CENTER CONDITIONS FOR A CUBIC SYSTEM WITH A BUNDLE OF TWO INVARIANT STRAIGHT LINES AND ONE INVARIANT CUBIC

Dumitru Cozma, Anatoli Dascalescu
Tiraspol State University, Chişinău, Republic of Moldova
dcozma@gmail.com, anatol.dascalescu@gmail.com

Abstract
For a cubic differential system with a bundle of two invariant straight lines and one invariant cubic it is proved that a fine focus is a center if and only if the first two Lyapunov quantities vanish.

Keywords: cubic differential system, center-focus problem, invariant algebraic curve, integrability.

2010 MSC: 34C05.

1. INTRODUCTION

In this paper we consider the real cubic system of differential equations
\begin{align*}
\frac{dx}{dt} &= y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x, y), \\
\frac{dy}{dt} &= -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x, y),
\end{align*}
(1)
where $P(x, y), Q(x, y) \in \mathbb{R}[x, y]$ are coprime polynomials. The origin $O(0, 0)$ is a singular point of a center or a focus type for (1). The purpose of this paper is to find verifiable conditions for $O(0, 0)$ to be a center.

It is known [1] that a singular point $O(0, 0)$ is a center for system (1) if and only if it has a holomorphic first integral of the form $F(x, y) = C$ in some neighborhood of $O(0, 0)$. There exists a formal power series $F(x, y) = \sum F_j(x, y)$ such that the rate of change of $F(x, y)$ along trajectories of (1) is a linear combination of polynomials $\{(x^2 + y^2)^j\}_{j=2}^{\infty}$:
\[
\frac{dF}{dt} = \sum_{j=2}^{\infty} L_{j-1}(x^2 + y^2)^j.
\]
Quantities $L_j, j = 1, \infty$ are polynomials with respect to the coefficients of system (1) called the Lyapunov quantities. The origin is a fine focus of order $m$ if $L_1 = L_2 = \ldots = L_{m-1} = 0$ and $L_m \neq 0$.

The origin is a center for (1) if and only if $L_j = 0, j = 1, \infty$. By the Hilbert basis theorem there exists a natural number $N$ such that the infinite system $L_j = 0, j = 1, \infty$ is equivalent with a finite system $L_j = 0, j = 1, N$. The number $N$ is known for quadratic systems ($N = 3$) and for cubic systems.
with only homogeneous cubic nonlinearities \((N = 5)\). If the cubic system (1) contains both quadratic and cubic nonlinearities, the problem of the center was solved only in some particular cases (see, for example [2, 3, 6, 7, 8, 9, 10, 11, 12, 15, 16]).

In this paper we solve the problem of the center for a cubic differential system (1) assuming that the system has two invariant straight lines and one invariant cubic passing through one singular point, i.e. forming a bundle. The paper is organized as follows. The results concerning relation between integrability, invariant algebraic curves and Lyapunov quantities are presented in Section 2. In Section 3 we find nineteen sets of conditions for the existence of a bundle of two invariant straight lines and one invariant cubic. In Section 4 we obtain conditions for the existence of a center and we prove the main result: a fine focus \(O(0, 0)\) is a center for cubic system (1) with a bundle of two invariant straight lines and one invariant cubic if and only if the first two Lyapunov quantities vanish.

2. IN Variant Algebraic Curves AND CENTERS

We shall study the problem of the center for cubic system (1) assuming that the system has irreducible invariant algebraic curves.

**Definition 2.1.** An algebraic curve \(\Phi(x, y) = 0\) in \(\mathbb{C}^2\) with \(\Phi \in \mathbb{C}[x, y]\) is said to be an invariant algebraic curve of system (1) if

\[
\frac{\partial \Phi}{\partial x} P(x, y) + \frac{\partial \Phi}{\partial y} Q(x, y) = \Phi(x, y)K(x, y),
\]

for some polynomial \(K(x, y) \in \mathbb{C}[x, y]\) called the cofactor of the invariant algebraic curve \(\Phi(x, y) = 0\).

For a real cubic system (1), \(\Phi = 0\) is a complex invariant algebraic curve with cofactor \(K\) if and only if \(\overline{\Phi} = 0\) is a complex invariant algebraic curve with cofactor \(\overline{K}\). Here conjugation of polynomials denotes conjugation of the coefficients of the polynomials.

If the cubic system (1) has sufficiently many invariant algebraic curves \(\Phi_j(x, y) = 0, \ j = 1, \ldots, q\), with cofactors \(K_j(x, y)\), then in most cases a first integral (an integrating factor) can be constructed in the Darboux form

\[
\Phi_1^{\alpha_1} \Phi_2^{\alpha_2} \cdots \Phi_q^{\alpha_q}
\]

and we say that the cubic system (1) is Darboux integrable.
The function (2), with \( \alpha_j \in \mathbb{C} \) not all zero, is a Darboux first integral (a Darboux integrating factor) for (1) if and only if

\[
\sum_{j=1}^{q} \alpha_j K_j = 0 \quad \left( \sum_{j=1}^{q} \alpha_j K_j = -\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} \right).
\]

The method of Darboux turns out to be very useful and elegant one to prove integrability for some classes of systems depending on parameters. These last years, interesting results which relate algebraic solutions, Lyapunov quantities and Darboux integrability have been published (see, for example, [4, 5, 6, 11, 17]). The cubic systems (1) which are Darboux integrable have a center at \( O(0,0) \).

By Definition 2.1, a straight line

\[ 1 + Ax + By = 0, \quad A, B \in \mathbb{C}, \quad (A, B) \neq (0,0) \]

is an invariant straight line of system (1) if there exists a polynomial \( K(x,y) \) such that the following identity holds

\[
A \cdot P(x,y) + B \cdot Q(x,y) \equiv (1 + Ax + By) \cdot K(x,y).
\]

Let the cubic system (1) have two invariant straight lines \( l_1, l_2 \) that are real or complex \((l_2 = \overline{l_1})\) intersecting at a point \((x_0, y_0)\). The intersection point \((x_0, y_0)\) is a singular point for (1) and has real coordinates. By rotating the system of coordinates \((x \rightarrow x \cos \varphi - y \sin \varphi, y \rightarrow x \sin \varphi + y \cos \varphi)\) and rescaling the axes of coordinates \((x \rightarrow \alpha x, y \rightarrow \alpha y)\), we obtain \( l_1 \cap l_2 = (0,1) \). In this case the invariant straight lines can be written as

\[
l_j \equiv 1 + a_j x - y = 0, \quad a_j \in \mathbb{C}, \quad j = 1, 2; \quad a_1 - a_2 \neq 0.
\]

As the point \((0,1)\) is a singular point for (1), then \( P(0,1) = Q(0,1) = 0 \). These equalities yield \( r = -f - 1, l = -b \). Substituting \( A = a_j, \quad B = -1 \) in (3) and identifying the coefficients of \( x^i y^j \), we find that the straight lines (4) are invariant for (1) if and only if \( a_j \) are the solutions of the system

\[
(a - 1)a_j^2 + (g - k)a_j - s = 0,
\]

\[
(f + 2)a_j^2 + (b - c - p)a_j - d - n - 1 = 0,
\]

\[
a_j^3 - ca_j^2 + (a - d + m - 2)a_j + g + q = 0.
\]

(5)

It is easy to see from (5) that when \( f = -2 \) and \( (a - 1)(a_1 - a_2) \neq 0 \), the system (1) can have two distinct invariant straight lines of the form (4) if and
only if the following coefficient conditions are satisfied

\[ f = -2, \quad k = (a - 1)(a_1 + a_2) + g, \quad l = -b, \quad s = (1 - a)a_1a_2, \]
\[ m = -a_1^2 - a_1a_2 - a_2^2 + c(a_1 + a_2) - a + d + 2, \quad r = 1, \]
\[ n = -d - 1, \quad p = b - c, \quad q = (a_1 + a_2 - c)a_1a_2 - g. \]  

The cofactors of (4) are \( K_j(x, y) = (g + (a - 1)a_j)x^2 - (a_j^2 - ca_j - d - 1)xy + x + (b - a_j)y^2 + a_jy \).

To investigation of the problem of the center for cubic differential systems with invariant algebraic curves are dedicated the works [6, 7, 8, 9, 10, 11, 12, 16]. In these papers, the problem of the center was completely solved for cubic systems with: four invariant straight lines; three invariant straight lines; two invariant straight lines and one invariant irreducible conic. The main results of these works are summarized in the following theorems:

**Theorem 2.1.** A singular point \( O(0, 0) \) is a center of cubic differential system (1) with:

(i) four invariant straight lines if and only if the first two Lyapunov quantities vanish \( (L_1 = L_2 = 0) \);

(ii) three invariant straight lines if and only if the first seven Lyapunov quantities vanish \( (L_j = 0, \ j = 1, \ldots, 7) \);

(iii) two invariant straight lines and one invariant conic if and only if the first four Lyapunov quantities vanish \( (L_j = 0, \ j = 1, \ldots, 4) \).

**Theorem 2.2.** Every center in the cubic differential system (1) with:

(i) four invariant straight lines comes from a Darboux first integral or a Darboux integrating factor;

(ii) three invariant straight lines comes from a Darboux integrating factor or a rational reversibility;

(iii) two invariant straight lines and one invariant conic comes from a Darboux first integral or a Darboux integrating factor.

The problem of the center was solved for cubic system (1) with two parallel invariant straight lines and one invariant cubic [12] of the form

\[ \Phi(x, y) \equiv x^2 + y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 = 0, \quad a_{ij} \in \mathbb{R}. \]
3. CONDITIONS FOR THE EXISTENCE OF A BUNDLE OF TWO INVARIANT STRAIGHT LINES AND ONE INVARIANT CUBIC

Let for cubic system (1) the set of conditions (6) hold. Then (1) has two invariant straight lines of the form (4) and the system can be written as

\[
\begin{align*}
\dot{x} &= y + ax^2 + cxy - 2y^2 + [(a - 1)(a_1 + a_2) + g]x^3 + (b-c)xy^2 + \\
&[d + 2 - a - a_1^2 - (a_1 + a_2)(a_2 - c)]x^2y + y^3 = P(x, y), \\
\dot{y} &= -x - a_1x^2 - dxy - by^2 + (a - 1)a_1a_2x^3 + (d + 1)xy^2 + \\
&[g + a_1a_2(c - a_1 - a_2)]x^2y + by^3 = Q(x, y).
\end{align*}
\]

(8)

Next for cubic system (8) we find conditions for the existence of one irreducible invariant cubic curve of the form (7) \((a_{03} = -1)\)

\[
\Phi(x, y) \equiv x^2 + y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 - y^3 = 0,
\]

(9)

passing through the same singular point \((0, 1)\), i.e. forming a bundle with the invariant straight lines (4), where \((a_{30}, a_{21}, a_{12}) \neq 0\) and \(a_{30}, a_{21}, a_{12} \in \mathbb{R}\).

By Definition 2.1, a cubic curve (9) is an invariant cubic for system (8) if there exist numbers \(c_{20}, c_{11}, c_{02}, c_{10}, c_{01} \in \mathbb{R}\) such that

\[
P(x, y) \frac{\partial \Phi}{\partial x} + Q(x, y) \frac{\partial \Phi}{\partial y} \equiv \Phi(x, y)(c_{20}x^2 + c_{11}xy + c_{02}y^2 + c_{10}x + c_{01}y).
\]

(10)

Identifying the coefficients of \(x^iy^j\) in (10), we find that

\[
c_{10} = 2a - a_{21}, \quad c_{01} = a_{12} - 2b, \quad c_{02} = 3b - a_{12}, \quad d = (3a_{21} - 2a - 1)/2, \\
g = (3a_{30} - 3a_{12} + 2b + 2c)/2, \quad c_{11} = (5a_{21} + 2ca_{12} - 2a_{12}^2 + 3 - 6a)/2, \\
c_{20} = 2c(a_{12}^2 + 2a_{21}) - 2a_{12}(a_{12}^2 + 3a_{30}) + 3a_{30} + a_{12}(2a - 11 + 2(a_1 + a_2) - 2a_1a_2 - 2c(a_1 + a_2)) + 6(b + c(a_1a_2 + 1) - a_1a_2(a_1 + a_2))/2
\]

and \(a_{30}, a_{21}, a_{12}\) are the solutions of the following system of equations

\[
\begin{align*}
F_{50} &\equiv -a_{30}(a_{12}^2 + 2a_{21}) + (a_{21}a_1a_2 - a_{12}a_{30})(a - 1) + \\
& a_{30}(a_{12}^2 + 3a_{12}a_{21} + 3a_{30}) + 3a_{30}(a_1 + a_2)(a_1a_2 + a - 1) + \\
& a_{12}a_{30}((a_1 + a_2)(c - a_1 - a_2) + a_1a_2) - 3ca_{30}a_1a_2 = 0, \\
F_{41} &\equiv a_{12}^2(a_{21}a_{21} + a_{30} - ca_{21}) + a_{30}^2(3a_{12} - 2c) + a_{30}(5a_{21} - ca_{12}) + \\
& 2a_{21}(a_1 + a_2)(a_1a_2 - 1 + a) - 2a_1a_2(ca_{21} - a_{12}(a - 1)) - \\
& (a_{12}a_{21} + 3a_{30})(a - 1 - a_1a_2 + (a_1 + a_2)(a_1 + a_2 - c)) = 0,
\end{align*}
\]

(11)

\[
\begin{align*}
F_{32} &\equiv a_{12}^2(a_1a_2 + a - 1)(a_1 + a_2) + a_1a_2(3 - 3a - ca_{12}) - \\
& c(a_{12}^2 + 3a_{12}a_{21} + 3a_{30}) + a_{12}^2 + 4a_{12}a_{21} + 4a_{12}a_{30} + 2a_{21}^2 - \\
& (a_1 + a_2)(a - 1 - a_1a_2 + (a_1 + a_2)(a_1 + a_2 - c)) = 0,
\end{align*}
\]
\[ F_{40} = 2a_1^2(a_{12} - c) + a_{21}(9a_{12} - a_{30} - 2b - 6c) - 2(b + c) + \\
a_{12}(5 - 2a + 2c(a_1 + a_2) - 2(a_1 + a_2)^2 + 2a_1a_2) + \\
4(a_1 + a_2)(a - 1) + 6a_1a_2(a + a_2 - c) + a_{30}(2a + 3) = 0, \]
\[ F_{31} = 2a(a_{21} + 2a_1a_2 - 1) + (a_{12} - a_{30})(8a_{12} - 4b) - a_{21}^2 + \\
2c(a_1 + 2a_2 - 3a_{12} + 3a_{30}) - 4(a_1 + a_2)^2 + 2a_{21} + 3 = 0, \]
\[ F_{22} = a_1^2(a_{12} - c) - a_{12}((a_1 + a_2)^2 - a_1a_2 - c(a_1 + a_2)) + \\
b(a_2 + 1) + a_1a_2(a_1 + a_2 - c) = 0. \]

Denote \( j_1 = a_{12}(a_1 + a_2) - 3a_1a_2 - a_{12}^2 - 2a_{21} \), \( j_2 = a_2^2 - a_1^2a_1a_2 - a_2a_21 - a_{30} \), \( j_3 = a_1^2 - a_2^2a_1 - a_1a_21 - a_{30} \), \( j_4 = 4a_1^2a_{30} - a_1^2a_{21}^2 + 18a_2a_{21}a_{30} - 4a_2^3 + 27a_{30}^2 \).

We shall study the consistency of (11) when \( (a - 1)(a_1 - a_2) \neq 0 \) and divide the investigation into five cases: \( \{ j_1 = 0 \} \), \( \{ j_2 = 0, j_1 \neq 0 \} \), \( \{ j_3 = 0, j_1j_2 \neq 0 \} \), \( \{ j_4 = 0, j_1j_2j_3 \neq 0 \} \), \( \{ j_1j_2j_3j_4 \neq 0 \} \).

### 3.1. CASE \( J_1 = 0 \)

Denote by \( e_1 = (2a_{12} - 6a_2)a_1^2 + (2a_{12} - 6a_1)a_2^2 + 11a_{12}a_{30} + (a_{12} - 3a_1 - 3a_2)a_{12}^2 - 6a_{30} \) the coefficient of \( c \) from \( F_{32} = 0 \). In this case \( j_1 = 0 \) yields \( a_{21} = (a_1a_{12} - 3a_1a_2 - a_{12}^2 + 2a_{21})/2 \).

#### 3.1.1. \( e_1 = 0 \)

Then \( a_{30} = [(2a_{12} - 6a_2)a_1^2 + (2a_{12} - 6a_1)a_2^2 + 11a_{12}a_{30} + (a_{12} - 3a_1 - 3a_2)a_{12}^2]/6 \) and \( F_{32} = f_1f_2f_3f_4 = 0 \), where \( f_1 = 3a_1 - a_{12} \), \( f_2 = 3a_2 - a_{12} \), \( f_3 = a_1 + 2a_2 - a_{12} \), \( f_4 = 2a_1 + a_2 - a_{12} \).

Assume that \( f_1 = 0 \). Then \( F_{41} \equiv F_{50} \equiv 0 \). If \( a_1^2 = 1/3 \), then \( F_{40} = 0 \) yields \( a_2 = a_1/3 \). In this case we obtain the following set of conditions

1) \( a = (13 - 9bc)/13 \), \( d = 3(3bc - 13)/13 \), \( g = (13b + c)/13 \), \( 27c^2 - 169 = 0 \), \( a_1 = (3c)/13 \), \( a_2 = c/13 \)

for the existence of an invariant cubic \( 13(x^2 + y^2)(y - 1) - cx(x^2 + 9y^2) = 0 \).

If \( a_1^2 \neq 1/3 \), then express \( b \) and \( a \) from the equations \( F_{22} = 0 \) and \( F_{40} = 0 \), respectively. We have \( F_{31} \equiv g_1g_2 = 0 \), where \( g_1 = 4a_1 + a_2 - c \), \( g_2 = 3a_1^4 - 4a_1^2a_2 + 6a_2^2 - 12a_1a_2 + 4a_2^2 + 1 \).

Suppose that \( g_1 = 0 \), then \( c = 4a_1 + a_2 \). In this case the right-hand-sides of (8) are not coprime polynomials and have a common factor \( a_2x - y + 1 \).

Suppose that \( g_1 \neq 0 \) and \( g_2 = 0 \). The equation \( g_2 = 0 \) admits the following parametrization \( a_1 = (u^2 - 1)/(2u) \), \( a_2 = (u^4 + 6u^2 - 3)/(8u) \). In this case we obtain the following set of conditions

2) \( a = [u^8 + 10u^6 - 28u^4 + 30u^2 + 3 - 8cu^3(u^2 - 3)]/[8u^2(3u^2 - 1)] \), \( b = [(u^4 + 22u^2 - 19 - 8cu)(u^2 - 3)(u^2 - 1)]/[16u(1 - 3u^2)] \), \( d = [u^8 + 37u^6 - 79u^4 + 71u^2 - 6 - 8cu^3(u^3 - 3)]/[8u^2(1 - 3u^2)] \), \( g = [(u^8 + 10u^6 + 44u^4 + 6u^2 + 3)(u^2 - 1) - 8cu^3(u^2 + 1)]/[16u^3(1 - 3u^2)] \), \( a_1 = (u^2 - 1)/(2u) \), \( a_2 = (u^4 + 6u^2 - 3)/(8u) \)

for the existence of an invariant cubic \( 8u^3(x^2 + y^2) + (u^2x - 2uy - x)^3 = 0 \).
Center conditions for a cubic system with a bundle of two invariant straight lines ...

The invariant cubic is $x^2 + y^2 + (a_1 x - y)(a_2 x - y)^2 = 0$.

Suppose that $j_3 = 0$. Then $a_{30} = [a_1(2a_1^2 - 3a_1a_{12} + 3a_1a_2 + a_{12}^2 - a_2a_{12})]/2$ and $F_{40} + F_{22} = 0$ gives $h_1h_2 = 0$, where $h_1 = 4a_1^2 - 5a_1a_{12} + 3a_1a_2 + a_{12}^2 - a_2a_{12} - 2$, $h_2 = 2a_1^3 - 3a_1^2a_{12} + 3a_1^2a_2 + a_1a_{12}^2 - a_1a_2a_{12} + 2a_{12} - 4a_2$.

If $h_1 = 0$, then $a_2 = (2 - 4a_1^2 + 5a_1a_{12} - a_{12}^2)/3(a_1 - a_{12})$. In this case we have the following set of conditions

$$a = a_1(a_1 - a_{12}), \ b = (2a_1^2 - a_1a_{12} - 1)/(3a_1 - a_{12}), \ c = 2(1-2a_1^2+4a_1a_{12}-a_{12}^2)/(3a_1-a_{12})$$

for the existence of an invariant cubic $x^2 + y^2 + (a_1 x - y)(a_2 x - y)^2 = 0$.

Let $h_1 \neq 0$ and $h_2 = 0$. We express $a_2$ and $b$ from the equations $h_2 = 0$ and $F_{40} = 0$. In this case the system of equations (11) has no real solutions.

The case $j_2 = 0$ can be reduced to $j_3 = 0$ if we replace $a_2$ by $a_1$.

Suppose that $j_2j_3 \neq 0$ and $j_4 = 0$. The equations $j_4 = 0$ admits the following parametrization $a_{30} = [h^2(4h - 27)]/(108v^3)$, $a_{12} = (h - 9)/v$, $a_1 = [11h^2 - 144h + 486 - 6a_2(h - 9)]/[6v(h - 9 - 3a_2)]$.

Let $h^2 - 36h + 12u^2 = 0$. This equation has the following parametrization $h = 36/(12u^2 + 5)$, $v = (36u)/(12u^2 + 5)$. In this case we get $F_{22} \equiv (12a_2u^2 + a_2 + 4u)(4a_2u + 12u^2 + 1) = 0$. 


We have the following set of conditions
\[ F(J) \text{ has no real solutions.} \]
and substitute into \( F_{30} = 0 \) and \( F_{31} = 0 \). The resultant of the polynomials \( F_{40} \) and \( F_{31} \) with respect to \( a_2 \) is
\[ Res(F_{40}, F_{31}, a_2) > 0. \]
The system of equations (11) has no real solutions.

### 3.2. CASE \( J_2 = 0, J_1 \neq 0 \)

In this case \( a_{40} = a_2(a_2^3 - a_1 a_2 - a_{21}) \). We express \( a \) from \( F_{32} = 0 \) and obtain \( F_{41} = f_1 f_2 f_3 f_4 = 0 \), where \( f_1 = a_1 + a_2 - c \), \( f_2 = 2a_1 a_2 - 3a_2^2 + a_{21} \),
\( f_3 = a_1^2 + 2a_1 a_2 - 3a_2^2 + 4a_{21} \), \( f_4 = a_1^2 - (a_1 + a_2)a_{12} + a_1 a_2 - a_1^2 - a_{21} \).

#### 3.2.1. \( f_1 = 0 \)
In this case \( a_{12} = c - a_1 \) and \( F_{50} \equiv 0 \). \( b \) from \( F_{31} = 0 \) and \( a_{12} \) from \( F_{40} = 0 \).

We have the following set of conditions
\[
\begin{align*}
7) & \quad a = \frac{[a_2(2a_2 - 2b - c)]}{2}, \quad d = 2(a - 1), \quad g = \frac{[6a_2(1 - a) - 2b + c]}{4}, \\
& \quad a_1 = \frac{(c - 2b)}{2} \\
\end{align*}
\]
for the existence of an invariant cubic
\[
2(x^2 + y^2) + (2a_2^2 - 2ba_2 - ca_2 - 2)x^2 + \\
(c + 2b - 2a_2)xy - 2y^2(y - ax2) = 0.
\]

#### 3.2.2. \( f_1 \neq 0, f_2 = 0 \)
In this case we find that \( a_{21} = a_2(3a_2 - 2a_{12}) \) and \( F_{50} \equiv 0 \). We express \( c \) from \( F_{22} = 0 \), \( b \) from \( F_{31} = 0 \) and \( a_{12} \) from \( F_{40} = 0 \).

We have the following set of conditions
\[
\begin{align*}
8) & \quad a = (a_{12} - 2a_2)(a_{12} + a_1 - ax_2 - c) + 1, \quad d = (-2a - 6a_1 a_2 + 9a_2^2 - 2), \\
& \quad g = (3a_2 a_2^2 - 3a_1 - 6a_2^2 + 2b + 2c)/2, \quad c = [(a_{12} + ax_2 - a_{12})(a_1 a_2 - ax_2) + \\
& \quad b(2a_2 a_2 - 3a_2^2 - 1)]/[a_{12} - ax_2)(a_{12} - ax_2)], \quad b = [(2a_2 - ax_2)ax_2 + 3a_2^2 - 2a_2(ax_2 - 2a)](a_{12} - ax_2)/[4(a_2^2 - 4a_1 ax_2 + a_2^2 + 1)], \\
& \quad a_{12} = (4a_2^2 + 8a_1 a_2^2 - 9a_2^2 - 6a_2^2 - 1)/4(a_1 - a_2)(a_2^2 + 1) \\
\end{align*}
\]
for the existence of an invariant cubic curve
\[
4(ax_1 - a_2)(a_2^2 + 1)(x^2 + y^2) + \\
((4a_2^2 - 8a_1 ax_2 - a_2^4 + 2a_2^2 - 1)x + y(4a_2 - 4a_1 a_2^2 - 4a_2^2 + 4a_2^2))(ax_2 - y)^2 = 0.
\]

#### 3.2.3. \( f_1 f_2 \neq 0, f_3 = 0 \)
In this case \( a_{21} = (3a_2^2 - a_1^2 - 2a_1 a_2)/4 \) and \( F_{50} \equiv 0 \). If \( a_1^2 + 2a_1 a_2 - 3a_2^2 - 4 = 0 \), then we obtain the following two sets of conditions:
\[
\begin{align*}
9) & \quad a = c + 4, \quad b = 0, \quad d = -(c + 6), \quad g = c + 3, \quad a_1 = -2, \quad a_2 = 0. \\
\end{align*}
\]
The invariant cubic is
\[
x^2 + y^2 - y(x + y)^2 = 0.
\]

#### 10) \( a = 4 - c, \quad b = 0, \quad d = c - 6, \quad g = c - 3, \quad a_1 = 2, \quad a_2 = 0.
\]
The invariant cubic is
\[
x^2 + y^2 - y(x - y)^2 = 0.
\]
If \( a_1^2 + 2a_1 a_2 - 3a_2^2 - 4 \neq 0 \), then we express \( c \) from the equation \( F_{22} = 0 \) and calculate the resultant of the polynomials \( F_{40} \) and \( F_{31} \) with respect to
b. We have $Res(F_{40}, F_{31}, b) = 2g_1g_2g_3g_4g_5g_6$, where $g_1 = a_1^3 - 12a_2a_1^2 + 2a_2^2(8a_1a_2 + 15a_2^2 - 4) + 4a_1(16a_1 - 8a_1a_2^2 - 7a_2^2 - 4a_2) - 64a_1^2 + 16a_1a_2^2 + 9a_2^2 + 24a_2^2 + 16$, $g_2 = a_1^2 - 6a_2a_1 + 5a_2^2 - 4$, $g_3 = a_1 - a_1$, $g_4 = a_1 - a_2$, $g_5 = (a_1 - a_2)^2 + 4$, $g_6 = a_2^2 + 1$ and $g_3g_5g_6g_7 \neq 0$.

Let $g_1 = 0$. This equation admits the following parametrization

$$a_{12} = a_2 + u, \quad a_2 = [64a_1(a_1 - u) - (u^2 - 4)]/[8(2a_1 - u)(u^2 + 4)].$$

In this case $F_{40} = F_{31} = h_1h_2h_3 = 0$, where $h_1 = (8a_1 - 4u)^2 + (u^2 + 4)^2 \neq 0$, $h_2 = 4a_1u^2 + 4bu^2 + 16b - 3u^4 - 4u$, $h_3 = 16a_1 - u^2 - 12u$.

If $h_2 = 0$, then $a_1 = (3u^3 - 4bu^2 - 16b + 4u)/(4u^2)$ and we have the following set of conditions

$$11) \quad a = (8b - bu^2 - 2u)/(8b - 2u), \quad d = (24b^2 - bu(u^2 + 18) + 9u^2)/[u(4b - u)],$$

$$c = [4b^2(u^2 + 8) - bu(7u^2 + 16) + u^2(u^2 + 1)]/[u^2(u - 4b)], \quad g = [(8b + u^2 - 2u)(8b - u^2 - 2u)(3u^2 + 4)]/[32u^2(u - 4b)], \quad a_1 = (3u^3 - 4bu^2 - 16b + 4u)/(4u^2), \quad a_2 = (u^4 - 64b^2 + 32bu - 4u^2)/[4u^2(u - 4b)].$$

for the existence of an invariant curve $16u^2(u - 4b)(x^2 + y^2) + (64b^2x + 16bu^2y - 32bxu - u^4x - 4uy^2 + 4u^2x)(ux - 2y)^2 = 0$.

If $h_2 \neq 0$ and $h_3 = 0$, then $a_1 = (u^3 + 12u)/16$ and we obtain

$$12) \quad a = [32bu + 3(u^2 + 4)]/[8u(u - 4)], \quad d = (4 - 4a - 3u^2)/4, \quad c = (128bu + u^6 + 32a^4 - 176u^2 + 128)/[16a(u^2 - 4)], \quad g = (32bu^3 + 128bu + 5u^6 - 20u^4 - 16u^2 + 64)/[32u(u^2 - 4)], \quad a_1 = (u^3 + 12u)/16, \quad a_2 = (u^2 - 4)/(4u).$$

The invariant cubic is $16u(x^2 + y^2) + (u^2x - 4x - 4uy)(ux - 2y)^2 = 0$.

Assume that $g_1 \neq 0$ and let $g_2 = 0$. The equation $g_2 = 0$ admits the parametrization $a_2 = (u^2 - 4)/(4u), \quad a_{12} = (5u^2 - 4)/(4u)$ and $F_{31} \neq 0$.

$3.2.4. \ \ f_1f_2f_3 \neq 0, \ \ f_4 = 0$. In this case we get $a_{21} = a_1^2 - a_1a_2 + a_1a_2 + a_2^2 + 2a_2$ and $F_{30} \equiv 0$. We express $c$ from $F_{22} = 0$ and calculate the resultant of the polynomials $F_{40}$ and $F_{31}$ with respect to $b$. We have $Res(F_{40}, F_{31}, b) = -2e_1 \cdots e_5$, where $e_1 = a_1^2 - a_1a_2 + a_1a_2^2 + a_2 - a_2^2 + 1$, $e_2 = a_1^2 - a_1a_2 - a_1a_2 + a_1a_2 - a_2^2 - 1$, $e_3 = (a_1 + a_2 - a_2)^2 + 1 \neq 0$, $e_4 = (a_1 - a_2)(a_1 - a_1) \neq 0$, $e_5 = (a_1^2 + 1)(a_2^2 + 1) \neq 0$.

If $e_1 = 0$, then $a_{12} = (a_1^2 + a_2 - a_2^2 + 1)/(a_1 - a_2)$ and $F_{31} \neq 0$.

If $e_1 \neq 0$, $e_2 = 0$, then $a_{12} = (a_1^2 - a_1a_2 - a_2^2 - 1)/(a_1 - a_2)$ and $F_{31} = 0$ yields $b = (a_1^2 - 2a_1a_2 - 1)/[2(a_1 - a_2)]$. In this case we obtain

$$13) \quad a = [a_1(a_2^2 + 1)]/(a_1 - a_2), \quad b = (a_2^2 - 2a_1a_2 - 1)/[2(a_1 - a_2)], \quad c = (a_1^2 - 2a_2 - 1)/(a_1 - a_2), \quad d = [2a_2(a_1a_2 + 1)]/(a_1 - a_2), \quad g = [a_2(3a_1^2a_2 + 2a_1 + a_2)]/[2(a_1 - a_2)].$$

The invariant cubic is $(a_2 - a_1)(x^2 + y^2) + ((a_1a_2 + 1)x + (a_1 - a_2)y)(a_1x - y)(a_2x - y) = 0$. 
3.3. CASE J₃ = 0, J₁J₂ ≠ 0

This case can be reduced to j₂ = 0 if we replace a₂ by a₁. We obtain the sets of conditions 7) – 13).

3.4. CASE J₄ = 0, J₁J₂J₃ ≠ 0

In this case the equation j₁ = 0 admits the following parametrization
\[ a_{12} = (h - 9)/v, \quad a_{30} = (4h^3 - 27h^2)/(108v^3), \quad a_{21} = (5h^2 - 36h)/(12v^2). \]

We express a from \( F_{32} = 0 \), then \( F_{41} = f_1f_2j_2j_3 = 0 \), where
\[ f_1 = h - 6, \quad f_2 = 6(a_1 + a_2 - c)v + 7h - 54. \]

3.4.1. \( f_1 = 0 \). In this case we find that \( h = 6, F_{50} = 0 \) and
\[ F_{22} = (a_1v + a_2v - cv - 3)(a_1v + 3)(a_2v + 3) + bv(v^2 - 3) = 0. \]

3.4.1.1. \( a_1 = (-3)/v \). If \( v^2 = 3 \), then \( F_{31} = 0 \) yields \( b = 0 \) and we obtain the following two sets of conditions:

14) \[ a = (22 + 3√3c)/9, \quad b = 0, \quad d = -(40 + 3√3c)/9, \quad g = (3c + 4√3)/3, \quad a_1 = -√3, \quad a_2 = (-√3)/9. \]
The invariant cubic is \( 3√3(x^2 + y^2) - (x + √3y)^3 = 0. \)

15) \[ a = (22 - 3√3c)/9, \quad b = 0, \quad d = (3√3c - 40)/9, \quad g = (3c - 4√3)/3, \quad a_1 = √3, \quad a_2 = √3/9. \]
The invariant cubic is \( 3√3(x^2 + y^2) + (x - √3y)^3 = 0. \)

If \( v^2 \neq 3 \) and \( b = 0 \), then \( F_{22} = 0 \) and \( F_{31} = 0 \) yields \( c = (4a_2^3v^2 - 14a_2v - v^2 - 19)/[4v(a_2v + 1)] \). We get the following set of conditions

16) \[ a = (4a_2v^3 - 2a_2v + 3v^2 - 3)/[4v^2(a_2v + 1)], \quad b = 0, \quad c = (4a_2^2v^2 - 14a_2v - v^2 - 19)/[4v(a_2v + 1)], \quad d = (-6a_2v^3 - 16a_2v - 5v^2 - 15)/[4v^2(a_2v + 1)], \quad g = (4a_2^2v^4 + 4a_2v^3 - 6a_2v - v^4 - v^2 - 6)/[4v^3(a_2v + 1)], \quad a_1 = (-3)/v, \quad F_{30} = 4a_2^3v^4 + 12a_2v^3 + 4a_2v - v^4 + 6v^2 + 3 = 0 \]
for the existence of an invariant cubic \( v^3(x^2 + y^2) - (x + vy)^3 = 0. \)

3.4.1.2. \( a_2 = (-3)/v \). This case can be reduced to the previous one replacing \( a_2 \) by \( a_1 \). We obtain the sets of conditions 14), 15) and 16).

3.4.1.3. \( (a_1v + 3)(a_2v + 3) \neq 0 \). We reduce the equations \( F_{40} = 0 \) and \( F_{31} = 0 \) by \( c \) from \( F_{22} = 0 \) and calculate the resultant of the polynomials \( F_{40} \) and \( F_{31} \) with respect to \( b \). We obtain \( Res(F_{40}, F_{31}, b) = -2v_1g_2(a_1v + 3)(a_2v + 3), \) where \( g_1 = 4a_2^2v^4 + 12a_1v^3 + 4a_1v - v^4 + 6v^2 + 3 \) and \( g_2 = 4a_2^2v^4 + 12a_2v^3 + 4a_2v - v^4 + 6v^2 + 3. \)

Assume that \( g_1 = 0 \). This equation admits the following parametrization
\[ a_1 = (1 + 6u^2 - 3u^3)/(8u^3), \quad v = (2u)/(u^2 - 1). \]
In this case \( F_{31} = h_1h_2 = 0, \) where \( h_1 = 8a_2u + u^4 + 6u^2 - 3, \quad h_2 = 6a_2u^3 - 2a_2u + 16bu^3 + 9u^4 - 12u^2 + 3. \)
If $h_1 = 0$, then $a_2 = (3 - u^4 - 6u^2)/(8u)$ and we have

(17) $a = (u^8 + 12u^6 - 10u^4 + 12u^2 + 1 + 8c(u^2 - 1)u^3)/(16u^4), \ b = (8cu^3(3u^4 - 10u^2 + 3) + 3u^{10} + 53u^8 - 270u^6 + 270u^4 - 53u^2 - 3)/(128u^5), \ d = -(8cu^3(u^2 - 1) + u^8 + 30u^6 - 38u^4 + 30u^2 + 1)/(16u^4), \ g = (24cu^3(u^4 + 2u^2 + 1) + 3u^{10} + 29u^8 + 90u^6 - 90u^4 - 29u^2 - 3)/(128u^5), \ a_1 = (1 - 3u^4 + 6u^2)/(8u^3), \ a_2 = (3 - u^4 - 6u^2)/(8u)$.

The invariant cubic is $8u^3(x^2 + y^2) - (u^2x + 2uy - x)^3 = 0$.

If $h_1 \neq 0$ and $h_2 = 0$, then $a_2 = (12u^2 - 16bu^3 - 9u^4 - 3)/(2u(3u^2 - 1))$ and we get the following set of conditions

(18) $a = (12u^2 - 3u^4 - 1)/8u^2, \ b = (8cu^3(1 - 3u^2) - 75u^6 + 97u^4 - 21u^2 - 1)/(64u^5), \ d = (u^2 - 3u^4 - 4)/(4u^2), \ g = (8cu^3(5u^2 + 1) - 12u^8 + 105u^6 - 83u^4 - 9u^2 - 1)/(64u^5), \ a_1 = (1 - 3u^4 + 6u^2)/(8u^3), \ a_2 = (8cu^3 + 13u^4 - 12u^2 - 1)/(8u^3)$

for the existence of an invariant cubic $8u^3(x^2 + y^2) - (u^2x + 2uy - x)^3 = 0$.

Assume $g_1 \neq 0$ and let $g_2 = 0$. This case can be reduced to the previous one replacing $a_2$ by $a_1$. We obtain the set of conditions (17) and (18).

3.4.2. $f_1 \neq 0, f_2 = 0$. In this case we have $c = (6va_1 + 6va_2 + 7h - 54)/(6v)$ and $F_{41} \equiv F_{50} \equiv 0$. Denote $a_1a_2 = w$ and $a_1 + a_2 = z$. We express $w$ from $F_{22} = 0$ and find that

$$w = [6v(5h^2 - 36h + 12u^2) + h^2(36 + 2vz - 2h) - 18h(vz + 2)]/(2hv^2).$$

The resultant of the polynomials $F_{40}$ and $F_{31}$ with respect to $z$ is

$$R_{\text{es}}(F_{40}, F_{31}, z) = -24hv(5h^3 - 36h^2 + 12v^2 + 288v^3)/(16h^2 - 216h + 9v^2 + 729)(h^2 + 36v^2)^2.$$

The equation $R_{\text{es}}(F_{40}, F_{31}, z) = 0$ yields $b = h(36h - 5h^2 - 12v^2)/(288v^3)$ and the system of equations (11) $F_{40} = 0, F_{31} = 0$ has real solutions only if $z = (4h^3 - 27h^2 + 108vh^2 - 972v^2)/(108v^3)$. In this case we obtain the following set of conditions

(19) $a = (4h^2 - 27h + 18v^2)/(18v^2), \ b = h(36h - 5h^2 - 12v^2)/(288v^3), \ c = (4h^3 - 27h^2 + 234hv^2 - 1944v^2)/(108v^5), \ d = (29h^2 - 216h - 108v^2)/(72v^2), \ g = (65h^3 - 432h^2 + 540hv^2 - 3888v^2)/(864v^5), \ a_1 = [h^2(4h - 27) + 108v^2(h - 9) - 108a_2v^3]/(108v^3), \ a_2 = (27 - 4h)h^2 + 108v^2(9 - h)a_2 + h^3(72 - 11h) + hv^2(2592 - 360h) - 432v^4 = 0$

for the existence of an invariant cubic $108v^3(x^2 + y^2) + (4hx - 27x - 3vy)(hx + 6vy)^2 = 0$.

3.5. CASE $J_1J_2J_3J_4 \neq 0$

We express $a$ from $F_{32} = 0$ and substitute into the equations of (11). Calculating the resultant of the polynomials $F_{50}$ and $F_{41}$ with respect to $c$ we
obtain that \( \text{Res}(F_{50}, F_{41}, c) = j_1 j_2 j_3 j_4 \neq 0 \). In this case the system of algebraic equations (11) is not consistent.

In this way we have proved the following theorem.

**Theorem 3.1.** Let \((a - 1)(a_1 - a_2) \neq 0\). The cubic differential system (8) has a bundle of two invariant straight lines (4) and one irreducible invariant cubic (9) passing through a singular point \((0, 1)\) if and only if one of the sets of conditions (1) – (9) holds.

4. **CENTERS IN CUBIC SYSTEMS WITH A BUNDLE OF TWO INVARIANT STRAIGHT LINES AND ONE INVARIANT CUBIC**

**Lemma 4.1.** The following four sets of conditions are sufficient conditions for the origin to be a center for system (1):

(i) \( a = (12u^2 - u^4 - 3)/(8u^2), b = (4u^2 - u^4 - 3)/(8u), c = (u^4 + 16u^2 - 17)/(8u), d = (u^2 - 4u^4 - 3)/(4u^2), f = -2, g = (3u^6 - 5u^4 + 5u^2 - 3)/(16u^3), k = (6u^6 - u^8 + 24u^4 - 38u^2 + 9)/(64u^3), l = -b, m = (5u^6 + 7u^4 - 65u^2 + 29)/(32u^2), n = -d - 1, p = b - c, r = 1, q = (72u^4 - 3u^8 - 22u^6 - 74u^2 + 27)/(64u^3), s = (u^{10} + u^8 - 26u^6 + 54u^4 - 39u^2 + 9)/(128u^4); \)

(ii) \( c = [2b(7a - 6)]/(3a - 2), d = 2(a - 1), f = -2, g = b(1 - a), k = b(a - 1), l = -b, m = (4 - 5a)/2, n = 1 - 2a, p = b - c, q = [b(7a^2 - 9a + 2)]/(3a - 2), r = 1, s = (a^2 - a)/2, (3a - 2)^2 + 16(a - 1)b^2 = 0; \)

(iii) \( a = (8b - bu^2 - 2u)/(8b - 2u), d = (24b^2 - bu(u^2 + 18) + 3u^2)/(4bu - u^2), c = [4b^2(u^2 + 8) - bu(7u^2 + 16) + u^2(u^2 + 2)]/[u^2(u - 4b)], f = -2, g = [(8b + u^2 - 2u)(8b - u^2 - 2u)(3u^2 + 4)]/[32u^2(u - 4b)], r = 1, k = (a - 1)(a_1 + a_2) + g, s = (1 - a)a_1a_2, q = (a_1 + a_2 - c)a_1a_2 - g, m = -a_1^2 - a_1a_2 - a_2^2 + c(a_1 + a_2) - a_1 - a_2, n = -d - 1, p = b - c, a_1 = (3a^3 - 4bu^2 - 16b + 4u)/(4u^2), a_2 = (u^4 - 64b^2 + 32b^2 - 4u^2)/(4u^2 - 4b - u); \)

(iv) \( a = (108 - u^2)/72, b = (u^3 - 36u)/432, l = -b, p = b - c, r = 1, f = -2, c = (2592 - u^4 - 252u^2)/(432u), d = (-5u^2 - 36)/72, n = -d - 1, g = (432 - u^4 + 24u^2)/(288u), k = (u^6 - 3888u^2 + 93312)/(31104u), m = [(u^4 + 81u^2 - 324)(u^2 - 36)]/(1296u^2), q = [(u^4 + 168u^2 - 432)(u^2 - 36)]/(20736u), s = [(u^2 + 108)(u^2 - 36)^2]/373248. \)

**Proof.** In each of the cases (i) – (iv), the cubic system (1) has two invariant straight lines and one invariant cubic. The system is Darboux integrable and has an integrating factor of the form \( \mu = \Phi(t_1^2 + t_2^2) \).

In the case (i): \( l_1 = (u^2 - 1)x + 2u(1 - y), l_2 = (u^4 + 6u^2 - 3)x + 8u(1 - y), \Phi = 8u^2(x^2 + y^2) - (u^2x - 2u^2y - x)^3, \alpha_1 = (-1)/2, \alpha_2 = 0, \alpha_3 = (-3)/2. \)
The following four sets of conditions are sufficient conditions for the origin to be a center for system (1):

(i) \( a = (3 - 2a_1a_2 - a_3^2)/2, \quad b = l = 0, \quad c = 2a_1 + 3a_2, \quad d = 2a - 5, \quad f = -2, \quad g = a_1(3a_2^2 + 1)/2, \quad k = (a_2 - 2a_1^2a_2 + 2a_1 - a_3^2)/2, \quad m = (2a_1 + 6a_1a_2 + 3a_3^2 - 3)/2, \quad n = -d - 1, \quad p = b - c, \quad q = a_1(-2a_1a_2 - 7a_3^2 - 1)/2, \quad r = 1, \quad s = (1 - a)a_1a_2; \)

(ii) \( a = a_1(h - 2a_1), \quad b = (a_1h - a_3^2 - 1)/h, \quad c = 2(a_1^2 + 2ha_1 - h^2 + 1)/h, \quad d = 2a - 2, \quad f = -2, \quad g = (6a_1h - 3a_2^2h^2 + 2a_1^2 + 4a_1h - h^2 + 2)/(2h), \quad k = (2a_1h - 8a_1^2 + 5a_1^2h^2 - 10a_1h - 2a_1h^3 + 4a_1h + h^2 - 2)/(2h), \quad r = 1, \quad m = (6a_1^3 + a_1^2h - 4a_1h^2 + 6a_1 + h^3 - 2h)/h, \quad n = -d - 1, \quad p = b - c, \quad q = (9a_1^2h^2 - 8a_1^4 - 6a_1^3h - 10a_1^2 - 2a_1h^3 + h^2 - 2)/(2h), \quad l = -b, \quad a_{12} = 3a_1 - h, \quad s = a_1(4a_1^2 - 3a_2^2h^2 + 6a_1^2 + a_1h^3 - a_1h - h^2 + 2)/h; \)

(iii) \( a = [a_2(2a_2 - 2b - c)]/2, \quad d = 2(a - 1), \quad f = -2, \quad g = [6a_2(1-a) - 2b + c]/4, \quad k = (2a_2 - 2a_2^2 - 4ab + 2ac + 2b - c)/4, \quad l = -b, \quad m = (c^2 - 4b^2)/4, \quad n = 1 - 2a, \quad p = b - c, \quad q = [6(a - 1)a_2 - 4ab + 2ac + 2b - c]/4, \quad r = 1, \quad s = [a_2(2ab - ac - 2b + c)]/2; \)

(iv) \( a = [a_1(2a_2^2 + 1)]/(a_1 - a_2), \quad b = (a_1^2 - 2a_1a_2 - 2)/(a_1 - a_2), \quad f = -2, \quad c = (a_1^2 - 2a_1 - 1)/(a_1 - a_2), \quad d = [2a_2(a_1a_2 + 1)]/(a_1 - a_2), \quad l = -b, \quad g = [a_2(3a_1a_2 + 2a_1 + a_2)]/[2(a_1 + a_2)], \quad k = (a - 1)(a_1 + a_2) + g, \quad s = (1 - a)a_1a_2, \quad m = -a_1 - a_1a_2 - a_2^2 + c(a_1 + a_2) - a + d + 2, \quad r = 1, \quad n = -d - 1, \quad p = b - c, \quad q = (a_1 + a_2 + c)a_1a_2 - g. \)

Proof. In each of the cases (i)–(iv), the cubic system (1) has a bundle of two invariant straight lines and one invariant cubic. The system is Darboux integrable and has a first integral of the form \( I_{1}^{a_1}I_{2}^{a_2}\Phi^{a_3} = C. \)

In the case (i): \( l_1 = 1 + a_1x - y, \quad l_2 = 1 + a_2x - y, \quad \Phi = x^2 + y^2 + (a_1x - y)(a_2x - y)^2, \quad a_1 = -1, \quad a_2 = -2, \quad a_3 = 1. \)

In the case (ii): \( l_1 = 1 + a_1x - y, \quad l_2 = (2a_1^2 + a_1h - h^2 + 2)x - hy + h, \quad \Phi = x^2 + y^2 + (a_1^2x^2 + (3a_1 - h)(xy - a_1x^2) - a_1xy - x^2 - y^2)(y - a_1x), \quad a_1 = 0, \quad a_2 = -1, \quad a_3 = 3. \)
In the case (iii): \( l_1 = (c - 2b)x - 2y + 2, \ l_2 = 1 + a_2x - y, \ \Phi = 2(x^2 + y^2) + ((2a_2^2 - 2ba_2 - ca_2 - 2)x^2 + (c + 2b - 2a_2)xy - 2y^2)(y - a_2x), \ \alpha_1 = -1, \ \alpha_2 = 0, \ \alpha_3 = 1. \)

In the case (iv): \( l_1 = 1 + a_1x - y, \ l_2 = 1 + a_2x - y, \ \Phi = (a_2 - a_1)(x^2 + y^2) + ((a_1a_2 + 1)x + (a_1 - a_2)y)(a_1x - y)(a_2x - y), \ \alpha_1 = 0, \ \alpha_2 = -1, \ \alpha_3 = 1. \) \( \Box \)

**Theorem 4.1.** Let the cubic system (1) have a bundle of two invariant straight lines (4) and one invariant cubic (9). Then a singular point \( O(0,0) \) is a center if and only if the first two Lyapunov quantities vanish.

**Proof.** To prove the theorem, we compute the first two Lyapunov quantities \( L_1 \) and \( L_2 \) in each sets of conditions 1)–19) obtained in Section 3 by using the algorithm described in [11]. In the expressions for \( L_j \) we will neglect the denominators and non-zero factors.

In Case 1) the first Lyapunov quantity is \( L_1 = 117bc + 36c^2 - 338. \) If \( b = 2(169 - 18c^2)/(117c), \) then Lemma 4.1, (i), \( u^2 = 1/3. \)

In Case 2) the vanishing of the first Lyapunov quantity gives \( c = (u^4 + 16u^2 - 17)/(8u). \) We are in conditions of Lemma 4.1, (i).

In Case 3) the first Lyapunov quantity vanishes. Then Lemma 4.2, (i).

In Case 4) the first Lyapunov quantity is \( L_1 = 3a_2^2 - 4a_2 - 1. \) If \( L_1 = 0, \) then Lemma 4.1, (ii).

In Case 5) the vanishing of the first Lyapunov quantity gives \( a = 2/3. \) We are in conditions of Lemma 4.2, (ii), \( a_2^2 = 1/3, \ h = 4/(3a_1). \)

In Case 6) the first Lyapunov quantity vanishes. Then Lemma 4.2, (ii).

In Case 7) the first Lyapunov quantity vanishes. Then Lemma 4.2, (iii).

In Case 8) the first Lyapunov quantity is \( L_1 = (a - 1)(a_1 - a_2)j_1 \neq 0. \)

Therefore the origin is a focus.

In Cases 9) and 10) we obtain \( L_1 = (c+3)(c+4) \neq 0 \) and \( L_1 = (c-3)(c-4) \neq 0, \) respectively. In these cases the origin is a focus.

In Case 11) the first Lyapunov quantity vanishes. We are in conditions of Lemma 4.1, (iii).

In Case 12) the first Lyapunov quantity is \( L_1 = (32bu + u^4 - 8u^2 + 16)(16b + u^3 - 4u)(u^2 + 4) \neq 0. \) Therefore the origin is a focus.

In Case 13) the first Lyapunov quantity vanishes. Then Lemma 4.2, (iv).

In Case 14) the vanishing of the first Lyapunov quantity gives \( c = -2\sqrt{3}. \)

The second Lyapunov quantity is \( L_2 \neq 0. \) Therefore the origin is a focus.

In Case 15) the vanishing of the first Lyapunov quantity gives \( c = 2\sqrt{3}. \) In this case \( L_2 \neq 0 \) and the origin is a focus.

In Case 16) the equation \( F_{10} = 0 \) admits the following parametrization

\[ v = (z^2 - 1)/(2z), \ a_2 = (z^2 - 6z + 1)(z + 1)^2 + 4z^2)/(2z + 1)^3(z - 1) \]

and the first Lyapunov quantity is \( L_1 = (z^2 + 4z + 1)(z^2 + 1)(z - 1)^4 \neq 0. \) Therefore the origin is a focus.
In Case 17) the first Lyapunov quantity is
\[ L_1 = 1536b^2u^6(u^2 + 1)^2(u^2 - 1) - 8bu^3(11u^4 - 18u^2 + 11)(3u^2 - 1)(u^2 + 1)^2(u^2 - 3) + (3u^2 - 1)^3(u^2 + 1)^2(u^2 - 3)^2(u^2 - 1)^3. \]
The equation \( L_1 = 0 \) has real solutions and \( L_2 \neq 0 \). Therefore the origin is a focus.

In Case 18) the origin is a focus as
\[ L_1 = 8cu^3 + 17u^4 - 16u^2 - 1 \neq 0. \]

In Case 19) the first Lyapunov quantity is
\[ L_1 = 36u^2 - h(17h - 108). \]
The equation \( L_1 = 0 \) admits the following parametrization
\[ v = (108u)/(17u^2 - 36), \]
\[ h = (108u^2)/(17u^2 - 36). \]
Then Lemma 4.1, (iv). \( \blacksquare \)

References


