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INTRINSIC VERSUS NUMERICAL CHAOS IN DISCRETE MODELS USED FOR THE STUDY OF $1\frac{1}{2}$ DEGREES-OF-FREEDOM HAMILTONIAN SYSTEMS

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Abstract The discrete models associated with Hamiltonian systems are obtained using the symmetric mapping techniques derived from Hamilton-Jacobi theorem. A local criterion for determining the optimal mapping step for the preservation of the dynamical characteristic (regular or chaotic) of an orbit is proposed. A global criterion is also derived. These criteria are applied for the study of a Hamiltonian system modeling the magnetic field configuration in tokamak.

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1. INTRODUCTION

Many important models in astronomy, plasma physics, fluid dynamics, mechanics are one-degree-of-freedom Hamiltonian systems subjected to time periodic perturbation. These systems, known as one and half degrees of freedom, are generically non-integrable. Their dynamics is not entirely regular (periodic orbits or quasiperiodic orbits lying on invariant tori) nor entirely chaotic (the chaotic orbits densely fill regions with positive measure in the phase space). Both dynamical regimes are connected in a complicate layer where regular and chaotic motion can not mix.

In order to understand this complex dynamics, the Poincaré map (the first return map) is a useful tool. It generates a two dimensional discrete dynamical system whose study gives relevant information about the original system. The exact Poincaré map can not be determined because this is equivalent with solving analytically the system. Usually it is approximated by using various techniques.

There are two philosophically different ways to realize this approximation of the Poincaré map: by performing a numerical integration with small step or by using the mapping techniques with large step.

An accurate discrete model must preserve the global properties of the continuous system (symplecticity, invariants of the motion, time-reversal or other specific symmetries) and the local dynamical characteristic of the orbits (regular or chaotic).

In order to obtain a correct approximation of the continuous system by numerical integration we must use a symplectic integrator, because the standard numerical methods introduce non-Hamiltonian perturbations that lead to a completely different long-time behavior. An excellent survey on the structure preserving integration algorithms is [1].

The main goal of mapping models is to replace the Poincaré maps of the original system by iterative maps. Important results about this topic can be found in [2]. By using these techniques the maps are constructed in the symplectic form, hence they preserve the most important property of the original system. They run much faster then the small step of numerical integration, but the main advantage is that the mapping models have better accuracy in the study of the chaotic dynamics due to the fact that the accumulation of the round-off errors is reduced.

In this paper we will focus on the symmetric mapping models derived from Hamilton-Jacobi method. We are especially interested in observing how the local dynamical characteristic of the orbits (regular or chaotic) is influenced by the mapping step.

Theoretically, by choosing a smaller mapping step, the results are closer to the continuous case. But for the study of long chaotic orbits to use small steps can be dangerous because the results are drastically influenced by to the accumulation of round-off errors.

In some cases the large mapping step creates artificial chaotic dynamics. This is the numerical chaos, generated by the imperfection of the discrete system. In other cases the chaotic dynamics is an intrinsic property of the original system and it is obviously reflected in the mapping model with large step. In this paper we propose a method for determining the optimal mapping step such that the dynamical characteristic (regular or chaotic) of an orbit is preserved. This method is used for obtaining a global criterion of accuracy for mapping having time-reversal symmetry.

The paper is organized as follows: in Section 2 some basic considerations related to Hamilton-Jacobi method are presented, Section 3 points out a direct method for identifying the chaotic orbits in discrete dynamical systems, Section 4 presents a criterion for determining the local optimal mapping step, the global criterion of accuracy is presented in Section 5, a summary and conclusions are given in Section 6.

2. THE MAPPING TECHNIQUE BASED ON HAMILTON-JACOBI METHOD

This mapping techniques was proposed in [3] by S. S. Abdullaev.

The Hamiltonian of the continuous systems for which this technique can be applied, written in action angle coordinates (I, θ), is

$$H\left(I, \theta, t\right) = H_0\left(I\right) + \sum_{m,n} H_{m,n}\left(I\right) \cdot \cos(m\theta - nt).$$

Because the perturbation is 2π periodic in t, the Poincaré cross-section (S) corresponds to $t = 2\pi$. The mapping model is generated by

$$TT_N : (S) \to (S)$$

$$TT_N = \underbrace{T_N \circ T_N \circ ... \circ T_N}_{N\,times}. \tag{1}$$

The map T_N, $T_N\left(I_k, \theta_k\right) = \left(I_{k+1}, \theta_{k+1}\right)$ having the step $2\pi/N$, is obtained from Hamilton-Jacobi method [3]. It is the composition of three area-preserving maps:

$$T_N : \begin{cases} I_k^* = I_k - \frac{\partial S^{(k)}}{\partial \theta_k}\,; \quad \theta_k^* = \left(\theta_k + \frac{\partial S^{(k)}}{\partial I_k^*}\right) \\ I_k^{**} = I_k^*\,;\ \theta_k^{**} = \left(\theta_k^* + \frac{2\pi}{N} H_0'\left(I_k^{**}\right)\right) \\ I_{k+1} = I_k^{**} + \frac{\partial S^{(k+1)}}{\partial \theta_{k+1}}\,;\ \theta_{k+1} = \left(\theta_k^{**} - \frac{\partial S^{(k+1)}}{\partial I_{k+1}}\right) \end{cases}$$

In the previous formula $S^{(k)} = S^{(k)}\left(I_k, \theta_k\right)$ is the value of the generating function

$$\begin{gathered} G\left(\mathrm{I}, \theta, t, t_0\right) = -\left(t - t_0\right) \textstyle\sum_{m,n} H_{mn}\left(I\right)\left[a\left(x_{mn}\right) \cdot\right. \\ \left. \cdot \sin(m\theta - nt) + b\left(x_{mn}\right) \cdot \cos(m\theta - nt)\right] \end{gathered}$$

taken at the moments $t = t_k = 2k\pi/N$, $a\left(x\right) = \left(1 - \cos x\right)/x$, $b\left(x\right) = \sin x/x$ and $x_{mn} = \left(mH_0'\left(I\right) - n\right)\left(t - t_0\right)$. In the symmetric case t_0 is taken exactly in the middle of the interval $[t_k, t_{k+1}]$. The non-symmetric models are obtained for other values of t_0. Maps obtained for $t_0 = t_k$ or $t_0 = t_{k+1}$ are commonly used because they can be easily manipulated, but only the symmetric mapping models preserve the time reversal symmetry of the initial system.

In [3] the accuracy of the mapping model was compared with the accuracy of the maps obtained by numerical integration. From the analyzed examples it was concluded that the mapping with a time step comparable with the period perturbation have the same accuracy as the symplectic integrator (fifth order Runge Kutta method) with the integration step two or three orders smaller.

In what follows we will deal with symmetric Hamilton-Jacobi mapping models because they preserve the main global properties of the Hamiltonian system (symplecticity and reversibility in time)

In order to exemplify our assertions, we will consider a non-integrable Hamiltonian system that describes the magnetic configuration in tokamaks (devices used for obtaining the thermo-controlled nuclear fusion). Because the tokamaks are toroidal devices it is natural to use toroidal coordinates (r, θ, ζ) in order to describe the magnetic field (ζ is the toroidal angle and (r, θ) are the poloidal coordinates in a circular poloidal section. Instead of the poloidal radius r the toroidal flux $\psi = r^2/2$ is commonly used because (ψ, θ) represent a pair of canonical variables [4]. The Hamiltonian of the system,

$$H(\psi, \theta, \zeta) = H_0(\psi) + H_1(\psi, \theta, \zeta)$$

is the sum of the ideal poloidal flux

$$H_0(\psi) = \frac{1}{4} \int (2 - \psi)\left(\psi^2 - 2\psi + 2\right) d\psi$$

and of the perturbation term $H_1(\psi, \theta, \zeta) = -\varepsilon \frac{\psi}{\psi+1} (\cos(\theta - \zeta) + \cos(\theta - 2\zeta))$ which is 2π periodic in ζ. In this case, ψ represent the action variable and the variable ζ is interpreted in analogy with a "time variable". The Poincaré section $(S) : \zeta = 2\pi$ is a vertical poloidal section of the device. The Poincaré map $T : (S) \to (S)$, $T(\psi_k, \theta_k) = \left(\psi_{k+1}, \theta_{k+1}\right)$, has a physical interpretation: the pair (ψ_k, θ_k) indicates the re-intersection of the magnetic field line with (S) after k toroidal turns.

The map obtained using the symmetric Hamilton-Jacobi method will be named the HJS-tokamap.

In figure 1 is presented the phase portrait of the HJS-tokamap obtained from (1) for $N = 1$. The θ coordinate is represented on the horizontal axis and the ψ coordinate is represented on the vertical axis.

The stochasticity parameter is $\varepsilon = 0.09$.

The main characteristics of the Hamiltonian dynamics (regular and chaotic orbits surrounding a complicate skeleton of island chains) can be observed. This is a phase portrait typical for systems generated by area-preserving maps.

3. A DIRECT METHOD FOR IDENTIFYING THE CHAOTIC ORBITS IN DISCRETE DYNAMICAL SYSTEMS.

The criterion we propose in order to verify if an orbit is regular or chaotic is related to a specific property of the chaotic dynamics: the sensitive dependence on initial conditions.

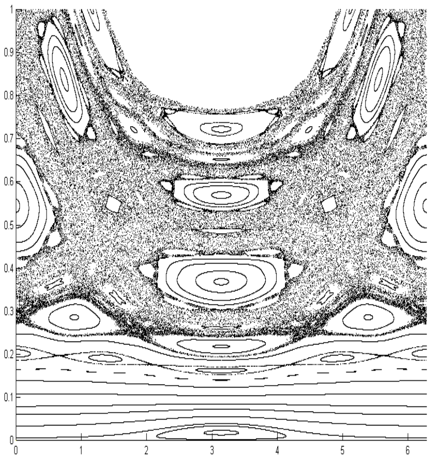

Fig. 1. Phase portrait of the HJS-tokamap TT_N for $N = 1$

If the point $A = (I, \theta)$ is mapped one time forward and then one time backward the obtained point, $B = \left(I', \theta'\right)$, theoretically coincides with A because $B = TT_N^{-1}\left(TT_N\left(A\right)\right)$. In numerical simulations it almost never happens, due to the accumulation of the round-off errors automatically generated by the computer. The points A and B are closed and smaller distance between A and B indicates a better computational accuracy.

For a fixed initial condition $A_0 = (I_0, \theta_0)$ we denote by $A_k = (I_k, \theta_k)$ the point obtained through k iterations of TT_N. The p-forward orbit of A_0 will be $O^+\left(A_0\right) = \{A_0, A_1, A_2, ..., A_p\}$.

By iterating k times the point A_p through TT_N^{-1} we obtain the point $B_k = \left(I'_k, \theta'_k\right)$ and the p-backward orbit of A_p is $O^-\left(A_p\right) = \{A_p, B_1, B_2, ...B_p\}$.

The value $d\left(k\right) = dist\left(A_k, B_{p-k}\right)$ may be interpreted as the error at the k iteration.

We define the local forward-backward error of order p by

$$FBE_N\left(I_0, \theta_0, p\right) = \frac{1}{p-1}\sum_{k=1}^{p-1} d\left(k\right). \tag{2}$$

If the orbit $O^+\left(I_0, \theta_0\right)$ is stable (this is the case with elliptic periodic points or quasiperiodic orbits) the values $d\left(k\right)$ are small and regularly increase in time. The values of $FBE_N\left(I_0, \theta_0, p\right)$ are also small.

In the case of the chaotic orbits the values of $d\left(k\right)$ are much larger (because they are amplified by the sensitive dependence on initial conditions)

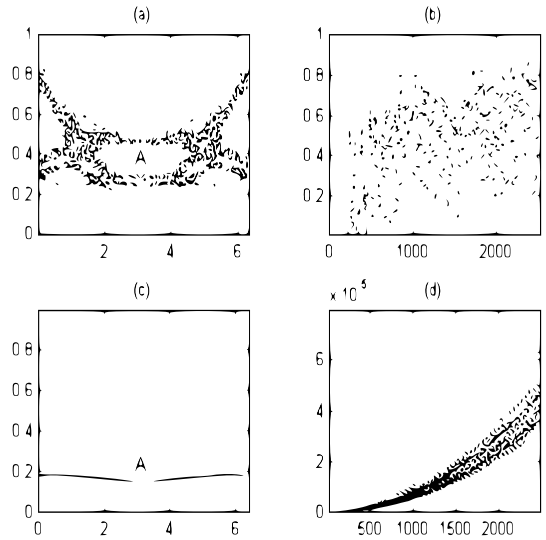

Fig. 2. HJS-tokamap for $\varepsilon = 0.09$ and $N = 1$ (a) Chaotic orbit, (b) Distribution of errors along the chaotic orbit, (c) Regular orbit, (d)Distribution of errors along the regular orbit

and irregular and $FBE_N\,(I_0, \theta_0, p)$ is several order larger than in the regular case.

In order to study the stability of an orbit we compute its forward-backward error, $FBE_N\,(I_0, \theta_0, p)$. Small values of $FBE_N\,(I_0, \theta_0, p)$ indicate a stable (regular) orbit and higher values indicate an unstable (chaotic) orbit.

In what follows we apply the previous considerations for the HJS-tokamap model, corresponding to $\varepsilon = 0.09$ and $N = 1$.

The value of the forward-backward errors of the orbits presented in figure 2 a and figure 2 c are $FBE_N\,(0.5, \pi, 1000) = 0.42486390559089$ and $FBE_N\,(0.15, \pi, 1000) = 0.0000145653379$.
The conclusion is that the orbit of $(0.5, \pi)$ is chaotic and the orbit of $(0.15, \pi)$ is regular.

These results are confirmed by the phase portraits presented in figure 2 a and figure 2 c. The values of the errors along the orbits are presented in figure 2 b, respectively figure 2 d.

The main ingredient in the method we propose is the instability of the chaotic dynamics, instability that amplifies the inherent computational errors made automatically by any computer in long simulations.

4. THE CRITERION FOR DETERMINING THE LOCAL OPTIMAL STEP

For a given Hamiltonian system we consider the sequence of discrete dynamical systems obtained generated by the maps TT_N obtained from (1) and a fixed initial condition $A_0 = (I_0, \theta_0)$.

The local optimal mapping step, might correspond to the smaller $N = N_{optimal}$ for which the dynamical characteristics (regular or chaotic) of the orbit of A_0 in the system generated by $TT_{N_{optimal}}$ and in the initial system coincide.

Because it is difficult, quite impossible, to know the type of the orbits in the continuous system, we replace the natural condition presented in the previous proposition by a criterion which involves the stabilization of the mapping procedures when N increases.

For a fixed A_0, length of the orbit "p" and acceptable error $E << 1$ we denote by $N_{optimal}$ the smaller natural value such that

$$\left|FBE_{N_{optimal}}\left(I_0, \theta_0, p\right) - FBE_{N_{optimal}+1}\left(I_0, \theta_0, p\right)\right| < E \tag{3}$$

The local optimal step is defined by

$$LOS\left(I_0, \theta_0, p\right) = 2\pi / N_{optimal}. \tag{4}$$

It is the largest mapping step for which the orbit starting from A_0 is chaotic (respectively regular) in the discrete system generated by TT_N and TT_{N+1}.

It depends on the initial point of the orbit, on its length and on the acceptable error.

In order to exemplify the determination of a local mapping step we consider the HJS-tokamap model corresponding to $\varepsilon = 0.09$, and the orbit starting from $A_0 = (0.5, \pi)$ and having the length $p = 1000$ and the admissible error $E = 10^{-5}$.

In this case

$$FBE_1\left(0.5, \pi, 1000\right) = 0.74754382044656 FBE_2\left(0.5, \pi, 1000\right)$$

$$= 2.710810800775104 \cdot 10^{-7}$$

$$FBE_3\left(0.5, \pi, 1000\right) = 1.663737798529739 \cdot 10^{-6}$$

$$FBE_4\left(0.5, \pi, 1000\right) = 2.512265885961251 \cdot 10^{-6}.$$

From the considerations presented in Section 3 it results that the orbit of A_0 is chaotic in the system generated by TT_1, and regular in the systems generated by TT_2, TT_3 and TT_4. It means that its chaoticity was a numerical one,

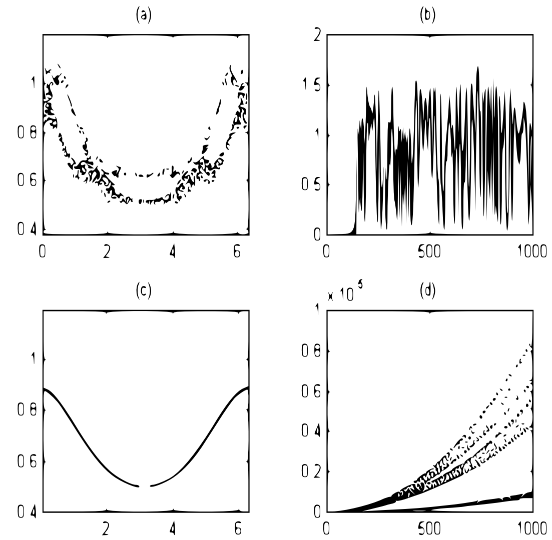

Fig. 3. HJS-tokamap for $\varepsilon = 0.09$. Forward orbit of $A_0 = (0.5, \pi)$ in $TT_1 - TT_4$ and the distribution of errors along iterations

introduced by the imperfection of the discrete system. Because

$$|FBE_2\,(0.5, \pi, 1000) - FBE_3\,(0.5, \pi, 1000)\,| \approx$$
$$\approx 0.00000139265672 < \mathrm{E}$$

the optimal mapping step is $LOS\,(0.5, \pi, 1000) = \pi$, corresponding to $N = 2$.

The previous considerations are reflected in figure 3 . The forward orbit of $A_0 = (0.5, \pi)$ in the system generated by TT_1 is plotted in figure 3 a with black points and the backward orbit of A_{1000} is plotted in the same figure with red points. The values of $d\,(k)$, $k = \overline{1, p}$ are presented in figure 3 b.

Figure 3 c represents the orbits of the same initial data for N=2 (black points), N=3 (blue points) and N=4 (magenta points). All of them are regular, quite superposed (for this reason the black points are not visible in the figure), which means that the mapping procedure was stabilized. Figure 3 d shows the values of $d\,(k)$, $k = \overline{1, p}$ (black points for $N = 2$, blue points for $N = 3$ and magenta points for $N = 4$).

Figure 3 c shows that round-off errors do not drastically influence the location of the regular orbits, even if they increase along the orbit.

It is interesting to observe the dependence of the local forward-backward error on the length of the orbit.

In figure 4 is presented the evolution of $|FBE_1\,(0.5, \pi, n) - FBE_2\,(0.5, \pi, n)|$ when the length of the orbit is $n \in \{100 \cdot k,\ \ k = \overline{1, 50}\}$

Fig. 4. HJS-tokamap for $\varepsilon = 0.09$. Evolution of forward-backward errors for $N = 1$

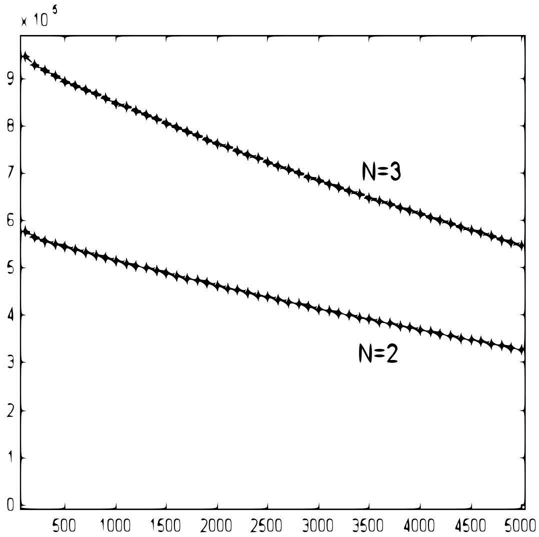

Fig. 5. HJS-tokamap for $\varepsilon = 0.09$. Evolution of forward-backward errors for $N = 2$ and $N = 3$

In figure 5 are plotted the values of the forward-backward errors corresponding to $N = 2$ and $N = 3$ along the orbit of the same initial point

It results that, for the orbit we studied, the forward-backward error does not depends drastically on the length of the orbit, hence the local optimal mapping step is the same for the short as for the long orbit.

5. A CRITERION FOR DETERMINING THE GLOBAL OPTIMAL MAPPING STEP

The local optimal mapping step depends essentially on the initial condition of the analyzed orbit.

In order to determine the global mapping step many local mapping step must be considered.

Because the orbits of the discrete dynamical systems generated by (1) are symmetric to the line $\theta = \pi$ we will focus on the orbits starting from this line.

In the computation we consider n orbits starting from equally distanced points $P_i = (I_i, \pi)$, $i = \overline{1, n}$. The common length of the orbits is p.
We denote the local forward-backward errors by

$$IA_N\,(i) = \left|FBE_N\,(I_i, \pi, p) - FBE_{N+1}\,(I_i, \pi, p)\right|.$$

We define the global forward-backward error of order $N$of the system by

$$IA_N = \frac{1}{n} \cdot \sum_{i=1}^{n} IA_N\,(i)$$

For affixed acceptable error $E << 1$ the global optimal mapping step corresponds to the smaller N such that

$$ER_N = |IA_N - IA_{N+1}| < E.$$

As an example we consider the HJS-tokamap with the stochasticity parameter $\varepsilon \approx 0.09$. We consider $n = 100$ orbits starting from equally distanced points in the annulus $0 \leq \psi \leq 1$. The common length of the orbits is $p = 1000$. The acceptable error is $E = 10^{-5}$.

The global forward-backward errors are

$ER_1 = 0.27225103288908$, $\mathrm{ER}_2 = 0.14524599193235$

$\mathrm{ER}_3 = 0.04151835374529$, $\mathrm{ER}_4 = 0.00000478886236$.

In this case the optimal mapping step corresponds to

$$N_{optimal} = 4.$$

This result is completed by the information obtained from figure 6 , where the forward-backward errors for the previously considered orbits are plotted. The black line, corresponding to $N = 1$ locate the chaotic orbits . For $N = 2$ the red line show that the orbits starting from $\psi < 0.8$ were stabilized (it is the case of the orbits we studied in the previous sections). The blue line and the magenta line, corresponding to $N = 3$ respectively $N = 4$, show that all the orbits were stabilized for.

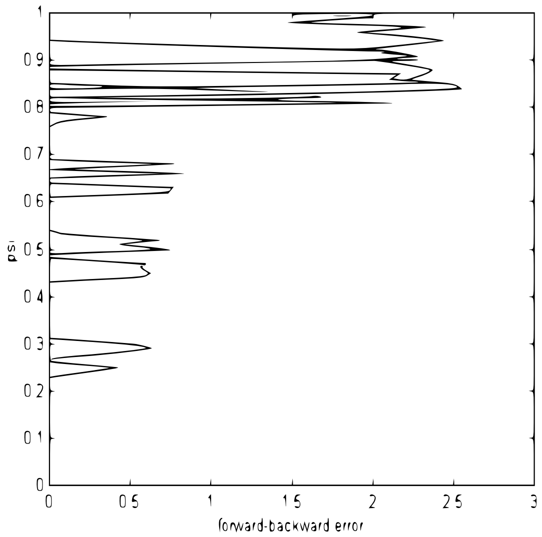

Fig. 6. HJS-tokamap for $\varepsilon = 0.09$. Global forward-backward errors for $N \in \{1, 2, 3, 4\}$

The conclusion of our calculation is that the discrete dynamical system that reflects accurately the properties of the initial Hamiltonian system is obtained by considering the mapping step $\Delta\zeta = \pi/2$, corresponding to $N = 4$. It results also that the phase portrait we presented in figure 1 is far from the real phase portrait of the Hamiltonian system. An accurate approximation of the real phase portrait is given in figure 7 .

This phase portrait shows that the dynamics of the system is regular, which it natural because the Hamiltonian of the system was obtained by very small perturbation (the maximal value of the perturbation is 0.045, and the perturbation around the orbit is around 0.027).

6. CONCLUSION

In this paper we studied the accuracy of some symmetric mapping models associated to $1\frac{1}{2}$ degrees-of-freedom Hamiltonian systems. These models are obtain through Hamilton-Jacobi method. We observed the influence of the mapping step on the local dynamical characteristic of the orbits (regular or chaotic). We proposed a direct method for identifying the chaotic orbits in discrete dynamical systems and we exemplified it using the Hamilton- Jacobi symmetric tokamap.

We observed that a large mapping step creates artificial chaotic dynamics and we proposed a method for determining the optimal mapping step such that the dynamical characteristic (regular or chaotic) of an orbit is preserved.

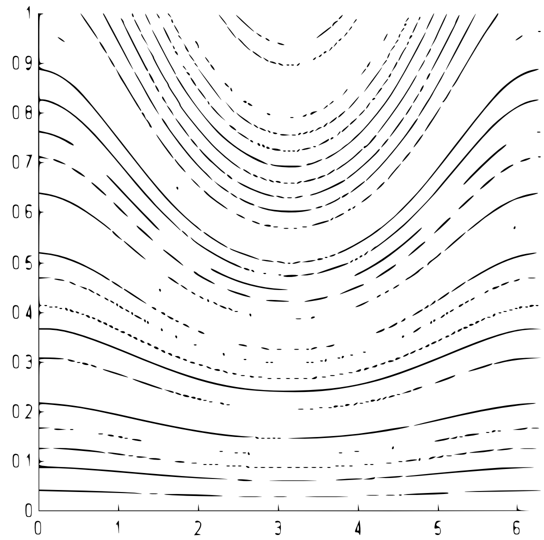

Fig. 7. Phase portrait of the HJS-tokamap TT_N for $N = 4$

This method was used for obtaining a global criterion of accuracy for mapping having time-reversal symmetry.

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