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ON CHARACTERIZATION OF BESSEL SYSTEMS

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Abstract In some the previous work, we have studied an affine system of Walsh type generated by a periodic function in connection with a multishift in Hilbert space. In this paper, we give a new method for characterization of Bessel system. This method is based on the consideration of the question:under which necessary and sufficiently conditions on the function φ an affine system of functions of the Walsh type $\{\varphi_n\}_{n\geq 0}$ to be Bessel system in the space $L^2(0,1)$?Finally, some examples are given to explain our representation method.

Keywords: affine system of Walsh type, Rademacher system, Walsh-Paley system, Cuntz algebra Θ_n, Bessel system.

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1. INTRODUCTION

The new notion of affine system of Walsh type was introduced, studied and proved results about orthogonalizing and completion with preservation of structure of affine system by Terekhin P.A.[1]. Our results work in [2] about affine system of Walsh type can be classified in to three sections results:first, on the basis of the functional analytic structure of a multishift in a Hilbert space, which is a generalized analogue of the operator(simple, one-side)shift and closely related to the representations of the Cuntz C^*-algebra, the definition of an affine system of functions of the Walsh type was given, second, various criteria and signs of the completeness of affine systems of functions were given, finally, the minimality of the affine system is established as well as an explicit form of the biorthogonally conjugate system of functions was indicated and its completeness was established. Mironov V.A., Sarsenbi A.M., and Terekhin P.A.,[10] studied an affine Bessel sequences in connection with the spectral theory and the multishift structure in Hilbert space. They constructed a non-Besselian affine system $\{u_n(x)\}_{n=0}^{\infty}$ generated by continuous periodic function $u(x)$. Their results were based on Nikishin's example concerning convergence in measure, also they showed that affine systems $\{u_n(x)\}_{n=0}^{\infty}$ generated by any Lipchitz function $u(x)$ are Besselian.

Definition 1.1. *Let H be a Hilbert space, and*

$$W_0, W_1 : H \to H$$

isometric operators operating in space H. Let's say that the two isometrics W_0 and W_1 define the structure of multishifts, if there is a vector $e \in H$ such that:

$$W_{\alpha_1} \ldots W_{\alpha_{k-1}} e, \alpha_v \in \{0,1\}, 0 \leq v \leq k-1, k \geq 0,$$

forms an orthonormal basis of the space H.

Remark 1.1. *The concepts of multishift was introduced and studied in many works [3–5].*

Suppose that, the function $\varphi(s)$, $s \in \Re$, (where $\Re$ is a real number space), satisfied the condition:

$$\varphi(s) \in L^2[0,1], \int_0^1 \varphi(s)ds = 0, \varphi(s+1) = \varphi(s),$$

and let $L_0^2 = L_0^2(0,1)$ be a space such functions (where, L_0^2 is the space of square - integral and having a zero integral), as well as, we denote a linear operators in this space as:

$$W_0\varphi(s) = \varphi(2s), W_1\varphi(s) = r(s)\varphi(2s), \tag{1}$$

where $r(s)$ is the periodic function:Haar-Rademacher-Walsh.
For any $n \in N$, using the binary representation, $n = \sum_{v=0}^{k-1} \alpha_v 2^v + 2^k$ we set:

$$\varphi_n(t) = \varphi_\alpha(t) = \varphi_{kj}(t) = W^n\varphi(t) = W^\alpha\varphi(t) = W_{\alpha_1} \ldots W_{\alpha_k}\varphi(t),$$

where,
$k = 0, 1, ...; j = 0, 1, ..., 2^{k-1}, \alpha = (\alpha_1, \ldots, \alpha_k) \in \Omega, \Omega = \bigcup_{k=0}^{\infty} \{0,1\}^k$ Besides, we set $\varphi_0(t) \equiv 1$,

$$W_{\alpha_1} \ldots W_{\alpha_k},$$

denote the product of the operators:the operator W_{α_k} acts first, W_{α_1} acts last, and the empty product is set the equal to the identity operator I. For any function $\varphi \in L_0^2$, we have:

$$\varphi_\alpha(t) = W^\alpha\varphi(t) = W_{\alpha_0} \ldots W_{\alpha_{k-1}}\varphi(t) =$$

$$\varphi(2^k t) r^{\alpha_{k-1}}(2^{k-1}t) \ldots r^{\alpha_0}(t) = \varphi(2^k t) \prod_{v=0}^{k-1} r_v^{\alpha_v}(t),$$

where, $r_k(t) = r(2^k t), k = 0, 1, ...$ is Rademacher system .

Definition 1.2. *The system* $\{\varphi_n\}_{n\geq 0} = \{W^\alpha \varphi\}$ *is the affine system of Walsh type of the function* φ *without the constant* $\varphi_0(t) \equiv 1$.

If the generating function select $\omega(t) = r(t)$, then the system $\{\omega_n\}_{n=0}^{\infty}$ will the classical system of Walsh-Paley system. Walsh functions (without constant $\omega_0(t) \equiv 1$):

$$\omega_n(t) = \omega_\alpha(t) = W^\alpha \omega(t) = W_{\alpha_0} \dots W_{\alpha_{k-1}} \omega(t) = r_k(t) \prod_{v=0}^{k-1} r_v^{\alpha_v}(t),$$

forms an orthonormal basis of the space $H = L_0^2(0,1)$, therefore according to the definition(1.1) operators:

$$W_0 \varphi(t) = \varphi(2t), W_1 \varphi(t) = r(t) \varphi(2t),$$

define the structure of multishift[2].

Definition 1.3. [6]. *The Walsh-Paley system,* $\omega = (\omega_n, n \in N)$ *is defined as:if* $n = \sum_{k=0}^{\infty} n_k 2^k \in N \cup \{0\}$ *has binary coefficient* $(n_k, k \in N \cup \{0\})$*, then*

$$\omega_n = \prod_{k=0}^{\infty} r_k^{n_k}, \tag{2}$$

where,

$$r(x) = \begin{cases} 1, & x \in (0, 1/2) \\ -1, & x \in (1/2, 1) \end{cases}$$

$r(x+k) = r(x)$, $x \in (0,1)$, $k \in N$ *and* $r_k(x) = r(2^k x)$, $x \in \Re$, $k \in N \cup \{0\}$, *where* $r(x)$ *is the Rademacher functions.*

Definition 1.4. *A system (sequence)* $\{\varphi_n\}_{n\in N}$ *in Hilbert space* H *is called a Bessel system, if there exists a positive constant* B *for which*

$$\sum_{n=1}^{\infty} |(g, \varphi_n)|^2 \leq B \|g\|^2, \forall g \in H. \tag{3}$$

Definition 1.5. [7]. *Let* $n \geq 0$*, the Cuntz algebra* Θ_n *is the* C^**-algebra generated by some isometries* $(S_i)_{i\in Z_n}$ *satisfying the Cuntz relations:*

$$S_i^* S_j = \delta_{ij} I, \sum_{i \in Z_n} S_i^* S_j = I, \tag{4}$$

where, $i, j \in Z_n$.

It should be noted that the extensions of operators $\{W_0, W_1\}$ to the space $L^2(0,1)$ of periodic function $\varphi(t)$ are defined by:

$$V_0\varphi(t) = \varphi(2t), V_1\varphi(t) = r(t)\varphi(2t). \tag{5}$$

From equation(1.4), we have representation of the Cuntz algebra Θ_2, which satisfy the Cuntz relations:

$$V_i^* V_j = \delta_{ij} I,$$

$$V_0V_0^* + V_1V_1^* = I.$$

Thus, the operators structure of the multishift $\{W_0, W_1\}$ is a restriction to the subspace $L^2(0,1)$ of the representation $\{V_0, V_1\}$ in the space $L^2(0,1)$ of the Banach C^*-algebra of Cuntz Θ_2.

2. THE MAIN RESULTS WITH EXAMPLES

Lemma 2.1. *The system* $\left\{\varphi_{k,j}\right\}_{j=0}^{2^{k-1}}$ *(k-fixed) is orthogonal block system.*

Proof The system $\left\{\varphi_{k,j}\right\}_{j=0}^{2^{k-1}} = \{W^\alpha\varphi\}_{\alpha\in\Omega}$. This is implies to that:

$$W^\alpha\varphi \in W^\alpha H.$$

Since,

$$W^\alpha H \perp W^\beta H, \alpha \neq \beta, |\alpha| = |\beta| = k.$$

Also,

$$W^\alpha\varphi \in W^\alpha H,$$

$$W^\beta\varphi \in W^\beta H.$$

Then, we have:

$$\left(W^\alpha\varphi, W^\beta\varphi\right) = 0.$$

From above, we have that: $\{W^\alpha\varphi\}_{|\alpha|=k}$ is orthogonal block system.

Lemma 2.2. *For all* $\alpha, \beta \in \Omega$*, we have:*

$$\left(\omega_\alpha, \varphi_\beta\right) = \begin{cases} (\omega_\alpha, \varphi), & if \alpha = \beta\gamma \\ 0, & o.w. \end{cases}$$

Proof. Write the Fourier-Walsh series of the function φ as:

$$\varphi = \sum_{\gamma\in\Omega} (\varphi, \omega_\gamma)\, \omega_\gamma.$$

Also, we have:

$$\varphi_\beta = W^\beta \varphi = \sum_{\gamma\in\Omega} (\varphi, \omega_\gamma) W^\beta \omega_\gamma = \sum_{\gamma\in\Omega} (\varphi, \omega_\gamma) \omega_{\beta\gamma}.$$

On other hand

$$\varphi_\beta = \sum_{\gamma\in\Omega} \left(\varphi_\beta, \omega_\alpha\right) \omega_{\beta\alpha}, \alpha = \beta\gamma.$$

The coefficient of the Fourier-Walsh series are unique. Also, if $\alpha = \beta\gamma$ for some $\gamma \in \Omega$, then

$$\left(\varphi_\beta, \omega_\alpha\right) = (\varphi, \omega_\gamma).$$

It should be noted that, if α can not be expressed as $\beta\gamma, \forall\gamma \in \Omega$, then

$$\left(\varphi_\beta, \omega_\alpha\right) = 0.$$

∎

Theorem 2.1. *Let* $\varphi \in L^2(0,1)$,*supp*$\varphi \subset [0,1]$, $\int_0^1 \varphi(t)dt = 0$. *If the inequality:*

$$\sum_{k=0}^{\infty} \left(\sum_{j=0}^{2^{k-1}} |(\varphi, \omega_{kj})|^2 \right)^{1/2} = c < \infty.$$

Then the affine system of Walsh type $\{\varphi_n\}_{n\geq 0}$ *is Bessel system with Bessel constant* $B = max\{1, c\}^2$.

Proof. Write the Fourier-Walsh series of the function φ as:

$$\varphi = \sum_{\alpha\in\Omega} x_\alpha \omega_\alpha,$$

and write the polynomial of affine system $\{\varphi_n\}_{n\geq 1}$ finite sum as:

$$P = \sum_{\beta\in\Omega} c_\beta \varphi_\beta.$$

We consider for $k = 0, 1, \ldots$, the Walsh-Paley polynomials can be represented as:

$$P_k = \sum_{|\alpha|=k} x_\alpha \sum_{\beta\in\Omega} c_\beta \omega_{\beta\alpha}.$$

The system $\{\omega_{\beta\alpha} : |\alpha| = k(k - fixed), \beta \in \Omega\}$ is orthogonal system.

$$\omega_{\beta\alpha} = \omega_{\beta'\alpha'}, \left|\alpha'\right| = k, \beta'\alpha' \in \Omega, \alpha = \alpha', \beta = \beta', \beta' \in \Omega.$$

Now, if $\beta\alpha = \beta^{'}\alpha^{'}$, then:

$$|\alpha| + |\beta| = \left|\alpha^{'}\right| + \left|\beta^{'}\right|, |\alpha| = \left|\alpha^{'}\right| and\, |\beta| = \left|\beta^{'}\right|, \alpha = \alpha^{'} and \beta \ = \beta^{'}.$$

We can count:

$$\|P_k\| = \left(\sum_{|\alpha|=k,\beta\in\Omega} |x_\alpha c_\beta|^2\right)^{1/2} = \left(\sum_{|\alpha|=k} |x_\alpha|^2\right)^{1/2} \left(\sum_{\beta\in\Omega} |c_\beta|^2\right)^{1/2}.$$

And

$$\sum_{k=0}^{\infty} \|P_k\| = \left(\sum_{\beta\in\Omega} |c_\beta|^2\right)^{1/2} . \sum_{k=0}^{\infty} \left(\sum_{|\alpha|=k} |x_\alpha|^2\right)^{1/2} < \infty.$$

We calculate:

$$(P, \omega_\gamma) = \sum_{\beta\in\Omega} c_\beta \left(\varphi_\beta, \omega_\gamma\right) = \sum_{\alpha,\beta:\gamma=\beta\alpha} c_\beta \left(\varphi, \omega_\alpha\right) = \sum_{\alpha,\beta:\gamma=\beta\alpha} x_\alpha c_\beta,$$

(By using Lemma(2.2)).

$$\left(\sum_{k=0}^{\infty} P_k, \omega_\gamma\right) = \sum_{k=0}^{\infty} (P_k, \omega_\gamma) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} x_\alpha \sum_{\beta\in\Omega} c_\beta \left(\omega_{\beta\alpha}, \omega_\gamma\right) = \sum_{\alpha,\beta:\gamma=\beta\alpha} x_\alpha c_\beta.$$

Since, $(\omega_{\beta\alpha}, \omega_\gamma) = \delta_{\beta\alpha,\gamma}$.
From the above, we have the following induction:$P = \sum_{k=0}^{\infty} P_k$!
Now:

$$\|P\| \leq \sum_{k=0}^{\infty} \|P_k\| = \sum_{k=0}^{\infty} \left(\sum_{|\alpha|=k} |x_\alpha|^2\right)^{1/2} \left(\sum_{\beta\in\Omega} |c_\beta|^2\right)^{1/2},$$

we have:

$$\left\|\sum_{\beta\in\Omega} c_\beta \varphi_\beta\right\| \leq \|\varphi\|^* \left(\sum_{\beta\in\Omega} |c_\beta|^2\right)^{1/2},$$

where, $\|\varphi\|^* = \sum_{k=0}^{\infty} \left(\sum_{|\alpha|=k} |x_\alpha|^2\right)^{1/2}$.
It is equivalent to Bessel inequality:

$$\left(\sum_{\beta\in\Omega} \left|(g, \varphi_\beta)\right|^2\right)^{1/2} \leq \|\varphi\|^* \|g\|,$$

$$\left(\sum_{k=0}^{\infty}|(g,\varphi_n)|^2\right)^{1/2} \leq \left(\left(\int_0^1 g(t)dt\right)^2 + \sum_{\beta\in\Omega}\left|(g,\varphi_\beta)\right|^2\right)^{1/2},$$

$$\leq max\left\{1, \|\varphi\|^*\right\}.\|g\|^2,$$

$$\sum_{n=0}^{\infty}|(g,\varphi_n)|^2 \leq B\|g\|^2, B = max\left\{1, \|\varphi\|^*\right\}^2.$$

Then, we have:if

$$\|\varphi\|^* = \sum_{k=0}^{\infty}\left(\sum_{|\alpha|=k}|x_\alpha|^2\right)^{1/2} < \infty.$$

Then the affine system of Walsh type $\{\varphi_n\}_{n\geq 0}$ is Bessel system. ∎

Remark 2.1. *Theorem (2.1) in this paper is an analog of some results obtained by the authors in [8–10].*

We are going to give some examples to apply theorem (2.1). These examples are based on consideration that:H^∞ is the Banach algebra of analytic functions on the open unit disk and $G(H^\infty)$ is the group of invertible elements of the algebra H^∞. Note that for ζ to be belong to $G(H^\infty)$, it is necessary and sufficient that the function $\zeta(z)$ be analytic on the disk $(|z| < 1)$ and that the following inequalities be valid:

$$0 < inf\,|\zeta(z)|\,, sup\,|\zeta(z)| < \infty.$$

Let $\varphi \in R(H)$, where $R(H)$ is the space of Rademacher. Let $R(H) = \overline{span}[r_k]$be linear closure of the span Rademacher system $\{r_k\}_{k=0}^{\infty}$. The space $R(H)$ invariant with respect to W_0 and the multishift operator $R(H)$ as:

$$r_k = W_0^k r, k = 0, 1, \ldots.$$

And

$$\varphi(t) = \sum_{k=0}^{\infty} a_k r_k, \sum_{k=0}^{\infty}|a_k|^2 < \infty. \tag{6}$$

We assign the analytic function:

$$\phi(z) = \sum_{k=0}^{\infty} a_k z^k. \tag{7}$$

In the unite disk $D = (|z| < 1)$ of Hardy space $H^2(D)$, with the coefficient a_k from equation(2.6). This mapping is an isometric isomorphism of $R(H)$ on to

Hardy space $H^2(D)$, and the restriction of W_0 to $R(H)$ is unitary equivalent by this mapping to the operator of multiplication by z, i.e. is a shift operator.

Theorem 2.2. [11].*Let $\{\omega_n\}_{n\geq 0}$ be the Walsh system, $\{r_k\}_{k\geq 0}$ be the Rademacher system and*

$$\varphi = \sum_{k=0}^{\infty} a_k r_k, \sum_{k=0}^{\infty} |a_k|^2 < \infty.$$

If the analytic function

$$\phi(z) = \sum_{k=0}^{\infty} a_k z^k, |z| < 1,$$

belong to $G(H^\infty)$, then the affine system of Walsh type $\{\varphi_n\}_{n\geq 0}$ is Riesz bases in $L^2(0,1)$.

Example 2.1. *The analytic function*

$$\phi(z) = \sum_{k=0}^{\infty} a_k z^k, |z| < 1,$$

has no zero in the closed unit circle, then affine system of Walsh type $\{\varphi_n\}_{n\geq 0}$forms Riesz bases and since any Riesz bases is Bessel system, then affine system of Walsh type $\{\varphi_n\}_{n\geq 0}$forms Bessel system too. Indeed of Wiener theorem an absolutely convergent series Tayler follows that $\phi \in G(H^\infty)$.

Example 2.2. *Let $\varphi(t) = 1 - 2t$, $0 < t < 1$ and satisfy the following condition $\int_0^1 \varphi(t)dt = 0$. $\varphi \in R(H)$, this meaning that, the function φ can be representation as Rademacher system:*

$$\varphi = \sum_{k=0}^{\infty} a_k r_k = \sum_{k=0}^{\infty} \frac{r_k}{2^{k+1}}.$$

Then the corresponding analytic function as:

$$\phi(z) = \sum_{k=0}^{\infty} a_k z^k = \sum_{k=0}^{\infty} \frac{z^k}{2^{k+1}} = \frac{1}{2-z}.$$

It is observe that, $\phi \in G(H^\infty)$, then, we have the affine system of Walsh type $\{\varphi_n\}_{n\geq 0}$ forms Riesz bases and Bessel system.

Theorem 2.3. *Let $\{w_n\}_{n\geq 0}$ be the Walsh system, $\{r_n\}_{n=0}^{\infty}$ be the Rademacher system and $\varphi \in L_0^2, \varphi = \sum\limits_{k=0}^{\infty} a_k r_k$, $\sum\limits_{k=0}^{\infty} |a_k|^2 < \infty$. Then, affine system of Walsh* *type* $\{\varphi_n\}_{n\geq 1}$ *is Riesz bases iff*

$$0 < c_1 \leq |\phi(z)| \leq c_2 < \infty,$$

where,

$$\phi(z) = \sum_{k=0}^{\infty} a_k z^k, |z| < 1,$$

is analytic function.

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