j}} (|\zeta_{j}|) \) \( d \nu (\zeta_{j}) \), \( \zeta_{j} \in D, j = 1, \ldots, m. \)

**4. PROOFS OF THE MAIN THEOREMS**

In this section we prove our main results. Comparing with the simplest case of the unit ball we may note our proofs are based mainly on same type ideas, but technically our proofs are more complicated. The estimate of Bergman kernel from below on the Bergman ball is playing again a crucial role (see [40], [20] for some related recent results in the unit ball and bounded pseudoconvex domains).
Proof of Theorem 3.1. The basic estimates and facts for this proof and next theorem are 1) $\Phi$ is plurisubharmonic there

$$\Phi(z_2) = \left( \int \left| f(z_1, z_2) \right|^q (\delta(z_1))^\alpha \, d\nu(z_1) \right)^\beta$$

$\beta > 0$, $\alpha_1 > -1$, $z_2 \in D$, $q \in (0, \infty)$, $f \in H(D^2)$, $\chi = B_D(z, r)$ or $D$ and

$$\left( \frac{\int |f(z)| (\delta(z))^\alpha \, d\nu(z)}{|D^2|} \right)^\tau \leq C_0 \int \frac{|f(z)|^p (\delta(z))^{\alpha p+(k+1)p-(k+1)} \, d\nu(z)}{|D^2|},$$

where $0 < \tau \leq 1$, $\alpha > -1$, $f \in H(D^2)$ and $d\nu(z) = d\nu(z_1) \times d\nu(z_2)$, $f(z) = f(z_1, z_2)$, $\delta(z) = (\delta(z_1)) \times (\delta(z_1))$. The second estimate can be seen, for example, in [40]. The proof of first estimate can be see in [39].

The proof of Theorem 3.1 will be given for the case $n = 2$; the arguments for the case $n > 2$ are very similar (and so, they will be omitted).

The implication $2) \implies 1)$ follows in a standard way by using our remark after Lemma 2.7 and the function $e_z(\zeta) = \prod_{j=1}^m K_{\beta_j} (\zeta_j, z_j)$, $z = (z_1, ..., z_m)$, $\zeta = (\zeta_1, ..., \zeta_m) \in D^m$, for sufficiently large $\beta_j : \beta_j > \beta_0$, $\beta_j \in \mathbb{N}$, $j = 1, 2$ (see [15]).

Indeed we have to choose simply an appropriate $f$ test function to get what we need. Put $f = K_{\beta_1} \times K_{\beta_2}$ and we put this test function into estimate

$$\left( \int_D \left( \int_D |f(z_1, z_2)| \, d\mu_1(z_1) \right)^{\frac{p_2}{p_1}} \, d\mu_2(z_2) \right)^{\frac{1}{p_2}} \leq \leq C_1(\mu) \left( \int_D \left( \int_D |f(z_1, z_2)| \, d\nu_1(z_1) \right)^{\frac{p_2}{p_1}} \, d\nu_2(z_2) \right)^{\frac{1}{p_2}}$$

Note these type arguments were used many times in various embeddings in ball, polydisk, before. Note using theorem A we have for $\beta_j > \beta_0$, $j = 1, 2$

$$\left( \int_D \left( \int_D \left| K_{\beta_1}(z_1, q_1) \right|^p |K_{\beta_2}(z_2, q_2)|^p \, d\nu(z_1) \right)^{\frac{1}{p_2}} \, d\nu_2(z_2) \right)^{\frac{1}{p_2}} \leq \leq$$
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\[ \leq C_2 \left( \int_D |K_{\beta_2}(z_2, q_2)|^{p_2} \, d\nu(z_2) \right)^{\frac{1}{p_2}} \times \left( \int_D |K_{\beta_1}(z_1, q_1)|^{p_1} \, d\nu(z_1) \right)^{\frac{1}{p_1}} \leq C_3 \left( (\delta(q_2))^{\alpha_2} \right) \left( (\delta(q_1))^{\alpha_1} \right) ; \]

where \( \alpha_1 = \frac{-(\beta_1(p_1-1))}{p_1} \), \( \alpha_2 = \frac{-(\beta_2(p_2-1))}{p_2} \).

Then using lemmas and remarks above on \( r \)-lattices we get the following estimates from below for \( f = K_{\beta_1} \times K_{\beta_2} \)

\[ \left( \int_D \left( \int_D |f(z_1, z_2)|^{q_1} \, d\mu_1(z_1) \right)^{\frac{q_2}{q_1}} \, d\mu_2(z_2) \right)^{\frac{1}{q_2}} \geq C_4 \left( \int_{B(a_{k_1}, R)} \left( \int_{B(a_{k_2}, R)} |f(z_1, z_2)|^{q_1} \, d\mu_1(z_1) \right)^{\frac{q_2}{q_1}} \, d\mu_2(z_2) \right)^{\frac{1}{q_2}} \]

\[ \geq C_5 \left( \int_{B(a_{k_1}, R)} |K_{\beta_2}(z_1, q_1)|^{q_2} \, d\mu_2(z_2) \right)^{\frac{1}{q_2}} \times \left( \int_{B(a_{k_2}, R)} |K_{\beta_1}(z_2, q_2)|^{q_1} \, d\mu_1(z_1) \right)^{\frac{1}{q_1}} \]

\[ \geq C_6 \left( \prod_{j=1}^{2} \mu_j \left( B(a_{k_j}, R) \right) \times \delta^{\tau_j}(a_{k_j}) \right) , \]

for some \( \tau_j, j = 1, 2 \). The rest is clear.

Therefore, we turn to the proof of the implication 1) \( \Rightarrow \) 2). Choose on \( r \)-lattice for \( D \) domain \( \{a_k\} \) and a family of Kobayashi balls \( \{B_D(a_k, r)\} \).

Expanding the domain as the union of dyadic domains, we obtain

\[ I(z_2) = \left( \int_D |f(z_1, z_2)|^{q_1} \, d\mu_1(z_1) \right)^{\frac{1}{q_1}} \leq C_7 \left( \sum_{k_1=0}^{+\infty} \max_{z_1 \in B_D(a_{k_1}, r)} \left\{ |f(z_1, z_2)|^{q_1} (\delta(z_1))^{(k+1)\frac{q_2}{q_1}} (\delta(z_1))^{\frac{q_2}{q_1}\alpha_1} \right\} \right)^{\frac{1}{q_1}} . \]
Taking into account that \( \frac{p_1}{q_1} \leq 1 \), we have

\[
I(z_2) \leq C_9 \left( \sum_{k=0}^{+\infty} \max_{z_1 \in B_D(a_k, r)} \left\{ |f(z_1, z_2)|^{p_1} \left( \delta(z_1)^{k+1} \delta(z_1)^{\alpha_1} \right) \right\} \right)^{\frac{1}{p_1}}.
\]

Now, applying Lemma 2.7 and properties of \( r \)-lattices to the function \( f(z_1, z_2) \) for fixed \( z_2 \in D \) we obtain

\[
I(z_2) \leq C_9 \left( \int_D |f(z_1, z_2)|^{p_1} \left( \delta(z_1)^{\alpha_1} \right) d\nu(z_1) \right)^{\frac{1}{p_1}}.
\]

Raising to the power \( q_2 \) both sides of the last inequality and integrating over \( \mu_1 \), we obtain

\[
\left( \int_D [I(z_2)]^{q_2} d\mu_2(z_2) \right)^{\frac{1}{q_2}} \leq C_0 \left( \int_D \left( \int_D |f(z_1, z_2)|^{p_1} \left( \delta(z_1)^{\alpha_1} \right) d\nu(z_1) \right)^{\frac{q_2}{p_1}} d\mu_2(z_2) \right)^{\frac{1}{q_2}}.
\]

Again we using decomposition of the \( D \) domain as the union of dyadic domains \( \{B_D(a_k, r)\} \) and take into account (1):

\[
\left( \int_D [I(z_2)]^{q_2} d\mu_2(z_2) \right)^{\frac{1}{q_2}} \leq C_1 \left( \sum_{k=0}^{+\infty} \max_{z_2 \in B_D(a_{k_2}, r)} \left( \int_D |f(z_1, z_2)|^{p_1} \left( \delta(z_1)^{\alpha_1} \right) d\nu(z_1) \right)^{\frac{q_2}{p_1}} \mu(B_D(a_{k_2}, r)) \right)^{\frac{1}{q_2}} \leq C_1 \left( \sum_{k=0}^{+\infty} \max_{z_2 \in B_D(a_{k_2}, r)} \left( \int_D |f(z_1, z_2)|^{p_1} \left( \delta(z_1)^{\alpha_1} \right) d\nu(z_1) \right)^{\frac{q_2}{p_1}} \mu(B_D(a_{k_2}, r)) \right)^{\frac{1}{q_2}} \leq C_2 \left( \sum_{k=0}^{+\infty} \max_{z_2 \in B_D(a_{k_2}, r)} \left( \int_D |f(z_1, z_2)|^{p_1} \left( \delta(z_1)^{\alpha_1} \right) d\nu(z_1) \right)^{\frac{q_2}{p_1}} \mu(B_D(a_{k_2}, r)) \right)^{\frac{1}{q_2}}.
\]

Considering equality \( \frac{1}{q_2} = \frac{p_2}{q_2} \frac{1}{p_2} \) and \( \frac{p_2}{q_2} \leq 1 \), we have the estimate

\[
\left( \int_D [I(z_2)]^{q_2} d\mu_2(z_2) \right)^{\frac{1}{q_2}} \leq
\]
\[ \leq C_3 \left( \sum_{k=0}^{+\infty} \max_{z \in B_D(a, r)} \left( \int_D |f(z_1, z_2)|^{p_1} (\delta(z_1))^{\alpha_1} \, d\nu(z_1) \right) \right)^{\frac{p_2}{p_1}} \left( \delta(z_2)^{\alpha_2} (\delta(z_1))^{k+1} \right)^{\frac{1}{q_2}}. \]

To prove the theorem it remains to apply Lemmas 2.7 and 3.1. An important fact is that lemma 2.7 and properties of \( r \)-lattices can be applied to \( z_2 = 0 \):

\[ \int_D f(z_1, z_2) \left( \int_D |f(z_1, z_2)|^{p_1} (\delta(z_1))^{\alpha_1} \, d\nu(z_1) \right)^{\frac{p_2}{p_1}} d\nu(z_2). \]

since \( \Phi(z_2) \) is plurisubharmonic (see [39]); hence, our proof is complete.

**Proof of Theorem 3.2.** As in the proof of Theorem 2.1, the implication \( 2) \Rightarrow 1) \) is verified in a standard way.

We have the following estimate for test function \( \tilde{f} = K_{b_1} \times K_{b_2} \) for \( n = 2 \) case

\[ I = \left( \int_{D^2} |\tilde{f}(\zeta)|^q \, d\mu(\zeta) \right)^{\frac{1}{q}} \leq C_4(\mu) \left( \int_D \left( \int_D |\tilde{f}(z_1, z_2)|^{p_1} \, d\mu(z_1) \right)^{\frac{p_2}{p_1}} d\mu(z_2) \right)^{\frac{1}{q_2}}. \]

Then we have

\[ I \geq C_5 \left( \int_{B(a_1, r)} \int_{B(a_2, r)} |\tilde{f}(z_1, z_2)|^q \, d\mu(\zeta) \right)^{\frac{1}{q}} \geq C_6 (\mu (B_D(a_1, r) \times B_D(a_2, r))) \times (\delta (a_1))^{\tau_1} \times (\delta (a_2))^{\tau_2} \]

for some \( \tau_1 = \tau_1(q, \beta), \tau_2 = \tau_2(q, \beta) \).

The last estimate is based on estimate from below of Bergman kernel on Bergman balls. And also based on theorem A we have

\[ \|\tilde{f}\|_{A_\alpha^q} \leq C_7 \left( (\delta (a_1))^{\beta_1} (\delta (a_2))^{\beta_2} \right) \]

for some values of \( \beta_1, \beta_2 \):

\[ \beta_1 = \beta_1(p_2, p_1, \beta_1, \beta_2); \]
\[ \tau_j - \beta_j = \left( -\alpha + k + 1 \right) \frac{q}{p_j}, \quad j = 1, 2. \] This gives one part of theorem.

We proceed to the proof of 1) \( \Rightarrow \) 2). Again, we prove the theorem for \( m = 2 \), since for \( m > 2 \) similar arguments are valid. First suppose that

\[ q \geq p_1 \geq p_2. \quad (5) \]

Using the arguments provided in the proof of Theorem 3.1, we have

\[
I = \left( \int_{D^2} |f(z_1, z_2)|^q \, d\mu(z_1) d\mu(z_2) \right)^{\frac{1}{q}} \leq \frac{1}{q}.
\]

\[
I \leq C_0 \left( \int_{D} \left( \delta(z_1) \right)^{\alpha_1 p_1} \left( \delta(z_2) \right)^{\alpha_2 p_2} \right) \times \left( \int_{D} \left( \delta(z_1) \right)^{\alpha_1 p_1} \left( \delta(z_2) \right)^{\alpha_2 p_2} \right) \frac{1}{q}.
\]

Taking into account that \( p_1 \leq q \), we obtain

\[
I \leq C_0 \left( \int_{D} \left( \delta(z_1) \right)^{\alpha_1 p_1} \left( \delta(z_2) \right)^{\alpha_2 p_2} \right) \times \left( \int_{D} \left( \delta(z_1) \right)^{\alpha_1 p_1} \left( \delta(z_2) \right)^{\alpha_2 p_2} \right) \frac{1}{q}.
\]

Again, using similar arguments to the inner integral, we obtain the estimate

\[
I \leq C_{10} \left( \int_{D} \left( \delta(z_2) \right)^{\alpha_2 p_2} \left( \delta(z_2) \right)^{\alpha_1 p_1} \right) \times \left( \int_{D} \left( \delta(z_1) \right)^{\alpha_1 p_1} \left( \delta(z_1) \right)^{\alpha_1 p_1} \right) \frac{1}{p_1}.
\]

Taking into account that \( \frac{p_2}{p_1} \leq 1 \), as above, we obtain

\[
I \leq C_{11} \left( \int_{D} \left( \delta(z_2) \right)^{\alpha_2 p_2} \left( \int_{D} \left( \delta(z_1) \right)^{\alpha_1 p_1} \right) \frac{p_2}{p_1} \right) \frac{1}{p_2}.
\]
The theorem is proved under the condition (5). Now we turn to the case
\[ q \geq p_2 \geq p_1. \] (6)
If \((z_{k_1}, z_{k_2}) \in B_D(a_{k_1}, r) \times B_D(a_{k_2}, r)\) then
\[ |f(z_{k_1}, z_{k_2})|^q \leq \frac{C_{12}}{p_1} \left( \int_{B_D(z_{k_1}, r)} |f(\zeta_1, z_{k_2})|^{p_1} \, d\mu(\zeta_1) \right)^{\frac{q}{p_1}}, \]
where \(B_D(a_{k_1}, r)\) is an expansion of the dyadic domains at the same center \(B_D(a_{k_1}, r)\) and \(z_{k_2} \in B_D^*(a_{k_2}, r)\). Therefore, as in the proof of the first part:
\[ I = \left( \int_{D^2} |f(z_1, z_2)|^q \, d\mu(z_1) \, d\mu(z_2) \right)^{\frac{1}{q}} \leq \]
\[ \leq C_0 \left( \sum_{k_2=0}^{+\infty} \sum_{k_1=0}^{+\infty} |f(z_{k_1}, z_{k_2})|^q \, \mu(B_D(z_{k_1}, r)B_D(z_{k_2}, r)) \right)^{\frac{1}{q}} \]
\[ \leq C_1 \left( \sum_{k_2=0}^{+\infty} \sum_{k_1=0}^{+\infty} (\delta(z_{k_1}))^{\alpha_1} \frac{2^{k_1}}{p_1} (\delta(z_{k_2}))^{\alpha_2} \frac{2^{k_2}}{p_2} (\delta(z_{k_2}))^{(k_1+1) \frac{2^{k_1}}{p_1}} \times \right.
\[ \times \left( \int_{B_D^*(z_{k_1}, r)} |f(\zeta_1, z_{k_2})|^{p_1} \, d\mu(\zeta_1) \right)^{\frac{q}{p_1}} ^{\frac{1}{q}} \right) \leq \]
\[ \leq C_2 \left( \sum_{k_2=0}^{+\infty} \sum_{k_1=0}^{+\infty} (\delta(z_{k_2}))^{\alpha_2} \frac{2^{k_2}}{p_2} (\delta(z_{k_2}))^{(k+1) \frac{2^{k_1}}{p_1}} \left[ \int_{B_D^*(z_{k_1}, r)} |f(\zeta_1, z_{k_2})|^{p_1} \, d\mu(\zeta_1) \right] \right)^{\frac{p_2}{p_1}} \frac{1}{q}. \]

Using the inequality (5), we obtain
\[ IP^2 \leq C_2 \left( \sum_{k_2=0}^{+\infty} \sum_{k_1=0}^{+\infty} (\delta(z_{k_2}))^{\alpha_2} (\delta(z_{k_2}))^{(k+1)} \left[ \int_{B_D^*(z_{k_1}, r)} |f(\zeta_1, z_{k_2})|^{p_1} \, d\mu(\zeta_1) \right] \right)^{\frac{p_2}{p_1}} \frac{1}{q}. \]

3.1 and the fact that function in brackets is plurisubharmonic (see discussion above). We have
\[ IP^2 \leq C_2 \int_D \left( \sum_{k_1=0}^{+\infty} \left( \int_{B_D^*(z_{k_1}, r)} |f(\zeta_1, z_{k_2})|^{p_1} \right) \right)^{\frac{p_2}{p_1}} \frac{1}{q}. \]
\begin{align}
\times (\delta (\zeta_1))^{\alpha_1} d\mu (\zeta_1) \int_{B_D(\zeta_1,r)}^{|f (\zeta_1, z_2)|^p_1 \delta (\zeta_1)^{\alpha_1} d\mu (\zeta_1)} \times 
\times (\delta (\zeta_2))^{\alpha_2} d\mu (\zeta_2) \leq C_3 \int_{D} \left( \int_{D} |f (z_1, z_2)|^p_1 \delta (\zeta_1)^{\alpha_1} d\mu (z_1) \right)^{\frac{p_2}{p_1}} \times 
\end{align}

By the condition (6), $\frac{p_2}{p_1} = \alpha \geq 1$. Hence, $\sum_{k=0}^{+\infty} b_k^\alpha \leq \left( \sum_{k=0}^{+\infty} b_k \right)^\alpha$ for all $b_k \geq 0$, $k \in \mathbb{Z}_+$. Therefore, from (7) we obtain

$P^2 \leq C_2 \int_{D} \left( \sum_{k=0}^{+\infty} \int_{B_D(\zeta_1,r)}^{|f (\zeta_1, z_2)|^p_1 \delta (\zeta_1)^{\alpha_1} d\mu (\zeta_1)} \times 
\times (\delta (\zeta_2))^{\alpha_2} d\mu (\zeta_2) \leq C_3 \int_{D} \left( \int_{D} |f (z_1, z_2)|^p_1 \delta (\zeta_1)^{\alpha_1} d\mu (z_1) \right)^{\frac{p_2}{p_1}} \times 
\end{align}$

The theorem is completely proved.

References

Embedding theorems for weighted anisotropic spaces of holomorphic functions...


