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AN EXAMPLE OF HIDDEN ATTRACTOR LOCALIZATION

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Abstract For the localization of a self-excited attractor we can numerically compute the trajectory with initial data in a vicinity of an unstable equilibrium. For the localization of a hidden attractor it is difficult to find the corresponding initial data, due to the fact that its basin of attraction does not overlap with a small vicinity of equilibria.

This survey is devoted to the application of a numerical method for localization of hidden attractors for a system of differential equations.

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1. INTRODUCTION

The classification of attractors, from a computational point of view, can be made using as a criterion the simplicity of detection their basin of attraction. Taking into account this classification criterion, recently a concept of hidden and self-excited attractors was proposed ([2], [5], [7]).

Definition 1.1. *An attractor is called a hidden attractor if its basin of attraction does not intersect with small neighborhoods of equilibria, otherwise it is called a self-excited attractor.*

The basin of attraction for a hidden attractor is not connected with any equilibrium. For example, hidden attractors are attractors in systems with no equilibria or with only one stable equilibrium (a special case of the multistability: coexistence of attractors in multistable systems).

Self-excited attractors can be localized numerically by a standard computational procedure, in which after a transient process, a trajectory, started from a point of unstable manifold in the neighborhood of equilibrium, reaches an attractor and identifies it. The Lorenz system

$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= x(\rho-z)-y\\ \dot{z} &= xy-\beta z\end{aligned}$$

was the first well-known example of a visualization of chaotic attractor in a dynamical system corresponding to the excitation of chaotic attractor from unstable equilibria. For classical parameters ($\sigma = 10, \beta = 8/3, \rho = 28$), the Lorenz attractor is self-excited with respect to all three equilibria and could have been found using the standard computational procedure.

For localization of hidden attractors it is necessary to develop special procedures, since there are no similar transient processes leading to such attractors. Recently, such hidden attractors were discovered in Chua's circuits by special analytical-numerical algorithm.

In the works of Leonov G.A., Kuznetsov N.V., the methods of search of periodic solutions of multidimensional nonlinear dynamical systems were suggested. They combined the method of harmonic linearization with numerical methods, and applied bifurcation theory to build an algorithm to detect hidden attractors of systems (see [1]-[7]).

In this paper, we discuss hidden attractors for a system of differential equations. We applied the above-mentioned technique to demonstrate the existence of such attractors for certain values of parameters.

2. DESCRIBING THE ALGORITHM

The algorithm that detects the hidden attractors, that appears in Leonov's works ([1]-[7]) is described by the following steps:

Step 1. Consider a system with vector nonlinearity of the form

$$\frac{dx}{dt} = Px + \psi(x), x \in R^n \tag{1}$$

where P is a constant $n \times n - matrix$, $\psi(x)$ is a continuous vector function ($\psi(0) = 0$).

Step 2. For the search of periodic solution close to harmonic oscillation, define a matrix K such that the matrix $P0 = P + K$ has a pair of purely imaginary eigenvalues $\pm i\omega 0\,(\omega 0 > 0)$ and the rest of its eigenvalues have negative real parts.
Then system (1) can be rewritten as

$$\frac{dx}{dt} = P0x + \varphi(x) \tag{2}$$

where $\varphi(x) = \psi(x) - Kx$.

Step 3. We apply the method of harmonic linearization for the system

$$\frac{dx}{dt} = P0x + \varphi^0(x) = P0x + \varepsilon\varphi(x) \tag{3}$$

where $\varepsilon > 0$ is small, and we search for a stable nontrivial periodic solution $x^0(t)$ close to harmonic one.
To determine the initial data $x^0(0)$ of starting periodic solution, system (3) is transformed by linear nonsingular transformation S to the form

$$\begin{array}{rcl} \dot{y1} & = & -\omega0y2 + \varepsilon\varphi1(y1, y2, y3) \\ \dot{y2} & = & \omega0y1 + \varepsilon\varphi2(y1, y2, y3) \\ \dot{y3} & = & A3y3 + \varepsilon\varphi3(y1, y2, y3). \end{array}$$

Here $y1, y2$ are scalar values, $y3$ is $(n-2)$-dimensional vector, $\varphi1,\ \varphi2$ are certain scalar functions, $\varphi3$ is an $(n-2)$-dimensional vector function, $A3$ is a constant $(n-2)\times(n-2)$-matrix, all eigenvalues of which have negative real parts.

Introducing the describing function

$$\begin{array}{c} \phi(a) = \int_0^{\frac{2\pi}{\omega0}} (\varphi1(a\cos(\omega0t), a\sin(\omega0t), 0)\cos(\omega0t) \\ +\varphi2(a\cos(\omega0t), a\sin(\omega0t), 0)\sin(\omega0t))dt \end{array} \tag{4}$$

we have the next theorem ([6]).

Theorem 2.1. *If it can be found a positive $a0$ such that*

$$\phi(a0) = 0, \frac{d\phi(a)}{da}|_{a=a0} \neq 0, \tag{5}$$

then for sufficiently small ε there exists a periodic solution $x^0(t)$ with the initial data $x^0(0) = S(y1(0), y2(0), y3(0))^*$, where $y1(0) = a0 + O(\varepsilon), y2(0) = 0, y3(0) = On-2(\varepsilon)$, and $On-2(\varepsilon)$ is an $(n-2)$-dimensional vector such that all its components are $O(\varepsilon)$.

If the stability is regarded in the sense that for all solutions with the initial data sufficiently close to $x^0(0)$, the modulus of their difference with $x^0(t)$ is uniformly bounded for all $t > 0$, then for the stability $x^0(t)$ it is sufficient that the condition $\frac{d\phi(a)}{da}|_{a=a0} < 0$ is satisfied.

Step 4. Introduce a finite sequence of continuous functions

$$\varphi^0(x), \varphi^1(x), ..., \varphi^m(x)$$

in such a way that the graphs of neighboring functions φ^j and φ^{j+1}, slightly differed from one another, the function $\varphi^0(x)$ is small, and $\varphi^m(x) = \varphi(x)$.

In the described procedure the simplest and the most natural class of functions φ^j are the following functions:

$$\varphi^0(x) = \varepsilon\varphi(x), \varphi^1(x) = \varepsilon^1\varphi(x), ..., \varphi^{m-1}(x) = \varepsilon^{m-1}\varphi(x), \varphi^m(x) = \varepsilon^m\varphi(x)$$

where ε is a classical small positive parameter and, for example, $\varepsilon^j = j/m, j = 1, ..., m$.

Step 5. We localize the attractor of original system (2), by following numerically transformation of this periodic solution.
After computing the periodic solution $x^0(t)$ of system (3) two cases are possible: all the points of this stable periodic solution are located in the domain of attraction of stable periodic solution $x^1(t)$ of the system

$$\frac{dx}{dt} = P0x + \varphi^j(x) \tag{6}$$

with $j = 1$ or, when pass from (3) to system (6) with $j = 1$, we observe the instability bifurcation destroying periodic solution. In the first case it is possible to find $x^1(t)$ numerically, starting a trajectory of system (6) with $j = 1$ from the initial point $x^0(0)$. Starting from the point $x^0(0)$, after transient process the computational procedure reaches to the periodic solution $x^1(t)$ and computes it. In this case the interval $[0, T]$, on which the computation is carried out, must be sufficiently large. After the computation of $x^1(t)$ it is possible to obtain the following system (6) with $j = 2$ and to organize a similar procedure of computing the periodic solution $x^2(t)$, starting a trajectory which, with increasing t, approaches to periodic trajectory $x^2(t)$, from the initial point $x^2(0) = x^1(T)$. Proceeding this procedure and computing $x^j(t)$, using trajectories of system (6) with the initial data $x^j(0) = x^{j-1}(T)$, we either arrive at periodic solution of system (6) with $j = m$ (i.e., at original system (2)) either observe, at a certain step, the instability bifurcation destroying periodic solution.

3. CASE STUDY

Let us apply the above mentioned algorithm to the 3- dimensional system with scalar nonlinearity, represented by the equations:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= \alpha 1y + \alpha 2z + f(\beta 1x + \beta 2y + \beta 3z) \end{aligned}$$

Here the function

$$f(\sigma) = (\alpha + \beta\sigma^2)\tanh^{-1}(\sigma) \tag{7}$$

describes the nonlinear element of system, $\alpha, \beta, \alpha1, \alpha2, \beta1, \beta2, \beta3$ are parameters of the system.

Step 1. We can rewrite the system as a Lur'e system

$$\frac{dx}{dt} = Px + q\psi(r^*x) \tag{8}$$

where $x \in R^3, P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \alpha1 & \alpha2 \end{pmatrix}, q = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, r = \begin{pmatrix} \beta1 \\ \beta2 \\ \beta3 \end{pmatrix}, \psi(\sigma) = f(\sigma).$

Step 2. We introduce the coefficient k (the coefficient of harmonic liniarization), in such a way that the matrix $P0 = P + kqr^* = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ k\beta1 & k\beta2 + \alpha1 & k\beta3 + \alpha2 \end{pmatrix}$ has a pair of pure-imaginary eigenvalues $\pm i\omega0, \omega0 > 0$, and one negative real eigenvalue $-d < 0$. We have the transfer function of the system (8)

$$W(p) = r^*(P - pI)^{-1}q = \frac{\beta3p^3 + \beta2p + \beta1}{p(-p^2 + \alpha2p + \alpha1)}, \tag{9}$$

where p is a complex variable, I is a unit matrix. We use this transfer function to define the values of k and $\omega0$.
The number $\omega0 > 0$ is defined by the equation

$$ImW(i\omega0) = 0 \Longleftrightarrow \beta3\omega0^4 + (\alpha1\beta3 - \alpha2\beta2 - \beta1)\omega0^2 - \alpha1\beta1 = 0, \tag{10}$$

and k is defined by the formula

$$k = -[ReW(i\omega0)]^{-1} = \frac{\omega0^4 + (\alpha2^2 + 2\alpha1)\omega0^2 + \alpha1^2}{(\alpha2\beta3 + \beta2)\omega0^2 + \alpha1\beta2 - \alpha2\beta1} \tag{11}$$

Step 3. We rewrite system (8) in the form:

$$\frac{dx}{dt} = P0x + q\varphi(r^*x), \tag{12}$$

where $\varphi(\sigma) = \psi(\sigma) - k\sigma$.
Further let us change $\varphi(\sigma)$ by $\varepsilon\varphi(\sigma)$ and consider the existence of a periodic solution for system

$$\frac{dx}{dt} = P0x + \varepsilon q \varphi(r^* x). \tag{13}$$

The transfer function of this system is

$$W_1(p) = r^*(P0 - pI)^{-1} q = \frac{\beta 3 p^2 + \beta 2 p + \beta 1}{-p^3 + (\alpha 2 + k\beta 3)p^2 + (\alpha 1 + k\beta 2)p + k\beta 1}.$$

By the non-singular linear transformation $x = Sy$ system (13) is reduced to the form

$$\begin{array}{rcl} \dot{y}1 & = & -\omega 0 y2 + b1 \varepsilon \varphi(y1 + c3y3) \\ \dot{y}2 & = & \omega 0 y1 + b2 \varepsilon \varphi(y1 + c3y3) \\ \dot{y}3 & = & -dy3 + b3 \varepsilon \varphi(y1 + c3y3). \end{array}$$

The transfer function of this system is

$$W_2(p) = c^*(A - pI)^{-1} b = \frac{-(b1c1 + b2c2)p + (b2c1 - b1c2)\omega 0}{\omega 0^2 + p^2} - \frac{b3c3}{d + p},$$

where

$$b = \begin{pmatrix} b1 \\ b2 \\ b3 \end{pmatrix}, c = \begin{pmatrix} c1 \\ c2 \\ c3 \end{pmatrix}, A = \begin{pmatrix} 0 & -\omega 0 & 0 \\ \omega 0 & 0 & 0 \\ 0 & 0 & -d \end{pmatrix}.$$

From the equality $W_1(p) = W_2(p)$ we obtain the components of vectors b and c, and the real eigenvalue $-d$:

$$\begin{array}{rcl} b_1 & = & -\frac{\beta 2 \beta 3 k - \beta 3 \omega 0^2 + \alpha 2 \beta 2 + \beta 1}{-\beta 3^2 k^2 + 2\alpha 2 \beta 2 k + \alpha 2^2 + \omega 0^2} \\ b_2 & = & \frac{(-\beta 3^2 k - \alpha 2 \beta 3 - \beta 2)\omega 0^2 + \beta 1 \beta 3 k + \alpha 2 \beta 1}{(\beta 3^2 k^2 + 2\alpha 2 \beta 3 k + \alpha 2^2 + \omega 0^2)\omega 0^2} \\ b_3 & = & 1 \\ c_1 & = & 1 \\ c_2 & = & 0 \\ c_3 & = & \frac{\beta 3^3 k^2 + 2\alpha 2 \beta 3^2 k + \alpha 2^2 \beta 3 + \beta 2 \beta 3 k + \alpha 2 \beta 2 + \beta 1}{\beta 3^2 k^2 + 2\alpha 2 \beta 3 k + \alpha 2^2 + \omega 0^2} \\ d & = -(\beta 3 k + \alpha 2). & \end{array}$$

The elements of matrix S can be deduce from the equations $A = S^{-1} P0 S$, $b = S^{-1} q, c^* = r^* S$.
We obtain the transformation matrix

$$\begin{pmatrix} s11 & s12 & s13 \\ s21 & s22 & s23 \\ s31 & s32 & s32 \end{pmatrix},$$

where

$$\begin{aligned}
s11 &= \frac{-\beta 3\omega 0^2+\beta 1}{\beta 3^2\omega 0^4-2\beta 1\beta 3\omega 0^2+\beta 2^2\omega 0^2+\beta 1^2}\\
s12 &= \frac{\beta 2\omega 0}{\beta 3^2\omega 0^4-2\beta 1\beta 3\omega 0^2+\beta 2^2\omega 0^2+\beta 1^2}\\
s13 &= \frac{1}{\beta 3^2k^2+2\alpha 2\beta 3k+\alpha 2^2+\omega 0^2}
\end{aligned}$$

$$\begin{aligned}
s21 &= \frac{\beta 2\omega 0^2}{\beta 3^2\omega 0^4-2\beta 1\beta 3\omega 0^2+\beta 2^2\omega 0^2+\beta 1^2}\\
s22 &= -\frac{(-\beta 3\omega 0^2+\beta 1)\omega 0}{\beta 3^2\omega 0^4-2\beta 1\beta 3\omega 0^2+\beta 2^2\omega 0^2+\beta 1^2}\\
s23 &= \frac{\beta 3k+\alpha 2}{\beta 3^2k^2+2\alpha 2\beta 3k+\alpha 2^2+\omega 0^2}\\
s31 &= -\frac{(-\beta 3\omega 0^2+\beta 1)\omega 0^2}{\beta 3^2\omega 0^4-2\beta 1\beta 3\omega 0^2+\beta 2^2\omega 0^2+\beta 1^2}\\
s32 &= -\frac{\beta 2\omega 0^3}{\beta 3^2\omega 0^4-2\beta 1\beta 3\omega 0^2+\beta 2^2\omega 0^2+\beta 1^2}\\
s33 &= \frac{(\beta 3k+\alpha 2)^2}{\beta 3^2k^2+2\alpha 2\beta 3k+\alpha 2^2+\omega 0^2}.
\end{aligned}$$

The equation $\phi(a)=0$ is

$$\frac{\pi(4\beta a^4+2(3\alpha-\beta)a^2-2(3\alpha+\beta)+(3a^2\beta-3a^2k+6\alpha+2\beta)\sqrt{1-a^2})}{3a\omega 0\sqrt{1-a^2}}=0. \tag{14}$$

and we search for a solution $0<a0<1$ which verify the conditions of the theorem.

Consider system (7) with the parameters $\alpha 1=-10, \alpha 2=-1, \beta 1=-18, \beta 2=-1, \beta 3=-1, \alpha=0.0567, \beta=7$. Note that for the considered values of parameters there is one equilibria in the system: the zero equilibrium $O=(0;0;0)$.

Let us try to apply the algorithm and define an initial data for periodic oscillation.The equation (10) for this parameters has two positive solutions, and by (11) we obtain following starting frequencies and coefficients of harmonic: $\omega 0=\sqrt{15}, k=5$ and $\omega 0=2\sqrt{3}, k=2$.

For parameters $\omega 0=\sqrt{15}, k=5$ we can find the initial amplitude $a0=0.8380212233$ that satisfies the conditions of Theorem 2.1 ($\phi(a0)=0, \frac{d\phi(a)}{da}|_{a=a0}=-0.6960685111<0$).

Thus, one can obtain the initial data, namely $x(0)=-0.1047526529$, $y(0)=-0.5237632646, z(0)=1.571289794$ for the first step of the multistage procedure for localization of hidden attractor.

Step 4-5. For the value of parameter $\varepsilon 1=0.1$, after the transient process the computational procedure reaches the periodic solution $x1(t)$ (Figure 1).

Then by sequentially increasing the parameter εj and the computation of $xj(t)$ (see (Figures 1-4)), the hidden attractor is computed for the original system (3).

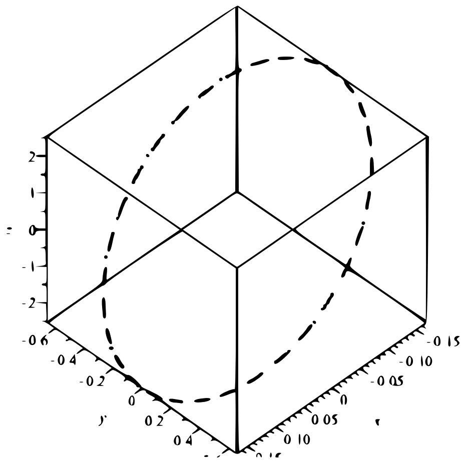

Fig. 1. Hidden attractor localization $\varepsilon = 0.1$

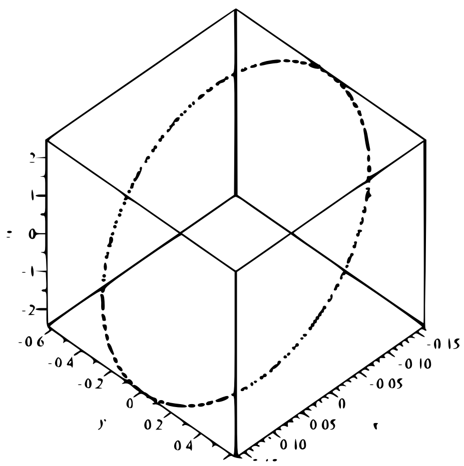

Fig. 2. Hidden attractor localization $\varepsilon = 0.4$

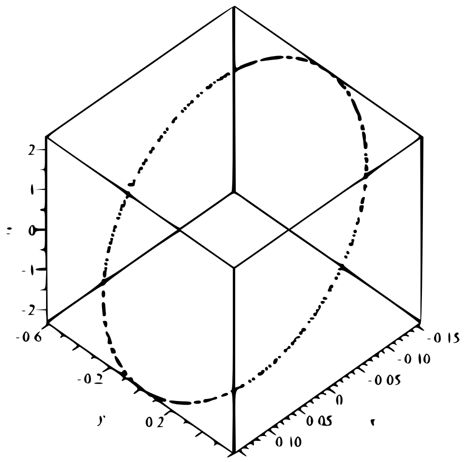

Fig. 3. Hidden attractor localization $\varepsilon = 0.7$

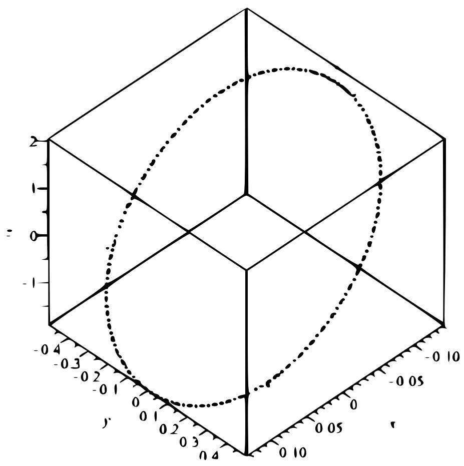

Fig. 4. Hidden attractor localization $\varepsilon = 1$

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