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A CASE-STUDY OF OPTIMAL PORTFOLIO COMPOSITION WITH INVESTMENTS LIMITS

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Abstract A portfolio composition, where the weights are supposed to depend on the investor's risk tolerance, is considered. The portfolio's certainty equivalent return has to be maximized. The mathematical model leads to a convex problem. The attached Kuhn-Tucker system is solved using the critical lines method.

Keywords: Kuhn-Tucker system, portfolio composition, critical lines.
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1. INTRODUCTION

Starting with the pioneering work of Markowitz from 1952 [9], the portfolio theory was studied by many researchers. Among those with recent contributions in this field we quote Jacobs, Levy and Markowitz [6], Bailey and López de Prado [1], Kwan [8], Norstad [11], Cumova, Moreno and Nawrocki [3], Marling and Emanuelsson [10], Calvo, Ivorra and Liern [2], Kan and Zhou [7].

The aim of our study is to illustrate how to compose an efficient portfolio with positive weights and investments limits for five companies.

Consider P_t the closing price of a stock at the end of time t and P_{t-1} the closing price of a stock at the end of earlier time $t-1$. It follows that $P_t - P_{t-1}$ is the price return of the stock at time t

The *discretely compounded rate of return* in the period $(t-1, t)$ is:

- $R_t = \frac{P_t - P_{t-1}}{P_{t-1}} = \frac{P_t}{P_{t-1}} - 1$, if the stock has not paid dividend;
- $R_t = \frac{P_t + D_t}{P_{t-1}} - 1$, if the stock paid dividend D_t

The *continuously compounded rate of return* in the period $(t-1, t)$ is defined as:

- $R_t = \ln \frac{P_t}{P_{t-1}}$, if the stock has not paid dividend;
- $R_t = \ln \frac{P_t + D_t}{P_{t-1}}$, if the stock paid dividend D_t.

The discretely compounded rate of return is slightly larger than the continuously compounded return rate. In the case when it is assumed that historical returns denote the distribution of the returns for the coming period, continuously compounded return is more appropriate.

In our study for experimental purpose five firms from New York Stock Exchange (NYSE) were selected, namely, Applied Materials (AMAT), Amazon (AMZN), Alibaba Group Holding Limited (BABA), Advanced Micro Devices, Inc. (AMD) and AT&T (T).

Data were downloaded from http://finance.yahoo.com searching the historical data for every company. They covered the period between February 2015 and April 2016. Monthly adjusted price are used. The observations were examined using Excel and Stata. The continuously compounded rates of return for the five firms were computed. They are given in Table 1.

Date	AMAT	AMZN	BABA	AMD	T
02.05.2016	0.1016	0.0635	0.0238	0.0863	-0.0096
01.04.2016	-0.0341	0.1053	-0.0268	0.2196	0.0033
01.03.2016	0.1155	0.0718	0.1385	0.2865	0.0583
01.02.2016	0.0722	-0.0605	0.0262	-0.0277	0.0244
04.01.2016	-0.0562	-0.1410	-0.1926	-0.2659	0.0608
01.12.2015	-0.0053	0.0165	-0.0340	0.1957	0.0217
02.11.2015	0.1183	0.0603	0.0030	0.1072	0.0048
01.10.2015	0.1324	0.2011	0.3518	0.2091	0.0424
01.09.2015	-0.0910	-0.0020	-0.1144	-0.0510	-0.0189
03.08.2015	-0.0701	-0.0444	-0.1696	-0.0642	-0.0453
01.07.2015	-0.1018	0.2112	-0.0489	-0.2179	-0.0090
01.06.2015	-0.0463	0.0113	-0.0822	0.0513	0.0280
01.05.2015	0.0220	0.0175	0.0942	0.0088	-0.0029
01.04.2015	-0.1310	0.1253	-0.0237	-0.1705	0.0734
02.03.2015	-0.1047	-0.0214	-0.0223	-0.1488	-0.0569
02.02.2015	0.0965	0.0698	-0.0455	0.1907	0.0486

Table 1. Continuously compounded rates of return.

Consider N stocks and let $R_{i,t}$ be the stock i's return rate at time $t, t = 1, \ldots, T$ and $i = 1, \ldots, N$. We recall some well-known formulae from statistics:

- The *mean* of periodic returns percentages for the stock i is

$$\overline{R}_i = \frac{1}{T}\sum_{t=1}^{T} R_{i,t}.$$

- The *variance* of the return rates for the company i is

$$\sigma_i^2 = \frac{1}{T}\sum_{t=1}^{T}(R_{i,t} - \overline{R}_i)^2,$$

while the s*ample variance*, used in our model (and denoted in the same way), is

$$\sigma_i^2 = \frac{1}{T-1}\sum_{t=1}^{T}(R_{i,t} - \overline{R}_i)^2.$$

- The *standard deviation* is $\sigma_i = \sqrt{\sigma_i^2}$.
- The *covariance* of the stocks i and j return rates is

$$cov_{ij} = \frac{1}{T-1}\sum_{t=1}^{T}(R_{i,t} - \overline{R}_i)(R_{j,t} - \overline{R}_j).$$

- The *correlation coefficient* of the stocks i and j return rates is $\rho_{ij} = \frac{cov_{ij}}{\sigma_i \sigma_j}$.

Some numerical characteristics of the return rates for the five firms considered by us are given in Table 2.

	AMAT	AMZN	BABA	AMD	T
Mean	0.0011	0.0428	-0.0077	0.0256	0.0140
Standard Error	0.0231	0.0230	0.0320	0.0427	0.0094
Median	-0.0197	0.0389	-0.0253	0.0301	0.0133
Standard Deviation	0.0925	0.0920	0.1279	0.1706	0.0376
Sample Variance	0.0086	0.0085	0.0163	0.0291	0.0014
Kurtosis	-1.6342	0.2295	3.4787	-1.1154	-0.5939
Skewness	0.1495	0.1194	1.3969	-0.1826	-0.2301
Range	0.2634	0.3522	0.5444	0.5524	0.1303
Minimum	-0.1310	-0.1410	-0.1926	-0.2659	-0.0569
Maximum	0.1324	0.2112	0.3518	0.2865	0.0734
Sum	0.0181	0.6843	-0.1227	0.4093	0.2233
Count	16	16	16	16	16

Table 2. Summary statistics.

Using the above formula for the covariance of the stocks i and j return rates, the covariance matrix of the stocks return rates for our case-study is obtained as follows:

	AMAT	AMZN	BABA	AMD	T
AMAT	0.008	0.0014	0.0071	0.0107	0.001
AMZN	0.0014	0.0079	0.0064	0.0047	0.0005
BABA	0.0071	0.0064	0.0153	0.0112	0.0011
AMD	0.0107	0.0047	0.0112	0.0273	0.0013
T	0.001	0.0005	0.0011	0.0013	0.0013

Let x_i be the weight of security i included in the portfolio, $i = 1, \ldots, N$. Thus, if S is the total amount of money which will be invested, then $x_i S$ will be invested in firm i. Obviously, $\sum\limits_{i=1}^{N} x_i = 1$.

The *portfolio expected return* is defined as:

$$E\left(R_p\right) = \sum_{i=1}^{N} x_i E\left(R_i\right),$$

where $E\left(R_i\right)$ is the expected return value of security i.

The *portfolio variance* is defined as:

$$\sigma_p^2\left(x\right) = \sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j cov_{ij}.$$

In our case-study we consider that the expected return of security i is the mean of periodic returns percentages, thus the portfolio expected return reads:

$$E\left(R_p\right) = 0.0011x_1 + 0.0428x_2 - 0.0077x_3 + 0.0256x_4 + 0.014x_5.$$

The portfolio variance reads:

$$\begin{aligned}\sigma_p^2\left(x\right) \;=\;& 0.008x_1^2 + 0.0079x_2^2 + 0.0153x_3^2 + 0.0273x_4^2 + 0.0013x_5^2 \\ &+0.0028x_1x_2 + 0.0142x_1x_3 + 0.0214x_1x_4 + 0.002x_1x_5 + 0.0128x_2x_3 \\ &+0.0094x_2x_4 + 0.001x_2x_5 + 0.0224x_3x_4 + 0.0022x_3x_5 + 0.0026x_4x_5.\end{aligned}$$

A simple computation shows that $\sigma_p^2(x)$ is a positive definite quadratic form, so it is a convex function.

2. THE MATHEMATICAL MODEL FOR OPTIMAL PORTFOLIO SELECTION WITH POSITIVE WEIGHTS AND INVESTMENTS LIMITS

Consider $r > 0$ the parameter that quantifies the investor's risk tolerance.

As the portfolio's certainly equivalent return is $E(R_p) - \frac{\sigma_p^2(x)}{r}$, we intend to minimize $\sigma_p^2(x) - rE(R_p)$. In addition, we suppose that the weight x_i is limited by c_i. Thus, the problem can be written as:

$$\begin{aligned} (\min) f(x) &= \sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j cov_{ij} - r\sum_{i=1}^{N} x_i E(R_i) \\ \sum_{i=1}^{N} x_i &= 1 \\ 0 &\leq x_i \leq c_i,\ i = 1, ..., N. \end{aligned}$$

As the function f is the sum between a positive definite quadratic form and a linear function, it follows that f is a convex function. The constraints are linear, so we have a convex differentiable programming problem, which can be solved using the Kuhn-Tucker theory.

The associated Lagrange-type function L reads:

$$L = \sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j cov_{ij} - r\sum_{i=1}^{N} x_i E(R_i) - \theta\left(\sum_{i=1}^{N} x_i - 1\right) - \sum_{i=1}^{N}\delta_i x_i + \sum_{i=1}^{N}\eta_i(x_i - c_i),$$

where $\theta \in R$ and $\delta_i \geq 0$, $\eta_i \geq 0$, $i = 1, ..., N$.

Thus, the Kuhn-Tucker system reads:

$$\begin{cases} \frac{\partial L}{\partial x_i} = 0,\ i = 1, ..., N \\ \sum\limits_{i=1}^{N} x_i = 1 \\ x_i \geq 0,\ i = 1, ..., N \\ x_i \leq c_i,\ i = 1, ..., N \\ \delta_i x_i = 0,\ i = 1, ..., N \\ \eta_i(x_i - c_i) = 0,\ i = 1, ..., N. \end{cases}$$

For our study $N = 5$ and we consider that all investments limits are $ci = 0.6$, $i = 1, \ldots, 5$. Thus, the Lagrange function is:

$$\begin{aligned} L &= 0.008x_1^2 + 0.0079x_2^2 + 0.0153x_3^2 + 0.0273x_4^2 + 0.0013x_5^2 + 0.0028x_1x_2 \\ &\quad + 0.0142x_1x_3 + 0.0214x_1x_4 + 0.002x_1x_5 + 0.0128x_2x_3 + 0.0094x_2x_4 \\ &\quad + 0.001x_2x_5 + 0.0224x_3x_4 + 0.0022x_3x_5 + 0.0026x_4x_5 \\ &\quad - r\,(0.0011x_1 + 0.0428x_2 - 0.0077x_3 + 0.0256x_4 + 0.014x_5) \\ &\quad - \theta\,(x_1 + x_2 + x_3 + x_4 + x_5 - 1) - \sum_{i=1}^{5} \delta_i x_i + \sum_{i=1}^{5} \eta_i (x_i - 0.6). \end{aligned}$$

The Kuhn-Tucker system reads:

$$\begin{cases} 0.016x_1 + 0.0028x_2 + 0.0142x_3 + 0.0214x_4 \\ \qquad +0.002x_5 - 0.0011r - \theta - \delta_1 + \eta_1 = 0, \\ 0.0028x_1 + 0.0158x_2 + 0.0128x_3 + 0.0094x_4 \\ \qquad +0.001x_5 - 0.0428r - \theta - \delta_2 + \eta_2 = 0, \\ 0.0142x_1 + 0.0128x_2 + 0.0306x_3 + 0.0224x_4 \\ \qquad +0.0022x_5 + 0.0077r - \theta - \delta_3 + \eta_3 = 0, \\ 0.0214x_1 + 0.0094x_2 + 0.0224x_3 + 0.0546x_4 \\ \qquad +0.0026x_5 - 0.0256r - \theta - \delta_4 + \eta_4 = 0, \\ 0.002x_1 + 0.001x_2 + 0.0022x_3 + 0.0026x_4 \\ \qquad +0.0026x_5 - 0.014r - \theta - \delta_5 + \eta_5 = 0, \\ x_1 + x_2 + x_3 + x_4 + x_5 = 1, \\ 0 \le x_i \le 0.6,\ i = 1, ..., 5 \\ \delta_i x_i = 0,\ i = 1, ..., 5 \\ \eta_i (x_i - 0.6) = 0,\ i = 1, ..., 5. \end{cases}$$

with $\theta \in R$ and $\delta_i \ge 0$, $\eta_i \ge 0$, $i = 1, ..., 5$.

This system consists both of equations and inequations and have 16 unknown variables, namely x_i, δ_i, η_i, $i = 1, \ldots, 5$ and θ. Obviously, the solutions depend on the values of the parameter r.

Remark that the status of any security can be *in*, *out* or *up*. More precisely:

- The status is *in*, when $0 < x_i < c_i = 0.6$. In this case, $\delta_i = 0$ and $\eta_i = 0$.
- The status is *out*, when $x_i = 0$. In this case $\eta_i = 0$.
- The status is *up*, when $x_i = c_i = 0.6$. In this case, $\delta_i = 0$.

In order to solve the Kuhn-Tucker system for different values of the investor's risk tolerance r, we follow the lines in Kwan [8]. Thus, the initial portfolio consists of those securities which have the highest expected return and we follow the steps 1-6 below.

Step 1. The initial portfolio consists of AMZN and AMD, which have the highest expected return. Thus $x_2 = 0.6$, $x_4 = 0.4$, so AMZN is *up*, while AMD is *in*. It follows $\delta_2 = \delta_4 = 0$ and $\eta_4 = 0$. As AMAT, BABA and T are *out*, we get $x_1 = x_3 = x_5 = 0$, so $\eta_1 = \eta_3 = \eta_5 = 0$. Thus, the first 6 equations of the Kuhn-Tucker system have the solution:

$$\begin{aligned} &\delta_1 = -0.01724 + 0.0245r, \quad \delta_3 = -0.01084 + 0.0333r, \\ &\delta_5 = -0.02584 + 0.0116r, \quad \eta_2 = 0.01424 + 0.0172r, \\ &\theta = 0.02748 - 0.0256r. \end{aligned}$$

From the conditions $\delta_i \geq 0$, $i = 1, 3, 5$, it follows $r \geq 2.224$. The critical value $r = 2.224$ is obtained when $\delta_5 = 0$. As $r < 2.224$, δ_5 would become negative. But $\delta_5 \geq 0$, so δ_5 must be chosen zero at the next step.

Step 2. As $r < 2.224$, $\delta_5 = 0$, so $x_5 > 0$ and T must changes its status from *out* to *in*. In addition, it follows $\eta_5 = 0$. AMZN remains *up*, thus, $x_2 = 0.6$ and $\delta_2 = 0$. AMD remains *in*, so $\delta_4 = \eta_4 = 0$. AMAT and BABA are *out*, thus $x_1 = x_3 = 0$ and $\eta_1 = \eta_3 = 0$. The Kuhn-Tucker system has the solution:

$$\begin{aligned} &\delta_1 = -0.001 + 0.017r, \quad \delta_3 = 0.005 + 0.0262r, \\ &\eta_2 = -0.0074 + 0.0269r, \quad x_4 = -0.0969 + 0.223r, \\ &x_5 = 0.4969 - 0.223r, \quad \theta = 0.00164 - 0.014r. \end{aligned}$$

From the conditions $\delta_1 \geq 0, \eta_2 \geq 0$ and $x_4, x_5 \in [0, 0.6]$, it follows $r \geq 0.43$. Thus, the solutions from Step 2 are valid for $r \in [0.43, 2.224]$. The critical value $r = 0.43$ is obtained for $x_4 = 0$, so at the next step AMD must change the status.

Step 3. As $r < 0.43, x_4 = 0$, so AMD is now *out* and $\eta_4 = 0$. AMZN remains *up*, so $x_2 = 0.6$ and and $\delta_2 = 0$. T is *in*, while AMAT and BABA remain *out*. Thus, $x_1 = x_3 = 0$, $\delta_5 = 0$, $\eta_1 = \eta_3 = \eta_5 = 0$, and the solution is:

$$\begin{aligned} &\delta_1 = 0.00084 + 0.0129r, \quad \delta_3 = 0.00692 + 0.0217r, \\ &\delta_4 = 0.00504 - 0.0116r, \quad \eta_2 = -0.00824 + 0.0288r, \\ &x_5 = 0.4, \quad \theta = 0.00164 - 0.014r. \end{aligned}$$

From the conditions $\delta_4 \geq 0, \eta_2 \geq 0$, it follows $r \geq 0.286$. Thus, the solutions from Step 3 are valid for $r \in [0.286, 0.43]$.

Step 4. As $r < 0.286$, $\eta_2 = 0$, thus AMZN must change its status from *up* to *in*, so $0 < x_2 < 0.6$. It follows $\delta_2 = \eta_2 = 0$. As T remains *in*, $\delta_5 = \eta_5 = 0$. AMAT, BABA and AMD remain *out*, so it follows $x_1 = x_3 = x_4 = 0$ and

$\eta_1 = \eta_3 = \eta_4 = 0$. The Kuhn-Tucker system has the solution:

$$\delta_1 = -0.00036 + 0.0171r, \quad \delta_3 = 0. + 0.04312r,$$
$$\delta_4 = 0.0008 + 0.0031r, \quad x_2 = 0.0975 + 1.756r,$$
$$x_5 = 0.9024 - 1.7561r, \quad \theta = 0.0024 - 0.0168r.$$

From the conditions $\delta_1 \geq 0, x_2, x_5 \in [0, 0.6]$, it follows $r \geq 0.171$. As the critical value 0.171 is obtained when $x_5 = 0.6$, at the next step T must be *up*. The solution at Step 4 holds for $r \in [0.171, 0.286]$.

Step 5. As $r < 0.171$, T change its status from *in* to *up*, so $x_5 = 0.6$. Thus, $\delta_5 = 0$. AMZN remains *in*, so $\delta_2 = \eta_2 = 0$. AMAT, BABA and AMD remain *out*, so $x_1 = x_3 = x_4 = 0$ and $\eta_1 = \eta_3 = \eta_4 = 0$. The Kuhn-Tucker system has the solution:

$$\delta_1 = -0.0046 + 0.0417r, \quad \delta_3 = -0.00048 + 0.0505r,$$
$$\delta_4 = -0.0016 + 0.0172r, \quad \eta_5 = 0.00496 - 0.0288r,$$
$$x_2 = 0.4, \quad \theta = 0.00692 - 0.0428r.$$

From the constraints it follows that this solution holds for $r \geq 0.11$. As $r = 0.11$, the conditin $\delta_1 \geq 0$ is violated, so at the next step δ_1 must be zero, thus $x_1 > 0$.

Step 6. As $r < 0.11$, AMAT change its status from *out* to *in*. Thus, $\delta_1 = \eta_1 = 0$. T remains *up*, so $x_5 = 0.6$ and $\delta_5 = 0$. AMZN remains *in*, so $\delta_2 = \eta_2 = 0$. BABA and AMD remain *out*, so $x_3 = x_4 = 0$ and $\eta_3 = \eta_4 = 0$. The Kuhn-Tucker system has the solution:

$$\delta_3 = 0.002 + 0.0275r, \quad \delta_4 = 0.0027 - 0.0226r,$$
$$\eta_5 = 0.0025 - 0.0065r, \quad x_1 = 0.1755 - 1.5916r,$$
$$x_2 = 0.2244 + 1.5916r, \quad \theta = 0.0046 - 0.0221r.$$

This solution holds for $0 < r < 0.11$.

3. RESULTS AND COMMENTS

Synthesizing the results from the above section, the optimal portfolio composition for different values of the risk tolerance r is given in Table 3.

Risk tolerance	AMAT	AMZN	BABA	AMD	T
$r \geq 2.224$	0	0.6	0	0.4	0
$0.43 \leq r < 2.224$	0	0.6	0	$-0.0969 + 0.223r$	$0.4969 - 0.223r$
$0.286 \leq r < 0.43$	0	0.6	0	0	0.4
$0.171 \leq r < 0.286$	0	$0.0975 + 1.756r$	0	0	$0.9024 - 1.7561r$
$0.11 \leq r < 0.171$	0	0.4	0	0	0.6
$r < 0.11$	$0.1755 - 1.5916r$	$0.2244 + 1.5916r$	0	0	0.6

Table 3. Optimal weights depending on risk tolerance.

As the investor's risk tolerance r takes its critical values, the corner portfolios are obtained.

They are presented in Table 4, together with the corresponding portfolio expected return. As expected, for big risk tolerance, the expected return is also big.

Critical values of risk tolerance	AMAT	AMZN	BABA	AMD	T	Portfolio expected return
$r = 2.224$	0	0.6	0	0.4	0	0.03592
$r = 0.43$	0	0.6	0	0	0.4	0.03128
$r = 0.286$	0	0.6	0	0	0.4	0.03128
$r = 0.171$	0	0.4	0	0	0.6	0.02552
$r = 0.11$	0	0.4	0	0	0.6	0.02552
$r \to 0$	0.175	0.225	0	0	0.6	0.01822

Table 4. Corner portfolios.

In Figures 1-4, the optimal investment weights for AMAT, AMZN, AMD and T are represented as functions of the risk tolerance using [14]. In these figures, the dots correspond to the corner portfolios.

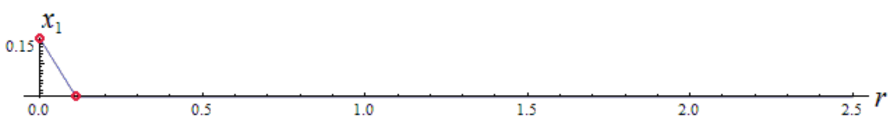

Fig. 1. Optimal AMAT investment weight as function of risk tolerance.

If the parameter r is eliminated between different optimal weights, the critical lines are obtained. Some of these critical lines are represented in Figures 5 and 6, namely those from the (x_5, x_4) and (x_5, x_2) planes. As in the previous figures, the dots correspond to the corner portfolios.

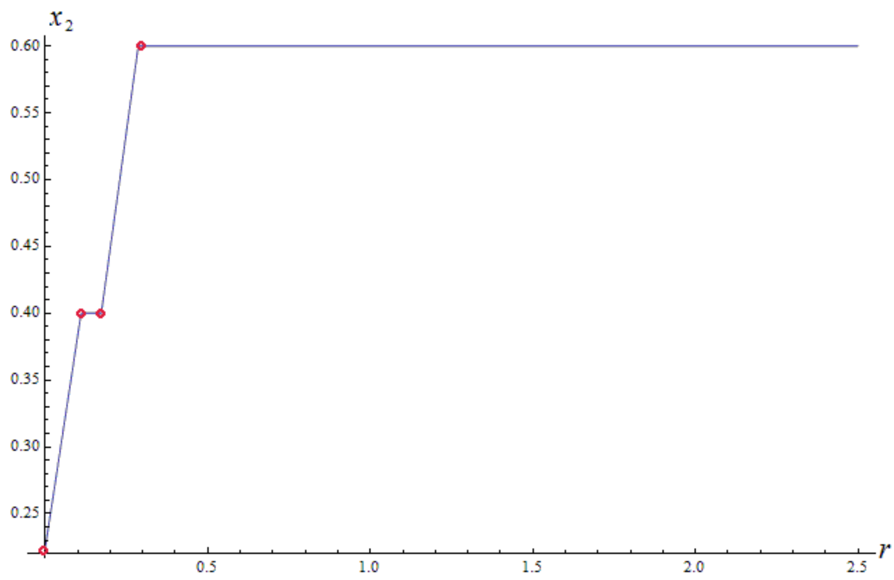

Fig. 2. Optimal AMZN investment weight as function of risk tolerance.

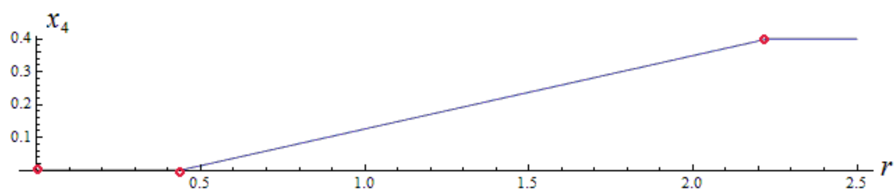

Fig. 3. Optimal AMD investment weight as function of risk tolerance.

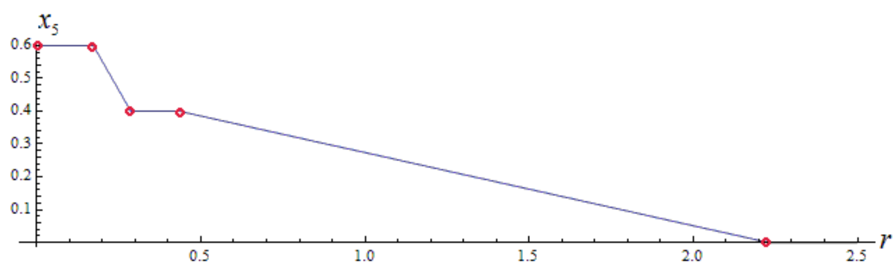

Fig. 4. Optimal T investment weight as function of risk tolerance.

Two diagrams illustrating the optimal portfolio composition for big and small risk tolerance are represented in Figures 7 and 8 respectively.

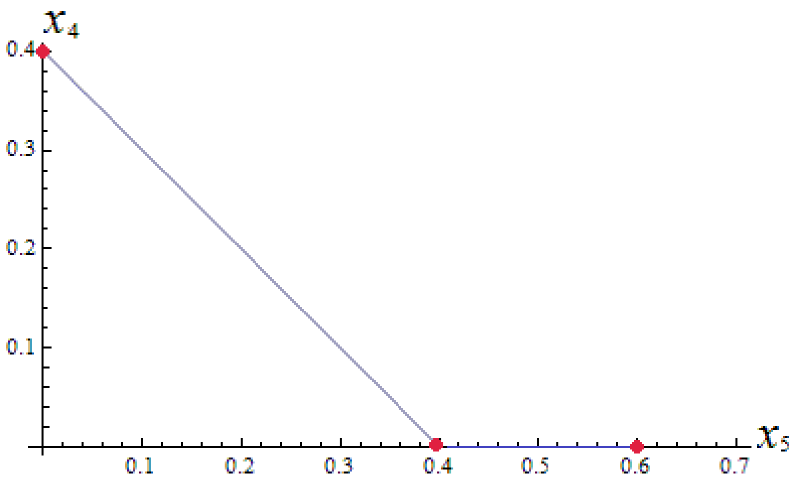

Fig. 5. Critical lines in the (x_5, x_4) -plane.

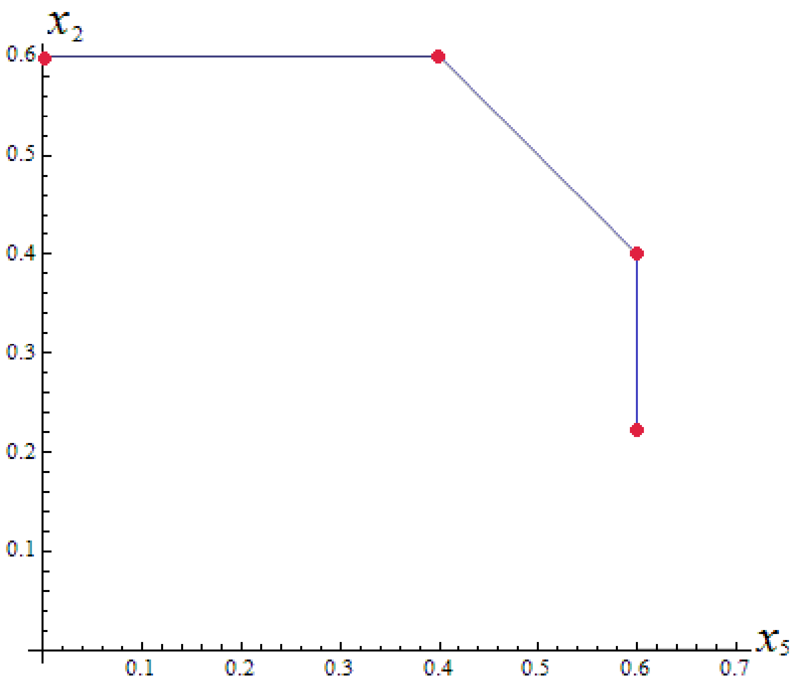

Fig. 6. Critical lines in the (x_5, x_2)-plane.

Finally, the variation of the portfolio expected return with respect to the risk tolerance is given in Table 5.

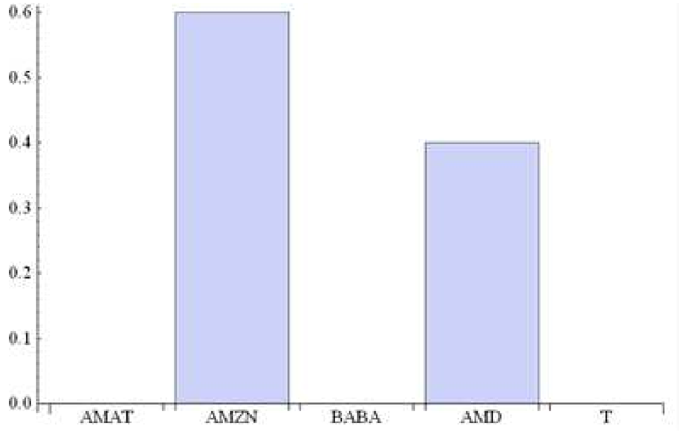

Fig. 7. Optimal portfolio composition for big risk tolerance.

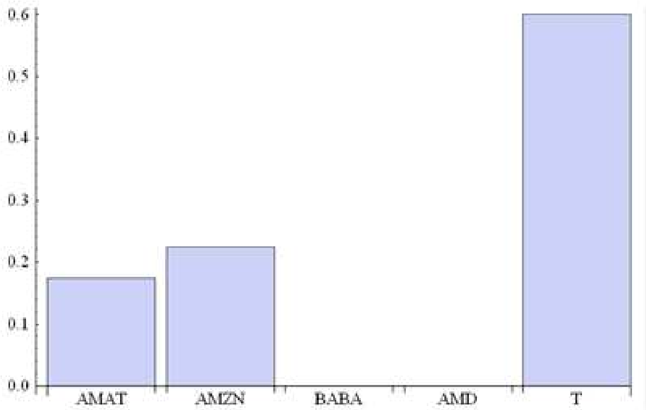

Fig. 8. Optimal portfolio composition for small risk tolerance ($r \to 0$).

Risk tolerance	Portfolio expected return
$0 < r < 0.11$	$0.01819 + 0.06636r$
$0.11 \leq r < 0.171$	0.02552
$0.171 \leq r < 0.286$	$0.01680 + 0.05057r$
$0.286 \leq r < 0.43$	0.03128
$0.43 \leq r < 2.224$	$0.03015 + 0.00258r$
$r \geq 2.224$	0.03592

Table 5. Portfolio expected return depending on risk tolerance.

The portfolio expected return as function of the risk tolerance is represented in Figure 9 as a polygonal line, where the dots correspond to the corner portfolios.

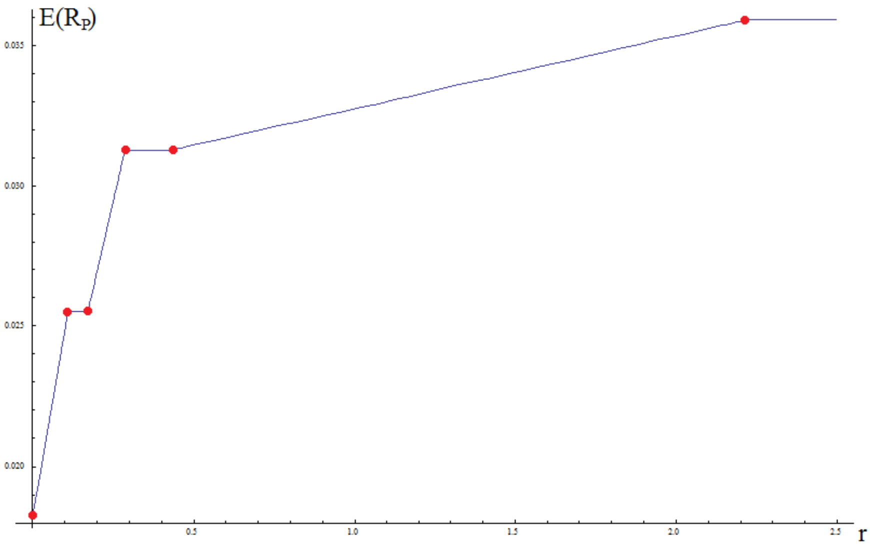

Fig. 9. The portfolio expected return as function of the risk tolerance.

4. CONCLUSIONS

For our case-study, 5 companies were chosen, namely AMAT, AMZN, BABA, AMD and T. We considered that the expected return of every security is the mean of periodic returns percentages obtained between February 2015 and April 2016. Of course, in real life, many other factors, such as the news concerning the firms or the changes of their management strategy influence these expected returns. Thus, our study is just a theoretical one. The aim was to find the optimal portfolio composition with positive weights and investments for every company limited at 60%, for different values of the investor's risk tolerance. We obtained a convex problem, which was solved using the Kuhn-Tucker theory. We found 5 critical values of the risk tolerance and the optimal weights as functions of this tolerance. In our conditions, no investments have to be made in BABA. For big risk tolerance, the optimal portfolio contains AMZN and AMD, while for small risk tolerance, T, AMZN and AMAT have to be selected.

References

[1] H.D. Bailey, M. López de Prado, *An Open-Source Implementation of the Critical-Line Algorithm for Portfolio Optimization*, Algorithms, **6**(2013), 169–196.

[2] C. Calvo, C. Ivorra, V Liern, *On the Computation of the Efficient Frontier of the Portfolio Selection Problem*, Journal of Applied Mathematics, **2012**(2012), Article ID 105616, 25 p.

[3] D. Cumova, D. Moreno, D. Nawrocki, *The Critical Line Algorithm for UPM-LPM Parametric General Asset Allocation Problem with Allocation Boundaries and Linear Constraints*, 2004. Retrieved from http://www90.homepage.villanova.edu/michael.pagano/DN

[4] F. Deari, P.V. Hudym, *Construction of Dynamic Investment Portfolio Management Model*, International Scientific Journal, Economic Sciences, Kiev, **4** (2015), 5–14.

[5] F. Deari, *Portfolio Composition: A Methodological Solution Using Lagrange Multiplier*, ICESOS'15, Sarajevo, 2015.

[6] B. Jacobs, K. Levy, H. Markowitz, *Portfolio Optimization with Factors, Scenarios, and Realistic Short Positions*, Operations Research, **53**(2005), No. 4, 586–599.

[7] R. Kan, GF Zhou, *Tests of Mean-Variance Spanning*, Annals of Economics and Finance, **13**-1(2012), 145–193.

[8] C.C. Kwan, *A Simple Spreadsheet-Based Exposition of the Markowitz Critical Line Method for Portfolio Selection*, Spreadsheets in Education (eJSiE), **2**(2007), No. 3, 30 p.

[9] H. Markowitz, *Portfolio Selection*, Journal of Finance, **7**(1952), No. 1, 77–91.

[10] H. Marling, S. Emanuelsson, *The Markowitz Portfolio Theory*, 2012. Retrieved from http://www.math.chalmers.se/ rootzen/finrisk/gr1_HannesMarling_Sara Emanuelsson_MPT.pdf.

[11] J. Norstad, *Portfolio Optimization, Part 2 Constrained Portfolios*, 2011. Retrieved from http://www.norstad.org/finance/portopt2.pdf

[12] M. Rubinstein, *Markowitz's Portfolio Selection: A Fifty-Year Retrospective*, Journal of Finance, **LVII**(2002), No. 3, 1041–1045.

[13] F.W. Sharpe, P. Chen, E.J. Pinto, W.D. McLeavey, *Asset Allocation*, In: Maginn, L. J., Tuttle, L. D., McLeavey, W. D., Pinto, E. J., Managing Investment Portfolios: A Dynamic Process, CFA Institute, USA, John Wiley & Sons Inc., 2007.

[14] Wolfram Research Inc. *Wolfram Mathematica 9*, 2012.