ROMAI J., v.13, no.2(2017), 117–141

A SURVEY ON THE BEST APPROXIMATION BY SPLINES

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Abstract As a part of *the Approximation Theory* and its *Applications* in *Vector Spaces*, approaches to achieve *the Best Approximation* by *Splines* are reviewed in this research work, starting from the background of *the Linear Spaces*, followed by *Hausdorff Locally Convex Spaces*, the Normed Linear Spaces and the particular case of *Hilbert* spaces. Special attention is given to the most significant *Best Approximation Problems* in *Separated Locally Convex Spaces* and to their straight connections with *the Vector Optimization*, that is, with *the general Efficiency*. Remarkable consideration is given to these problems in *H - locally convex spaces* where the splines introduced by an original method represent *the only best approximation simultaneous* and *vectorial solutions* as *optimal interpolation elements*. The survey covers *illustrative references*.

Keywords: locally convex space, best approximation, general efficiency, multifunction, spline function, H – locally convex space.

2010 MSC: Primary: 41A65; Secondary: 41A99.

1. INTRODUCTION

What does the Best Approximation mean? Which are its Descriptions, the Immediate Implications and the Applications? In the first place, we think that all of these questions mean to be "the sooner the better", closed, following a specified meaning, to the elements of all non - empty subsets placed in arbitrary but proper spaces, by the points of every possible other sets from any suitable another spaces and with respect to their "mathematical" structures. In fact, *any process of optimization represents a Best Approximation Problem* and *conversely*. The most important aim in any such as this subject of every scientific work is to establish appropriate conditions for the existence of the Best Approximation Elements, the corresponding possibilities of Approximation (sometimes established using the properties of the set containing all of them) or, better, the effective Identification of the *Optimal Solutions* and so on. Here we examine this matter in the background offered by the Hausdorff Locally Convex Spaces, having in view and starting sometimes for generalization from the corresponding cases in Topological Spaces and the particular position of the Normed Linear Spaces. Thus, following *Postolică, V., 1981*, [85], let (X, τ) be an arbitrary topological space and A be a non-empty subset. If for each $x \in X$ one denotes by $S(x)$ a base (fundamental system) of open neighborhoods and one considers $V_y(x) = \bigcap_{\substack{V \in S(x) \\ y \in V}} V$ for any $y \in X$, then $a_0 \in A$ is *a best approximation* of x by the elements of A with respect to $S(x)$ if $\bar{V}_{a_0}(x) \subseteq \bigcap_{a \in A} \bar{V}_a(x)$, where $\bar{B}$ represents the closure of every $B \subseteq (X, \tau)$. Cleary, this concept depends on the base of neighborhoods and it can be extended whenever every at least for the points $x \in X$ for which one looks to find and study the best approximation elements from a nonvoid subset $A \subset X$ it was attached a proper system of non-empty sets which allows the approaching to x from every element of A (see also, *Deutsch, F*, 1966, [27]). At the same time, it is easy to see that this modality exhibited for the first time in (*Postolică, V., 1981* [85]), corresponds in locally convex spaces to the best concomitant approximation with respect to every semi-norm, that is, as we shall see, with the best simultaneous approximation (*Isac, G., Postolică, V., 1993*, [45]) and in the metric spaces and the normed linear spaces, respectively, with the usual best approximation (*Goel, D., S., Holland, A., S.,B., Nasim, C., Sahney, B., N., 1974*, [34] *Holmes, R., B.,1972*, [37]; *Laurent, J., P., 1972* [54]; *Milman, P., 1977,* [61]; *Mohebi, H., 2002,* [65], *Narang, T., D., 1983*, [69], *Singer, I.*, 1967, [123], 1979, [124] and so on). Generally speaking, the *following ways were* and *they are used* in the *study* of the *best approximation* matters:

a) existence and uniqueness of the best approximation elements;

b) from topological point of view, properties of the sets containing the best approximation elements;

c) with respect to a family of functions;

d) many combinations between a) – c);

e) adequate numerical methods to identify or to approximate the best approximation elements.

The base of this research paper is represented by the book (*Isac, G., Postolică, V., 1993*, [45]) which was substantiated on our *Doctoral Thesis (1987)* and the corresponding references, completed by recent results, remarks and pertinent scientific works.

2. THE BEST APPROXIMATION IN LINEAR SPACES

Let X be a vector space and $\mathrm{F} = \{f_\alpha\}_{\alpha \in A}$ family of norms defined on X. The study of the Best Approximation of an element $x_0 \in X$, by the elements of a non - empty subset $G \subset X$, with respect to the family F, can be achieved by the following ways:

A. We can define a convenient norm H, using the family F and we can study the problem to find the elements $g_0 \in G$ such that $H(x_0 - g_0) \leq H(x_0 - g)$, for all $g \in G$. For example, if

$$\mathrm{F} = \{\|\cdot\|_i\}_{i=\overline{1,n}}\,,;$$

$n \in N^*$, it is possible to define H by $H(x) = \max\{\|x\|_i\}_{i=\overline{1,n}}$ or $H(x) = [\sum_{i=1}^{n} \|x\|_i^p]^{1/p}$ for every $x \in x$, with $1 < p < +\infty$ (see, for instance *Bacopoulos, A., Godini, G., Singer, I., 1976*, [4],*1980*, [6], *Dierieck, C., 1976,* [28], *Milman, P., 1977,* [61]). In particular, given $(X, \|\cdot\|)$ a normed linear space and $B \subset X$ a non-empty, bounded subset, then we can consider also the problem to find the elements g_0 in another non - empty subset $G \subset X$such that $\inf\limits_{g\in G} \sup\limits_{b\in B} \|b - g\| = \sup\limits_{b\in B} \|b - g_0\|$. It is obvious that each of the next problems has solutions whenever the set B is non-empty and compact, thanks to the usual Weierstrass theorem: $\min\limits_{b\in B} \|x_0 - b\|$ and $\max\limits_{b\in B} \|x_0 - b\|$ $(x_0 \in X)$. The first part of the above comment represents the Best Simultaneous Approximation Problem of the set B by the elements of G, studied for example, in *Goel, D., S., Holland, A., S.,B., Nasim, C., Sahney, B., N., 1974*, [34] and following some variants in *Milman, P., 1977,* [61], *Mohebi, H., 2002,* [65], *Narang, T., D., Ahuja, G. C., 1979*, [68], *Narang, T., D., 1983,* [69], *Singer,I., 1967,* [123], *1979*, [124], *Veeramani, P., 2002,* [132] and in other related papers. On the other hand, even if F is a family of real functions the Best Approximation Problem was tackled as follows.

B. The second way is to find the elements $g_0 \in G$ such that $f_\alpha(x_0 - g_0) \leq f_\alpha(x_0 - g)$, for all $g \in G$ and $\alpha \in A$. By this way, we have the Best Simultaneous Approximation of the element x_0 by G with respect to the family F. From this point of view, we may consider the Best Simultaneous Approximation Problem as a particular case of the Strong Vectorial Optimization. This remark has implied several interesting developments in the *Theory and Applications of Optimization* for *multivalued functions* (*Bitran, G., R., 1981*, [8], *Borwein, J., M., 1980,* [11],*1981,* [13], [14], *1983,* [15]; *Borwein, J., M., Wolkowicz,H, 1981*, [16], *Brumelle, Shelby, 1981*, [17], *Diereck, C., 1976*, [28], *Dolecki, S., 1980*, [29], *Jahn, J., 1983*, [48], *Kawasaki, H., 1981,* [50], *1982,* [51], *Luc, D. T., 1984*, [56], *Malivert, C., 1984,* [58], *Nayashi, M., Komiya, H., 1982,* [70], *Olech, C., 1969*, [74], *Ponstein, J., 1980*, [82], 1987, [83], *Singer, I., 1979*, [124], *Valadier, M., 1972,* [130], *Yu, P., L., 1974,* [134], *Zălinescu, C.,1978*, [137], *1980*, [138], *1983*, [140], *Zowe, J., 1975*, [141], *1978*, [142] and so on).

C. The third way is to find necessary, sufficient, necessary and sufficient or dual conditions for the existence of the elements $g_0 \in G$ for which there exists no $g \in G$ such that $f_\alpha (x_0 - g) \leq f_\alpha (x_0 - g_0)$, for every $\alpha \in A$, with at least one of these inequality being strict. So, we obtain the Best Vectorial Approximation of the element x_0 by G with respect to the family F (*Bacopoulos, A., Godini, G., Singer, I.,1976*, [4], *Bacopoulos, A.,Singer,I,1977*, [5], *Bacopoulos, A., Godini, G., Singer, I.,1980,* [6], *Postolică, V., 1983*,[45], *1984*, [91], *1985*, [93], *Singer, I., 1979*, [124] for further references). In all these cases, the corresponding projection maps are the *multifunctions*. This way shows also that *the Best Vectorial Approximation Problems are* special cases of the *Vectorial Optimization Programs* (*Bitran, G., R.,Magnanti, T., L.,1979*, [9], *Borwein, J., M., 1977*, [10], *1980*, [11], *1981*, [13], *Cesari, L., Suryanarayana, M., B., 1976*, [21], 1978, [22], [23], 1980, [24], *Corley, H.W., 1980*, [25], *1981*, [26], *Godini, G.,1978*, [33], *Isac, G., 1982*, [40], *1983,* [41],*1985*, [43], *1987,* [44],*1994*, [46], *Isac, G., Postolică, V., 1993*, [45], *2005,* [47], *Jahn, J., 1983*, [48], *Kawasaki, H., 1981*, [50],*1982*, [51], *Leitman, G., Yu, P., L., 1974,* [55], *Luc, D., T., 1984*, [56], *1989*, [57], *Minami, M., 1980*, [62],*1981*, [63], *Nagahisa, Y., 1981*, [66], *Nakayama, H., 1984*, [67], *Nayashi, M., Komiya, H., 1982*, [70], *Nieuwenhuis, J. W., 1980*, [71], *1982*, [72], *Penot, J., P., 1978*, [77], *Ponstein, J.,1988, [83],Pontini,C., 1990*, [84], *Postolică, V., 1984*, [91], *1986*, [94], [95], *1993*, [98], *1997*, [99], *1998*, [100], [101], *2002*, [104], *Postolică, V., Scarelli, A., 2000*, [102], *Robert, F., 1985*, [119], *Ruzica, S., Wiecek, M., M., 2003*, [120], *Tanino, T., Sawaragi, Y., 1979*, [127], *1980*, [128], and all recent connected research works) and, at the same time, a program which intends to find *the vectorial optimum points* for some *multifunctions*, in particular, the *critical points* for a class of *dynamical systems* or the *equilibria* for several set - valued maps, which needs the corresponding dual concepts as: *vectorial subdifferential*, *conjugate* and (*Borwein, J., M., 1983,* [15], *Borwein, J., M., Wolkowicz, H., 1981,* [16], *Brumelle, Shelby, 1981*, [17], *Gabor, Grzegorz, Quincampoix, Marc, 2003*, [32], *Isac, G., 1982,* [40], *Chapter 4 in Isac, G., Postolică, V., 1993*, [45], *Jahn, J., 1983*, [48], *Kawasaki, H., 1981,* [50], *1982*, [51], *Luc, D. T., 1989*, [57], *Malivert, C.,*1984, [58], *Postolică, V., 1986*, [94], [95], *1989,* [96], *1990,* [97], *1993*, [98], *1997,* [99], *1998*, [101], *Precupanu, T., 1980*, [115], *Tanino, T., Sawaragi, Y., 1979*, [127], *1980*, [128] and other significant scientific research works). Concerning the uniqueness of the best approximation elements it is known that, in an *arbitrary Banach space* X, *every element possesses at most a best approximation with respect to any non - empty convex subset*, *if* and *only if* X is *strictly convex (see for example, Theorem 3.7 and Remark 3.4 in Chapter 3 of Barbu, Viorel, Precupanu, T., 1986, [7]).* Thus, in every Banach space there is a strong connection between the *strict convexity* of the space and the *uniqueness* of the elements of the *best approximation* by *non - empty* and *convex subsets* with *respect* to the *norm* which generates its *topology*. Starting from the more useful definition of the strict convexity for locally convex spaces introduced by *Huffman, E., 1977*, [39]) and using our research papers *Postolică, V., 1983,* [87], 1985, [93], which contain also its extension for a family of real functions defined on a real or complex linear space, in Chapter 3 of *Isac, G., Postolică, V., 1993*, [45] were exhibited the basic relationships between the *uniqueness* of the *elements* of *best simultaneous (vectorial) approximation*, the *strict simultaneous (vectorial) convexity* and some connections between the *simultaneous strictly convexity* and the *vectorial strictly convexity*. Obviously, these settings can be extended to the families of functions defined on *linear spaces* and taking values in *ordered linear spaces*, under *proper hypotheses*.

3. THE BEST APPROXIMATION IN LOCALLY CONVEX SPACES

The Best Approximation Problem in any separated locally convex space with respect to the corresponding family of the semi-norms which generates the locally convex topology was studied at various methods.

The first of them lay in thinking the best approximation matter with respect to each of the appropriate semi - norm (not necessary considered with respect to the other semi - norms of the initial family which induces the Hausdorff locally convex topology), modality of study which is *equivalent* to consider the *Minkowski gauge* attached to some *non - empty, convex, absorbing and circled subset*. In all these cases, through the agency of the corresponding separated subspace one obtains a *best approximation problem* in a *normed linear space* (*Asimov, L., 1978*,[2], *Duffin, R., J., Karlowitz, L. A., 1968*,[30], *Laurent, J., P., 1972*,[54], *Otto, Ch., 1979*,[75], *Reich, S., 1978* [118], *Sahney, B., N. 1983*,[121], *Schock, E., 1972*, [122], *Singh, S., 1980*, [125], *Ubhaya, V., 1974*,[129]and so on).

Another mean of investigation, specific to the *metrizable locally convex spaces* and in direct connection with the first point of study specified in the second section of this research work consist in the use of the *cvasi* or *asymmetrical norms*, with their some particular projections in *normed modules* over *semifields* or *vectorial lattices* (see, for example, *Albinus, G., 1966*, [1], *Gusev, A. I., 1975,* [35], *Hirschfeld, R. A., 1958*, [36], *Nikolski, V., N., 1948*, [73], *Vasiliev, A., I., 1998,* [131], *Wulbert, D., F., 1966,* [133], *Zarnadze, D., N., 1980*, [135],*1992*, [136]and the other connected papers). Some generalizations were obtained by replacing the semi - norms with proper functions (*Pai, D., V., Govindarajulu, P., 1984*, [76], *Precupanu, A., Precupanu, T. 1987*, [110],*1998*, [111], *Precupanu, T., 1984*, [116], *Sahney, B. N.,1983*, [121]). As we shall see in the sequel, the *Best Vectorial Approximation* in *locally convex spaces* is a particular case of *Pareto type Optimization*, but we must also to remark that the Pareto Optimality with respect to the families of semi - norms induces some type of best approximation problems as in (*Aubin, J., P., 1971,* [3], *Caligaris, O., Oliva, P., 1981*, [19], *Censor, Y., 1977*, [20]).

The next considerations are devoted to the basic concepts concerning the *Best Approximation* in *Locally Convex Spaces* by summarizing significant results obtained in this field: *existence of the best (simultaneous vectorial or other kinds) approximation elements, uniqueness, connections* with the *general Efficiency* by *its particular branch* named *Pareto type Optimality* and the *multifunctions*, following: *Isac, G., Postolică, V., 1993*, [45], *Kramar, E., 1981*, [52], *Peressini, A., L., 1967*, [79], *Pietch, A., 1972*, [80], *Postolic, V., 1981*, [86], *1983*, [87], *1989*, [96], *1990*, [97], *1998*, [101], *2002*, [104], *Powell, M., J. D., 1981*, [109], *Precupanu, A., Precupanu, T., 2002*, [112], *Precupanu, T., 1968*, [113], *1969*, [114], *1980*, [115], *1984*, [116], *1992*, [117] and other such as these research works.

Let X be a Hausdorff locally convex space with the topology induced by a family $\mathrm{P} = \{p_\alpha : \alpha \in A\}$ of semi - norms, $x_0 \in X$ and G a non-empty subset. Throughout this paper we denote by $\emptyset$ the empty sets.

Definition 3.1. *(Isac, G., Postolică, V., 1993, [45], Postolică, V., 1983, [87]-[90]). An element $g_0 \in G$ is said to be a best simultaneous approximation of x_0 by the elements of G with respect to the family P (abreviated, g_0 is a* P *- b.s.a of x_0) if*

$$p_\alpha (x_0 - g_0) \leq p_\alpha (x_0 - g) \; for\, all\, g \in G\, and\, p_\alpha \in \mathrm{P}. \tag{1}$$

When, in addition, each element $x \in X$possesses at least one P - b.s.a in G, then G is called P-*simultaneous proximal* and we denote by $S(x_0, G)$ the set of all P - b.s.a elements of G for each $x_0 \in X$.

By analogy with the usual compactness hypothesis in normed linear spaces, the P - simultaneous proximality and, implicitly, the existence of the P - b.s.a elements is ensured using the next concept of approximate compacity given by

Definition 3.2. *(Efimov, I. V., Stekin, S. B., 1959,[31], Singh, S., Watson, B., Srivstana, P., 1997,[126]). A non-empty set $B \subset X$ is called approximatively compact with respect to the family*P *(abreviated B is* P*-approximatively compact) if for each family $\left\{(b_{\alpha i})_{i \in I} : \alpha \in A\right\}$ of nets in Bsuch that the net $(p_\alpha (x - b_{\alpha i}))_{i \in I}, x \in X$ converges to* $\inf \{p_\alpha (x - b) : b \in B\}$, $\forall \alpha \in A$, *there exists a subnet $(b_t)_{t \in T}$ of any net having its limit in B.*

This notion together with the continuity of the semi - norms leads immediately to

Theorem 3.1. *Any P-approximatively compact subset of X is P-simultaneous proximinal.*

The Theorem 2.3 in Chapter I of *Isac, G., Postolică, V., 1993,* [45], together with the results contained in the first paragraph of Chapter 2 from *Barbu, Viorel, Precupanu, T., 1986*, [7] and the Corollary 2.4 in *Zălinescu, C., 1983*, [140] generates the following dual characterization of non - empty, convex and simultaneous proximinal subsets.

Theorem 3.2. *(Isac, G., Postolică, V., 1993, [45]) A non-empty and convex subset G of X is P-simultaneous proximinal if and only if for every $(x_0, r_0) \in X \times R_+^A$ with $G \bigcap \{x \in X : p_\alpha (x_0 - x) \leq r_0 (\alpha), \forall \alpha \in A\} = \emptyset$, there exists α_0 in A and x_0^* in the dual space X^* of X such that*

$$r_0 (\alpha_0) < x_0^* (x_0) - p_{\alpha_0}^* (x_0^*) - \sup_{g \in G} x_0^* (g) . \tag{2}$$

If, in addition, the set G contains the origin, then (3.2) is equivalent with

$$r_0 (\alpha_0) < x_0^* (x_0) - p_{\alpha_0}^* (x_0^*) - p_{G^0} (x_0^*) \tag{3}$$

where $p_{\alpha_0}^*$ is the corresponding conjugate map of p_{α_0} and p_{G^0} is the Minkowski functional associated to the polar $G^0 = \{x^* \in X^* : x^* (x) \leq 1, \forall x \in G\}$ of G.

As we shall see, the *Best Simultaneous Approximation* is a strongly connected special instance of the *Best Vectorial Approximation*, which was initially introduced (*Isac, G., Postolică, V., 1993*, [45], *Postolică, V., 1983,* [87]) as a particular case of the Best Vectorial Approximation in comparison with a family of *real valued functions* defined on a linear space. But, both of them, can be obtained in the general context of the Topogical Ordered Vector Spaces, through the agency of the *general Efficiency*, with its particular case represented by the generalized Pareto type Optimality (eventually approximated as we considered in *Isac, G., Postolică, V., 2005*, [47] as follows. Thus, let Y be a (topological) real vector space ordered by a *convex cone* K, T a non - empty subset and $t_0 \in T$.

Definition 3.3. *(Postolică, V., 1989,[96],1998,[101],2002,[104],[105],Postolic, V., Scarelli, A., 2000,[102]) We say that t_0 is a minimal efficient point for T with respect to K, in notation, $t_0 \in MIN_K (T)$ if it satisfies one of the following equivalent condition:*
(i) $T \bigcap (t_0 - K) \subseteq t_0 + K$;
(ii) $K \bigcap (t_0 - T) \subseteq -K$;
(iii) $(T + K) \bigcap (t_0 - K) \subseteq t_0 - K$;
(iv) $K \bigcap (t_0 - T - K) \subseteq -K$.

It is clear that, whenever K is pointed, that is, $K \bigcap (-K) = \{0\}$, then $t_0 \in MIN_K (T)$ means $T \bigcap (t_0 - K) = \{t_0\}$ or, equivalently, $K \bigcap (t_0 - T) = \{0\}$. In a similar manner one defines the *maximal efficient* elements of T. In fact, $t_0 \in T$ is a *maximal efficient point* of T with respect to K in notation,

$t_0 \in MAX_K(T)$ *iff* it is a *minimal efficient element* for T with respect to $-K$, that is, $t_0 \in MIN_{-K}(T)$. Consequently, the *set of all efficient points* for T with respect to K is defined by $Eff_K(T) = MIN_K(T) \cup MAX_K(T)$.

In the language of the *multifunctions*, the above definition is equivalent with t_0 to be a fixed point for any of the following multifunctions: $F_i : T \to T$, $i = \overline{1,4}$,

$$F_1(x) = \left\{ t \in T : T \bigcap (t - K) \subseteq x + K \right\},$$

$$F_2(x) = \left\{ t \in T : T \bigcap (x - K) \subseteq t + K \right\},$$

$$F_3(x) = \left\{ t \in T : (T + K) \bigcap (t - K) \subseteq x + K \right\},$$

$$F_4(x) = \left\{ t \in T : (T + K) \bigcap (x - K) \subseteq t + K \right\},$$

that is, $t_0 \in F_i(t_0)$ for every $i = \overline{1,4}$, which means that t_0 is an *equilibrium point (Gabor Grzegorz; Quincampoix, Marc, 2003*, [32]) for each of the set - valued maps $t \to F_i(t)$, $i = \overline{1,4}$. If, in addition, K is pointed, that is, then $t_0 \in T$ is a minimal efficient point of T with respect to K if and only if one of the following equivalent relations hold:

(i) $T \bigcap (t_0 - K) = \{t_0\}$; (ii) $T \bigcap (t_0 - K \setminus \{0\}) = \emptyset$;
(iii) $K \bigcap (t_0 - T) = \{0\}$; (iv) $(K \setminus \{0\}) \bigcap (t_0 - T) = \emptyset$;
(v) $(T + K) \bigcap (t_0 - K \setminus \{0\}) = \emptyset$.

It is important to mention here that (i) means that t_0 is a *critical point* (*Isac, G.,1982*, [40]) for the generalized dynamical system $\Gamma : T \to T$ defined by $\Gamma(t) = T \bigcap (t - K)$, $t \in T$ and the *Duality for Vectorial Optimization Programs* with *multifunctions* was also studied in *Chapter 4* of *Isac, G., Postolică, V., 1993*, [45], *Postolică, V.*, *1984*, [91], *1985* ,[93], *1986*, [94], [95], *1989*, [96] and in the other related papers. Some generalizations of the Strong Optimization and Pareto Efficiency with respect to cones to their approximate variants in (topological) ordered vector spaces together with the immediate bilateral connections can be found in *Isac, G., Postolică, V., 2005,* [47].

Theorem 3.3. *(Isac, G., Postolică, V., 1993, [45], 2005, [47], Postolică, V. 1983, [87]-[90], 1998, [101]). If we denote the set of all strong $K-$minimal (ideal) elements of T by $S(T,K) = \{t_1 \in T,\ t - t_1 \in K,\ \forall t \in T\}$ and $S(T,K) \neq 0$, then $MIN_K(T) = S(T,K)$.*

A significant illustration of this coincidence result will be exhibited in **Section 5** of this Research Report. At the same time we remark that it is possible to have $S(T,K) = \emptyset$ and $MIN_K(T) = T$. Indeed, for example, if $Y =$

$\mathbb{R}^N$, $K = \mathbb{R}_+^N$ and $T = \left\{(x_i) \in X : x_i \geq 0 \text{ whenever } i \in N \text{ and } \sum_{i \in N} x_i = 1\right\}$, where N denotes the usual set of natural numbers, then $S(T, K) = \emptyset$ and $MIN_K(T) = T$.

A *Dual Characterization* for the *Efficient Points*, very useful in the *Scalarization* of the *Vector Optimization Programs* with the corresponding projection in the Best *Approximation Problems*, is contained in

Theorem 3.4. *(Isac, G., Postolică, V., 1993, [45], Postolică, V., 1998, [101]). If K is any closed convex cone in Y and T is every non-empty subset, then $t_0 \in MIN_K(T)$ if and only if for each $t \in T \backslash (t_0 + K)$, there exists y^* in the usual dual space Y^* of Y such that $y^*(t) < y^*(t_0)$ and $y^* \in K^0$, where K^0 denotes the well known polar cone of K defined by $K^0 = \{y^* \in Y^* : y^*(y) \leq 0 \; for \; all \; y \in K\}$.*

The following concept of convex cone in the background offered by any Hausdorff locally convex space was and it remains very important in the study of Pareto Efficiency.

Definition 3.4. *(Isac, G., 1982, [40], 1983, [41]). If $(X, \mathrm{P}) = \{p_\alpha : \alpha \in I\})$ is any separated locally convex space with the topology induced by a family* P *of semi - norms, being ordered by a convex cone K and having its topological dual space denoted by X^*, then K is named supernormal (nuclear) if for every semi-norm $p_\alpha \in \mathrm{P}$ there exists $x^* \in X^*$ such that $p_\alpha(x) \leq x^*(x)$ for all $x \in K$.*

The class of the *Nuclear Cones* introduced by Professor George Isac in *Points Critiques des Systèmes Dynamiques, Cones Nucléaires et Optimum de Pareto. Research Report, Royal Military College of St. Jean, Qubec, Canada, 1981*, published in *Isac, G.,1982*, [40], renamed by him as Supernormal Cones (*Isac, G.,1983*, [41]) and called by us and officially recognized as Isacs Cones (*Postolică, V., 2009*, [106]) for the Hausdorff separated Locally Convex spaces was initially imposed by the Theory and the Applications of the Pareto type Efficient Points (especially existence conditions based on Completeness instead of Compactness were decisive together with the main properties of the Efficient Points Sets). Many examples, important remarks, pertinent results, comments and related topics can be found in *Isac, G., Postolică, V., 1993*, [45], *Isac, G., 1994*, [46], *Postolică, V., 1993*, [98],*1997*, [99],*1998*, [101], *2001*, [103], *2002*, [104], [105], *2009*, [106], *2014*, [107], *2015*, [108] and so on.

Definition 3.5. *(Isac, G., Postolică, V., 1993, [45], Postolică, V., 1983, [87]-[90], 1984, [91], 1986, [94]-[95], 1989, [96]). $g_0 \in G$ is said to be a best vectorial approximation of x_0 by G with respect to* P *(abbreviated, go is* P *- b. v. a. of x_0) if*

$$(p_\alpha(x_0 - g_0)) \in MIN_{\underset{+}{I}} \{(p_\alpha(x_0 - g)) : g \in G\}, \tag{4}$$

where A denotes the power of the closed, convex and pointed cone $_+$.

All non - empty subsets of X which contain at least one P - b. v. a. element for each $x \in X$ will called P - *vectorial proximinal.* Whenever for each $x \in X \backslash G$ there exists no a P - b. s. a. (P - b. v. a.), then the set G is called P - *anti – simultaneous (vectorial) proximinal.* Evidently, *the vectorial anti – proximinality* ensures *the simultaneous anti – proximinality*, but the reverse implication is not generally valid as it can be seen in the example below. If one denotes by $V(x_0, G)$ the set of all P - b. v. a. elements of G for $x_0 \in X$, then it is obvious that $S(x_0, G) \subseteq V(x_0, G)$ and, taking into account **Theorem 3.3** it follows that $S(x_0, G) = V(x_0, G)$ whenever $S(x_0, G) \neq \emptyset$. In general, the *converse* of the above inclusion is *not valid* even in *Euclidean spaces.* For example, let $X = \mathbb{R}^n\ (n \in N^*, n \geq 2)$ endowed with the H - locally convex topology generated by the family $P_0 = \left\{p_i : i = \overline{1,n}\right\}$ of seminorms defined by

$p_k(x_1, x_2, \ldots, x_n) = |x_k|,\ \forall x = (x_1, x_2, \ldots, x_n) \in X,\ k = \overline{1,n}$

and for each real number α let

$$G_\alpha = \left\{(x_i) \in {}^n : \sum_{s=1}^{n} x_s = \alpha\right\}.$$

Then, $S(0, G_\alpha) = \emptyset$, $V(0, G_\alpha) = G_\alpha$ and each set G_α is P_0- *anti – simultaneous proximinal* without to be P_0- *anti – vectorial proximinal.*

Indeed, in the contrary case, let $x_0 = (x_0 i) \in R^n \backslash G_\alpha$ for which there exists $g_0 = (g_{0i}) \in G_\alpha$ a P_0 - b.s.a. Then, there exists $j = \overline{1,n}$ such that $x_{0j} \neq g_{0j}$. Suppose that $x_{0j} > g_{0j}$. There exists $n \in \mathbb{N}^*$ so that $x_{0j} - g_{0j} - \frac{1}{n} > 0$. Taking $g_1 = (g_{1i}) \in G_c$ defined by $g_{1h} = g_{1_j} - \frac{1}{n}$ if $h = j$, $g_{1h} = g_{1_j} + \frac{1}{n}$ for $h = j + 1(j-1)$ and $g_{1h} = g_{1i}$ if $i \neq j, j+1(j-1), h \neq j, j+1(j-1)$, the quality of P_0- b.s.a. of g_0 for x_0 by G_c fails. In a similar manner one acts for $x_{0j} < g_{0j}$. The last part of the announced conclusion is also easy to be verified. Accordingly to the system of seminorms defined in a similar manner for $X = \mathbb{R}^N$ and $G_\beta = \left\{(x_i) \in \mathbb{R}^N_+ : \sum_{i \in N} x_i = \beta\right\}(\beta > 0)$, it follows also that $S(0, G_\beta) = \emptyset$ and $V(0, G_\beta) = G_\beta$, $\forall \beta > 0$.

Also, through the agency of the *general efficiency*, ([107], [108]), **Definition 3.5** gives emphasis to an important connection between the *Best Approximation* and the *Potential Theory* using the *coincidence result of the efficient points sets* with *Choquet's boundaries* (*Bucur, I., Postolic, V., 1996*, [18], *Postolic, V., 2002*, [105] and their references). Moreover, in all such as these problems, it can be used special methods. Thus, for example, following the *polarity calculus* it is possible to establish the existence of the solutions by the *generated duality scalarization* (*Barbu Viorel, Precupanu, T., 1986*,[7], *Jahn, J., 1983,* [48], *Laurent, J. ,P., 1972*, [54], *Precupanu, T., 1980*, [115], *1992*, [117] and so on).

Between the Best Simultaneous Approximation and the Best Vectorial Approximation there exists other connected versions *(Precupanu, A., Precupanu, T., 2002*, [112]) :

I. *For every* $p_\alpha \in P$ *there exists* $g_\alpha \in G$ *with*

$$p_\alpha (x_0 - g_\alpha) \leq p_\alpha (x_0 - g), \forall g \in G;$$

II. *There exists* $\bar{p}_\alpha \in P$ *and* $\bar{g}_\alpha \in G$ *such that*

$$\bar{p}_\alpha (x_0 - \bar{g}_\alpha) \leq \bar{p}_\alpha (x_0 - g), \forall g \in G;$$

III. *There exists* $\alpha_0 \in I$ *and* $\bar{g}_0 \in G$ *so that*

$$p_\alpha (x_0 - \bar{g}_0) \leq p_\alpha (x_0 - g), \forall g \in G, \forall \alpha \geq \alpha_0;$$

IV. *There exists* $\alpha_0 \in I$ *and* $g_0 \in G$ *such that*

$$\max_{\alpha \in A} \inf_{g \in G} p_\alpha (x_0 - g) = \min_{g \in G} \sup_{\alpha \in A} p_{\alpha_0} (x_0 - g) = p_\alpha (x_0 - g_0).$$

Clearly, III is a particular case of Best Simultaneous Approximation with respect to a section of P, having special properties if one considers the directed families of semi – norms, that is, whenever any section generates the same locally convex topology.The immediate relationship between these variants was given in the next theorem.

Theorem 3.5. *(Precupanu, A., Precupanu, T., 2002, [112])***.**

(i) if there exist P - b. s. a. elements of x_0 *by* G, *then the solutions of all problems I-III coincide with them;*

(ii) whenever g_0 *is some* P - *b. s. a. element for* x_0 *by* G, *IV has solutions if and only if there exists* $\alpha_0 \in A$ *and a* P - *b. s. a.* g_0 *of* x_0 *by* G *such that* $p_{\alpha_0} (x_0 - g_0) = \sup_{\alpha \in A} p_\alpha (x_0 - g_0)$.

In all these cases, (α_0, g_0) *is a saddle point and* $p_{\alpha_0} (x_0 - y_0)$ *a saddle value for the minimax problem IV (Precupanu, T., 1992).*

By using (*Precupanu, T., 1984*, [116]) concerning the Global Optimality for a family of Optimization Programs having a lower - semicontinuous valued function, in (*Precupanu, A., Precupanu, T., 2002*, [112]) was given an identical characterization for the proximinality with the corresponding sense of I. Moreover, following the **Theorem 1.1** (*Precupanu, A., Precupanu, T., 1987*,[110],*1998*,[111]), in *Precupanu, A., Precupanu, T., 2002*, [112] was obtained a necessary and sufficient condition for the existence of the solutions of II and one analyzes two significant examples together with the next connection between the problems III and IV.

Theorem 3.6. *(Precupanu, A., Precupanu, T., 2002, [112]) (i) every solution* $(\alpha_0, g_0) \in A \times G$ *of IV is solution for III and* $p_\alpha(x_0 - g_0) = p_{\alpha_0}(x_0 - g_0)$;

(ii) conversely, any solution $(\alpha_0, g_0) \in A \times G$ *of III satisfying the above equality is solution of IV for the section of P induced by* α_0.

4. UNIQUENESS OF THE BEST APPROXIMATION ELEMENTS.

It is well known that in any Banach space every element possesses at most best approximation with respect to any non-empty subset iff is strictly convex (see, for instance **Theorem 3.7** and **Remark 3.4** in **Chapter 3** of *Barbu Viorel, Precupanu, T., 1986*, [7]. Starting from the more useful definition of strict convexity for locally convex spaces introduced by *Huffman, E., 1977*, [39] and using our contributions contained in **Chapter 3** of *Isac, G., Postolică, V., 1993,* [45]; *Postolică, V., 1983*, [87], *1985*, [93], we reconsider here the basic notions and results concerning the uniqueness of the elements of best simultaneous and vectorial approximation.

Let X be a Hausdorff locally convex space whose topology is generated by a family $\mathrm{P} = \{p_\alpha : \alpha \in A\}$ of semi-norms and M a non - empty subset of X.

Definition 4.1. *(Huffman, E., 1977, [39], Isac, G., Postolică, V., 1993, [45], Postolică, V., 1985, [93]), Postolică, V., 1985). The set M is called* P*-simultaneous strictly convex if for every* $x, y \in M$, $x \neq y$, $\frac{1}{2}(x+y) \in M$ *and* $p_\alpha(x) = p_\alpha(y)$ *for all* $p_\alpha \in \mathrm{P}$, *we have* $p_\alpha\left(\frac{x+y}{2}\right) \leq p_\alpha(x)$, $\forall \alpha \in A$ *with at least one of these inequalities being strict.*

An *ample class* of *simultaneous strictly convex separated locally convex spaces* is represented by the *H - locally convex spaces* indicated in the next **Section 5**. As a generalization to locally convex spaces of the specified result in Banach spaces, the following theorem shows the strong relationship between the simultaneous strictly convexity and the uniqueness of the best simultaneous approximation elements.

Theorem 4.1. *(Isac, G., Postolică, V., 1993, [45]). A Hausdorff locally convex space is simultaneous strictly convex with respect to a family P of semi-norms which induces the locally convex topology if and only if every element of the space has at most one P-best simultaneous approximation in any non-trivial and closed segment.*

Even if the uniqueness of the best vectorial approximation elements seems to hold only in strong optimization programs, under proper conditions (one of the reason for this fact being the important connections with the general Efficiency), we give here an appropriate concept of strict convexity which achieves it.

Definition 4.2. *(Isac, G., Postolică, V., 1993, [45] ;Postolică, V., 1983, [87]). X is* P*-vectorial strictly convex if whenever $x, y \in X$, $x \neq y$,*

$$(p_\alpha(x)) \lhd (p_\alpha(y)) \lhd (p_\alpha(x))$$

we have

$$\left(p_\alpha\left(\frac{x+y}{2}\right)\right) < (p_\alpha(x)) \text{ or } \left(p_\alpha\left(\frac{x+y}{2}\right)\right) < (p_\alpha(y)).$$

Example 4.1 Let $X = \mathbb{R}$, $a, b \in (0, +\infty)$ and the family P$= \{p_1, p_2\}$of semi-norms defined on X by $p_1(x) = a|x|$ and $p_2(x) = b|x|$, $x \in X$. If $x, y \in X$, $x \neq y$ and$(p_1(x), p_2(x)) \lhd (p_1(y), p_2(y)) \lhd (p_1(x), p_2(x))$, then $x = -y \neq 0$. Consequently, X is P-vectorial strictly convex. It is obvious that every vectorial strictly convex Hausdorff locally convex space is simultaneous strictly convex with respect to the same family of semi - norms which generates the locally convex topology, but, in general, the converse is not true (**Example 2.2** in **Chapter 3** of *(Isac, G., Postolică, V., 1993)*, [45]). Unfortunately, the vectorial strictly convexity is only a sufficient condition ensuring the uniqueness of the vest vectorial approximation points in any non-empty and convex subset (**Theorem 2.1** and **Remark 2.3** in **Chapter 3** of *(Isac, G., Postolică, V., 1993)*, [45]). In our opinion, *the above announced uniqueness of the best vectorial approximation elements is unusual, this problem being well defined only for the best simultaneous approximation, by virtue of the previous reasons.*

5. BEST APPROXIMATION IN A SPECIAL KIND OF SEPARATED LOCALLY CONVEX SPACES: THE H - LOCALLY CONVEX SPACES BACKGROUND AND THE CORRESPONDING SPLINES

We conclude this research paper with topics on the best approximation in H-locally Convex Spaces. So, the concept of H - locally space was introduced for first time by *Precupanu, T., 1968*, [113], *1969*, [114], studied also by *Krammar, E., 1981*, [52] and defined as any Hausdorff locally convex space $(X, \mathrm{P} = \{p_\alpha : \alpha \in A\})$ with any semi-norms p_α satisfying the parallelogram law:

$$p_\alpha^2(x+y) + p_\alpha^2(x-y) = 2\left[p_\alpha^2(x) + p_\alpha^2(y)\right], \forall x, y \in X.$$

Every such as this space is simultaneous strictly convex (**Theorem 1.1** in *Isac, G., Postolică, V., 1993*, [45], *Postolică, V., 1983*, [87], and concerning the simultaneous proximinality, the following result is valid.

Theorem 5.1. *(Precupanu, A., Precupanu, T., 2002, [112]). A non - empty, convex and complete subset* G *of any H - locally convex space* $(X, \mathrm{P} = \{p_\alpha : \alpha \in A\})$ *is P-simultaneous proximinal if and only if for every* $x_0 \in X$*, there exists a net* $(g_j)_{j \in J}$ *such that* $\lim_{j \in J} p_\alpha (x_0 - g_j) = \inf_{g \in G} p_\alpha (x_0 - g)$, $\forall p_\alpha \in P$.

We introduced the concept of Spline Function in arbitrary H - locally convex spaces (*Postolică, V., 1981*,[86]), and we established the basic properties of Approximation and Optimal interpolation for these Splines, with the appropriate extensions (*Isac, G., Postolică, V., 1993*, [45], *Postolică, V., 1981*, [86], *1990*, [97], *1998*, [101], *2001*, [103], *2002*, [104], [105] and other connected works). Our splines are natural generalizations in the H - locally convex spaces of the usual Abstract Splines which appear in the Hilbert's Spaces like the minimizing elements for an arbitrary semi norm subject to the restrictions given by a set of linear and continuous functionals.

Let (X, P $=p_\alpha : \alpha \in A\}$) be a H - locally convex space with each semi - norm p_a being induced by a scalar semi - product (.,.) an $\alpha \in I$ and M a closed linear subspace of X for which there exists a H - locally convex space ($Y, \mathrm{Q} = \{q_\alpha : \alpha \in I\}$) with every semi - norm $q_\alpha \in$ Q generated by a scalar semi-product $< .,. >_\alpha$ an $\alpha \in I$ and a linear possibly continuous operator $U : X \to Y$ such that

$$M = \{x \in X : (x, y)_\alpha =< Ux, Uy >_\alpha, \forall \alpha \in I\}.$$

The space of spline functions with respect to U was defined in *(Postolică, V., 1981*, [86]) as the U - orthogonal of M, that is

$M^\perp$=$\{x \in X$: $< Ux, Uz >_\alpha = 0, \forall z \in M, \alpha \in I\}$.

Clearly, $M^\perp$ is the orthogonal of M in the H - locally convex sense, that is, $(x, y)_\alpha = 0, \forall x \in M,\ y \in M^\perp,\ \alpha \in I.$

Let us consider the direct sum X'=$M \oplus M^\perp$ and for every $x \in X'$**,** we denote its projection onto $M^\perp$ by s_x. Then, taking into account the **Theorem 4** in (*Postolică, V., 1981*, [86]), it follows that this spline is a P- *best simultaneous U - approximation* of x with respect to $M^\perp$ since it satisfies pa *(x - sx)* $\leq$ *pa(x - y)* $\forall$ $y \in M^\perp$, $p_\alpha \in$ P. Moreover, following the **Theorem 3.3** of the present research work, the results given in **Chapter 3** of (*Isac, G., Postolică, V., 1993*, [45]) and the Theorem 3 in (*Postolică, V., 1981*, [86]), we have

Theorem 5.2. *(i) for every* $x \in X'$ *the only elements of best simultaneous and vectorial approximation with respect to any family of seminorms which generates the H - locally convex topology on X by the linear subspace of splines are the spline functions*s_x*. Moreover, if M and* $M^\perp$ *supply an orthogonal decomposiion for X, that is X=*$M \oplus M^\perp$*, then* $M^\perp$ *is simultaneous and vectorial proximinal;*

(ii) if $K=\mathbb{R}_+^I$, *then for each* $s \in M^{\perp}$, *every* $\sigma \in M^{\perp}$ *is the only solution of following optimization problem* $MIN_K(\{(q_\alpha(U(\eta - s))): \eta \in X$ *and* $\eta - s \in M\})$;

(iii) for every $x \in X'$ *its spline function* s_x *is the only solution for the next vectorial optimization problems:* $MIN_K(\{(q_\alpha(U(\eta - x))) : \eta \in M^{\perp}\})$, $MIN_K(\{(p_\alpha(x - y)) : y \in M^{\perp}\})$ *and* $MIN_K(\{(q_\alpha(Uy)) : y - x \in M\})$.

Finally, let us consider three examples in which we specify the expressions of the corresponding splines, following the paragraph 3 in **Chapter 1** of *Isac, G., Postolică, V., 1993*, [45] and *Postolică, V., 1981*, [86],*2001*, [103] and with the main property that M and $M^{\perp}$ realize orthogonal decompositions for all of them.

Example 5.1. Let X be a metric space, $Y \subseteq X$ a separable subspace, $B(Y)$ the σ - algebra of all Borel subsets in Y and $\mu : B(Y) \to \mathbb{R}_+$ a measure. We denote *the Hilbert space* of all *countable additive set functions* $F : B(Y) \to R$ of *bounded 2 - variation* with respect to μ (*Postolică, V., 1990*, [97]) by $BV_2^C(Y, \mu)$ and we consider a (possible closed) linear subspace Y_0 of $L^2(Y)$.

Therefore, Y_0 and its orthogonal $Y_0^{\perp} = \left\{ f \in L^2(Y) : \int\limits_Y f \cdot g d\mu = 0, \forall g \in Y_0 \right\}$ realize an *orthogonal decomposition* of the space $L^2(Y)$. Hence, for every function $f \in L^2(Y)$ we have $f = f_0 + f_0^{\perp}$ with $f_0 \in Y_0$ and $f_0^{\perp} \in Y_0^{\perp}$. If one considers (*Isac, G., Postolică, V., 1993*, [98], *Postolică, V., 2001*, [103]) $BV_2^C(Y, \mu)$ endowed with the topology of Hilbert space generated by the scalar product $(.,.)$ defined by $(F, G) = \int\limits_Y f \cdot g d\mu$ for every $F, G \in BV_2^c(Y, \mu)$ given by $F(A) = \int\limits_A f d\mu$, $G(A) = \int\limits_A g d\mu$, $\forall A \in B(Y)$ with $f, g \in L^2(Y)$, $L^2(Y)$ endowed with the usual topology generated also by the scalar product $\langle f, g \rangle = \int\limits_Y f_0^{\perp} \cdot g_0^{\perp} d\mu + \int\limits_Y f \cdot g d\mu$, $\forall f = f_0 + f_0^{\perp}$, $g = g_0 + g_0^{\perp} \in L^2(Y)$, $\left(f_0, g_0 \in Y_0, f_0^{\perp}, g_0^{\perp} \in Y_0^{\perp}\right)$ and the *linear continuous operator* $U : BV_2^C(Y) \to L^2(Y)$ is defined by $U(F) = f$ for every $F \in BV_2^C(Y, \mu)$ with $F(A) = \int\limits_A f d\mu, \forall A \in B(Y)$, $f \in L^2(Y)$, then

Theorem 5.3 (*Postolică, V., 2000, [104]*).

(i) For every set function $F \in BV_2^C(Y, \mu)$, $F(A) = \int\limits_A f d\mu, \forall A \in B(Y)$, *with* $f \in L^2(Y)$, *we have* $F = F_0 + F_0^{\perp}$ *where* $F_0(A) = \int\limits_A f_0 d\mu$, $F_0^{\perp}(A) = \int\limits_A f_0^{\perp} d\mu$, $A \in B(Y)$, $f = f_0 + f_0^{\perp}$, $f_0 \in Y_0$, $f_0^{\perp} \in Y_0^{\perp}$:

(ii) $M = \left\{ F \in BV_2^C(Y) : F(A) = \int\limits_A f d\mu, f \in Y_0, A \in B(Y) \right\}$

$$M^{\perp} = \left\{ G \in BV_2^C(Y) : G(A) = \int_A g d\mu, g \in Y_0^{\perp}, A \in B(Y) \right\}$$

realize an orthogonal decomposition of the Hilbert space $BV_2^C(Y,\mu)$ *that is,* $BV_2^C(Y,\mu) = M \oplus M^{\perp}$.

(iii) $\int\limits_Y |f_0|^2 \, d\mu = \inf \left\{ \int\limits_Y |f-g|^2 \, d\mu : g \in Y_0^{\perp} \right\}, \forall f = f_0 + f_0^{\perp}$ *with* $f_0 \in Y_0$ *and* $f_0^{\perp} \in Y_0^{\perp}$.

The next examples are devoted to the real splines with countable sets of knots, as significant examples of the general splines in H - locally convex spaces and generalizations for the usual, finite dimensional, piecewise polynomial splines. Thus, following *Postolică, 1981*, [86] and *Isac, G., Postolică, V., 1993*, [45], we come back on some pertinent examples given in *Postolică, 1981,* [86], [87], with the main aim to indicate also the proper Orthogonal Decompositions for the important H - Locally Convex Spaces. We specify the effective expressions of the Splines obtained using our described method, with Applications for the Appropriate Numerical Investigations.

Example 5.2. Let $X = H^m(\mathbb{R}) =$
$= \{f \in C^{m-1}(R) : f^{(m-1)}$ is locally absolutely continuous and $f^{(m)} \in L^2_{loc}(\mathbb{R})\}$, $m \geq 1$ endowed with the H-locally convex topology generated by the scalar semi-products

$(x,y)_k = \sum_{h=0}^{m-1}[x(h)(k)y(h)(k) + x(h)(-k)y(h)(-k)] + \int_{-k}^{k} x(m)(t)y(m)(t)dt$, $k = 0,1,2,\dots$ and $Y = L_2^{loc}(\mathbb{R})$ with the H - locally convex topology induced by the scalar semi-products $< x, y >_k = \int_{-k}^{k} x(t)y(t)dt, k = 0,1,2,\dots$. If $U : X \to Y$ is the derivation operator of order m, then

$$M = \{x \in H^m(\mathbb{R}) : x^{(h)}(\nu) = 0,\ \forall h = 0,...,m-1,\ \nu \in \mathbb{Z}\}$$

and

$$M^{\perp} = \{H^m(\mathbb{R}) : \int_{-k}^{k} s^{(m)}(t)x^{(m)}(t)dt, \forall x \in M, k = 0,1,2,...\}.$$

We proved in *Postolică, 1981*, [86] that
$M^{\perp} = \{s \in H^m(\mathbb{R}) : s_{/(\nu,\nu+1)}$is a polynomial function of degree $2m-1$ at most$\}$ and if $y = (y_\nu)$, $y' = (y'_\nu)$, $y'' = (y''_\nu)$, $y^{m-1} = (y_\nu^{m-1})$, are $m \in \mathbb{N}^*$ sequences of real numbers, then there exists *an unique spline* $S \in M^{\perp}$ satisfying the following conditions of interpolation: $S^{(h)}(\nu) = y^{(h)}(\nu)$ whenever $h = \overline{0, m-1}$ and $\nu \in \mathbb{Z}$. Moreover, we observed in the paragraph 3 of *Isac, G., Postolică, V., 1993*, [45], that any spline of function S such as this is defined by $S(x) = p(x) +$

$\sum_{h=0}^{m-1} c_1^{(h)}(x-1)_+^{2m-1}+\sum_{h=0}^{m-1} c_2^{(h)}(x-2)_+^{2m-1}+\cdots+\sum_{h=0}^{m-1} c_0^{(h)}(-x)_+^{2m-1}+\ldots$ where $u_+ = (juj + u)/2$ for every real number u, p is a polynomial function of degree $2m$-1 at most perfectly determined by the conditions $p^{(h)}(0) = y_0^{(h)}$ and $p^{(h)}(1) = y_1^{(h)}$ for all h=$\overline{0, m-1}$ and the coefficients $c_\nu(h)(h = \overline{0, m-1}$, n $\in \mathbb{Z}$) are successively given by the interpolation.

Therefore, for every function $f \in H^m(\mathbb{R})$, there exists an unique function denoted by $Sf \in M^\perp$ such that $S_f^{(h)}(\nu) = f^{(h)}(\nu), \forall h = \overline{0, m-1}$ and $n \in \mathbb{Z}$. Hence M and $M^\perp$ give, in this case, an orthogonal decomposition for the space $H^m(\mathbb{R}$.

Example 5.3. Let $X = Fm = \{f \in C^{m-1}(\mathbb{R}) : f^{(m-1)}$ is locally absolutely continuous and *f(m)* $\in L^2(\mathbb{R})\}$ with the H - locally convex topology induced by the scalar semiproducts $(x, y)_\nu = x(\nu)y(\nu) + \int_{\mathbb{R}} x^{(m)}(t)y^{(m)}(t)dt, \nu \in \mathbb{Z}, Y = L^2(\mathbb{R})$ endowed following the topology generated by the inner product$(x, y)_\nu = \int_{\mathbb{R}} x(t)y(t)dt, n \in Z$ and *U:X*$\to Y$ be the derivation operator of order *m*. Then, *M*= $\{x \in F_m : x(n) = 0 foralln \in \mathbb{Z}\}$ and $M^\perp = \{s \in F_m : \int_{\mathbb{R}} x^{(m)}(t)y^{(m)}(t) = 0$ for every $x \in M\}$.

In a similar manner as in **Example 5.1,** it may be proved that $M^\perp$ coincides with the class of all *piecewise polynomial functions* of order $2m$ (degree $2m$ - 1 at most) having their knots at *the integer points.* Moreover, for every function f in Fm there exists an unique spline function $S_f \in M^\perp$ which interpolates f on the set $\mathbb{Z}$ of all integer numbers, that is, *Sf* satisfies the equalities *Sf(n)=f(n)* for every $n \in \mathbb{Z}$, being defined by

$S_f(x) = p(x)+a_1(x-1)_+^{2m-1}+a_2(x-2)_+^{2m-1}+\cdots+a_0(-x)_+^{2m-1}+a_{-1}(-x-1)_+^{2m-1}+\ldots$ where u_+ has the same signification as in **Example 5.1**, the coefficients $a_\nu(\nu \in \mathbb{Z})$ are successively and completely determined by the interpolation conditions $Sf(\nu) = f(\nu), \nu \in \mathbb{Z} \setminus \{0, 1\}$ and p is a polynomial function satisfying the conditions *p(0)=f(0)* and *p(1)=f(1).* The uniqueness of S_f is ensured by the **Theorem 2** in *Postolică, 1981,* [86]. Thus, *M* and $M^\perp$ give an orthogonal decomposition of the space F_m and, as in the preceding example, $M^\perp$ is *simultaneous* and *vectorial proximinal* with respect to the family of semi - norms generated by the above scalar semiproducts. Our examples show that the abstract construction of splines can be used to solve also several frequent problems of interpolation and approximation, having the possibility to choose the spaces and the scalar semi - products. It is obvious that for a given (closed) linear subspace of an arbitrary H - locally convex space X such a H - locally convex space Y (respectively, a linear (continuous) operator *U:X*$\to Y$) would not exist. Otherwise, the problem of best vectorial approximation by the corresponding orthogonal space of any (closed) linear subspace M for the elements in the direct sum $M \oplus M^\perp$might be always reduced to the best simultaneous approximation. But, in general, such a possibility doesn't exist.

Finally, we must mention *the excellent contribution* to the study of *the spline functions* and *their applications for finite numbers of interpolatory knots* given by *the distinguished* and *regretted professor Gheorghe Micula* at the *"Babeş – Bolyai" "University of Cluj – Napoca.* His main aim was to use these splines *to solve differential*, *integral* and *partial differential equations* (see, for instance, *Micula, Gh., 1978*,[59], *Micula, Gh., Micula, Sanda, 1999*,[60]). Clearly, this survey can be completed any time with additional scientific contributions in this beautiful field of *Mathematics* named *the Best Approximation and its Applications.*

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