ROMAI J., v.14, no.2(2018), 67–87

# ON THE ASYMPTOTIC STRUCTURE OF THE STABILIZING SOLUTION OF A CLASS OF SINGULARLY PERTURBED RICCATI EQUATION OF STOCHASTIC CONTROL

Vasile Drăgan[1], Ioan-Lucian Popa[2], Samir Aberkane[3]

[1] *Institute of Mathematics "Simion Stoilow" of the Romanian Academy Bucharest, Romania and Academy of the Romanian Scientists, Romania*

[2] *Department of Exact Sciences and Engineering, "1 Decembrie 1918" University of Alba Iulia, Alba Iulia, Romania*

[3] *Université de Lorraine, CRAN, UMR 7039, Campus Sciences, Vandoeuvre-les-Nancy Cedex, France,*

[3] *CNRS, CRAN, UMR 7039, France*

Vasile.Dragan@imar.ro; lucian.popa@uab.ro; samir.aberkane@univ-lorraine.fr

**Abstract** An optimal control problem for a system of linear Itô differential equations with two fast time scales and Markovian jumping is considered. The asymptotic structure for the stabilizing solution satisfying a sign condition for the coupled stochastic algebraic Riccati equations is derived. Furthermore, a near optimal control whose gain matrices do not depend upon small parameters is discussed.

**Keywords:** singularly perturbed linear stochastic systems, Markovian jumping parameters, optimal control problem, Riccati equations of stochastic control.

**2010 MSC:** 93E20, 93B12, 34H15, 49N10.

## 1. INTRODUCTION

Systems described by singularly perturbed differential equations have been the subject of intensive study during the last fifty years. Recall here that a singularly perturbed system of differential equations contains small parameters $\epsilon$ as coefficients of the derivatives of some unknown functions of the system. Usually, in the deterministic case, such small parameters are neglected, thus one may associate two subsystems of lower dimensions which are independent of the small parameters, namely the fast subsystem (boundary layer subsystem) and the reduced subsystem (the slow subsystem). Some properties of the solutions of the original system are then deduced/approximated from the properties of the corresponding subsystems independent from the small parameters $\epsilon$. The interested reader can refer to the textbooks [2, 8, 12] and the references therein.

It is important to point out here that in the stochastic framework, such a *simple* neglection of the small parameter $\epsilon$ doesn't generally lead to satisfactory solutions. Hence, problems related to singularly perturbed stochastic systems could not be viewed as simple extensions of there deterministic counterparts. This makes the study of this class of systems a challenging (and relatively not fully investigated) topic. The problem of exponential stability for a class of singularly perturbed stochastic systems has been addressed in [9, 3] (for the linear case) and [17, 16] (for the nonlinear case). Linear-quadratic type control problems are addressed for instance in [5, 14] as well as $\mathcal{H}_\infty$-type control problems in [4].

The aforementioned works, both in deterministic and stochastic cases, have in common that they all consider the case with only one fast time scale. Very few results have been reported in the literature dealing with several fast time scales. We cite here [15] for the deterministic case and [6] for the stochastic framework. Pursuing our efforts in the study of singularly perturbed stochastic systems, we consider in this paper a stochastic optimal control problem described by a quadratic performance criterion and a linear controlled system modeled by a system of singularly perturbed Itô differential equations with two fast time scales and Markovian jumping. Our goal in this work is to analyse the asymptotic structure with respect to the small parameters $\varepsilon_j > 0$, $j = 1, 2$ associated to the two fast time scales of the stabilizing solution of the matrix Riccati equation associated to the optimal control problem under consideration. The results derived in this stochastic framework cannot be obtained mutatis-mutandis from the already existing ones in the deterministic case, as those from [1]. The knowledge of the asymptotic structure of the stabilizing solution of the Riccati equation allows us to avoid the ill conditioning of the numerical computations required for obtaining the gain matrix of the optimal control. Also, the analysis performed in this work may be used for the design of a near optimal control for many practical applications in which the values of the small parameters are not precisely known.

The remainder of the paper is organized as follows. In the next section we formulate the considered problem. The main results of the paper are presented in in Section 3 and we conclude the paper in Section 4. Due to page limitations, proofs of main results have not been included here.

## 2. THE PROBLEM

Let us consider the optimal control problem described by the controlled system

$$\varepsilon_k dx_k(t) = \left[\sum_{j=0}^{2} A_{kj}(\eta_t)x_j(t) + B_k(\eta_t)u(t)\right] dt + \varepsilon_k^{\delta}\left[\sum_{j=0}^{2} C_{kj}(\eta_t)x_j(t) + D_k(\eta_t)u(t)\right] dw(t)$$

$$x_k(0) = x_{k0}, \qquad k \in \{0, 1, 2\} \tag{1}$$

and the quadratic cost functional

$$J(x_0, u) = \mathbb{E}\int_0^{\infty}\left[\sum_{k,j=0}^{2} x_k^T(t)M_{kj}(\eta_t)x_j(t) + 2\sum_{j=0}^{2} x_j^T(t)L_j(\eta_t)u(t) + u^T(t)R(\eta_t)u(t)\right] dt \tag{2}$$

$M_{kj}(\eta_t) = M_{jk}^T(\eta_t)$, $0 \le j \le k \le 2$, $R(\eta_t) = R^T(\eta_t)$, where $x_j(t) \in \mathbb{R}^{n_j}$, $0 \le j \le 2$, are the state vectors and $u(t) \in \mathbb{R}^m$ are the control parameters; $\{w(t)\}_{t\ge 0}$ is a one dimensional standard Brownian motion defined on a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and $\{\eta_t\}_{t\ge 0}$ is a standard homogeneous Markov process taking values in the set $\mathfrak{N} = \{1, 2, ..., N\}$ and having the transition semigroup $P(t) = e^{Qt}$ with the generator matrix $Q = (q_{ij})$ whose elements satisfy $q_{ij} \ge 0$ if $i \ne j$ and $\sum_{k=1}^{N} q_{ik} = 0$ for all $i, j \in \mathfrak{N}$. Throughout the work $\{w(t)\}_{t\ge 0}$ and $\{\eta_t\}_{t\ge 0}$ are independent stochastic processes. $\mathbb{E}$ stands for the mathematical expectation.
In (1) $\varepsilon_0 = 1$ and for $k \ge 1$, $\varepsilon_k : [0, \varepsilon^*] \to [0, \infty)$ satisfy

$$\lim_{\varepsilon\to 0} \varepsilon_k(\varepsilon)/\varepsilon_{k-1}(\varepsilon) = 0. \tag{2}$$

We also assume that in (1), $\delta > \frac{1}{2}$. The class of admissible controls $\mathcal{U}_{adm}(x_0)$ consists of all measurable stochastic processes $u = \{u(t)\}_{t\ge 0}$ which are adapted to the filtration generated by the stochastic processes $\{w(t)\}_{t\ge 0}$ and $\{\eta_t\}_{t\ge 0}$ having the properties:

(a) $\displaystyle\int_0^{\infty} \mathbb{E}[|u(t)|^2]dt < \infty$

(b) $J(x_0, u) < \infty$

(c)

$$\lim_{t\to\infty} \sum_{j=0}^{2} \mathbb{E}[|x_j(t; x_0, u)|^2 | \eta_0 = i] = 0 \tag{4}$$

where $x_j(t; x_0, u)$, $0 \le j \le 2$ is the solution of the problem with given initial value (1) determined by the input $u$ and $\mathbb{E}[\cdot|\eta_0 = i]$ is the conditional expectation with respect to the event $\{\eta_0 = i\}$.

The linear quadratic optimization problem (LQOP) which we want to solve requires to finding a control $\tilde{u} \in \mathfrak{U}_{adm}(x_0)$ with the property that

$$J(x_0, \tilde{u}) = \min_{u \in \mathfrak{U}_{adm}(x_0)} J(x_0, u).$$

## 3. MAIN RESULTS

We shall write $A_{jk}(i), B_k(i), ...$ instead of $A_{jk}(\eta_t), B_k(\eta_t), ...$ every time when $\eta_t = i \in \mathfrak{N}$. We introduce the notations:

$$\begin{gathered}
\mathbb{A}(i) = \begin{pmatrix} A_{00}(i) & A_{01}(i) & A_{02}(i) \\ \frac{1}{\varepsilon_1} A_{10}(i) & \frac{1}{\varepsilon_1} A_{11}(i) & \frac{1}{\varepsilon_1} A_{12}(i) \\ \frac{1}{\varepsilon_2} A_{20}(i) & \frac{1}{\varepsilon_2} A_{21}(i) & \frac{1}{\varepsilon_2} A_{22}(i) \end{pmatrix} \\
\mathbb{C}(i) = \begin{pmatrix} C_{00}(i) & C_{01}(i) & C_{02}(i) \\ \varepsilon_1^{\delta-1} C_{10}(i) & \varepsilon_1^{\delta-1} C_{11}(i) & \varepsilon_1^{\delta-1} C_{12}(i) \\ \varepsilon_2^{\delta-1} C_{20}(i) & \varepsilon_2^{\delta-1} C_{21}(i) & \varepsilon_2^{\delta-1} C_{22}(i) \end{pmatrix} \\
\mathbb{B}(i) = \begin{pmatrix} B_0^T(i) & \frac{1}{\varepsilon_1} B_1^T(i) & \frac{1}{\varepsilon_2} B_2^T(i) \end{pmatrix}^T \in \mathbb{R}^{n \times m} \\
\mathbb{D}(i) = \begin{pmatrix} D_0^T(i) & \varepsilon_1^{\delta-1} D_1^T(i) & \varepsilon_2^{\delta-1} D_2^T(i) \end{pmatrix}^T \in \mathbb{R}^{n \times m} \\
\mathbb{M}(i) = \begin{pmatrix} M_{00}(i) & M_{01}(i) & M_{02}(i) \\ M_{01}^T(i) & M_{11}(i) & M_{12}(i) \\ M_{02}^T(i) & M_{12}^T(i) & M_{22}(i) \end{pmatrix} \\
\mathbb{L}(i) = \begin{pmatrix} L_0^T(i) & L_1^T(i) & L_2^T(i) \end{pmatrix}^T \in \mathbb{R}^{n \times m}
\end{gathered} \tag{4}$$

with $n = n_0 + n_1 + n_2$. With these notations (1) and (2) may be written in a compact form as follows:

$$dx(t) = (\mathbb{A}(\eta_t)x(t) + \mathbb{B}(\eta_t)u(t))dt + (\mathbb{C}(\eta_t)x(t) + \mathbb{D}(\eta_t)u(t))dw(t) \tag{5}$$

$$J(x_0, u) = \mathbb{E} \int_0^\infty [x^T(t)\mathbb{M}(\eta_t)x(t) + 2x^T(t)\mathbb{L}(\eta_t)u(t) + u^T(t)R(\eta_t)u(t)]dt \tag{6}$$

where $x(t) = \begin{pmatrix} x_0^T(t) & x_1^T(t) & x_2^T(t) \end{pmatrix}^T$, $x_0 = \begin{pmatrix} x_{00}^T & x_{10}^T & x_{20}^T \end{pmatrix}^T$. From (6) and (7) one sees that for each fixed value of $\varepsilon > 0$ the optimal control problem stated before is a standard stochastic LQ problem which was investigated in [11] and for a more general settings see also Chapter 6 from [7].
Applying the results derived in the afore mentioned references, it follows that the optimal control in the optimization problem described by (6) and (7) and the class of admissible controls $\mathcal{U}_{adm}(x_0)$ is

$$\tilde{u}(t) = \tilde{\mathbb{F}}(\eta_t)\tilde{x}(t) \tag{7}$$

where

$$\tilde{\mathbb{F}}(i) = -(R(i) + \mathbb{D}^T(i)\tilde{X}(i)\mathbb{D}(i))^{-1}(\mathbb{B}^T(i)\tilde{X}(i) + \mathbb{D}^T(i)\tilde{X}(i)\mathbb{C}(i) + \mathbb{L}^T(i)) \tag{8}$$

$\tilde{\mathbb{X}} = (\tilde{X}(1), ..., \tilde{X}(N))$ is the unique stabilizing solution of the algebraic Riccati equation of stochastic control SARE

$$\begin{aligned} &\mathbb{A}^T(i)X(i) + X(i)\mathbb{A}(i) + \mathbb{C}^T(i)X(i)\mathbb{C}(i) + \mathbb{M}(i) - \\ &- (X(i)\mathbb{B}(i) + \mathbb{C}^T(i)X(i)\mathbb{D}(i) + \mathbb{L}(i))(R(i) + \mathbb{D}^T(i)X(i)\mathbb{D}(i))^{-1} \times \\ &(X(i)\mathbb{B}(i) + \mathbb{C}^T(i)X(i)\mathbb{D}(i) + \mathbb{L}(i))^T + \sum_{j=1}^{N} q_{ij}X(j) = 0 \end{aligned} \tag{10}$$

satisfying the sign conditions

$$R(i) + \mathbb{D}^T(i)X(i)\mathbb{D}(i) > 0, \qquad 1 \le i \le N. \tag{10}$$

In [11] was proposed a method based on a semidefinite programming (SDP) for numerical computation of the stabilizing solution of SARE which satisfies (11), while in [7] was proposed an iterative procedure for computing of this solution.
From (5) one sees that the presence of the small parameters $\varepsilon_k$ in the coefficients of the system (6) favors the appearance of the stiffness phenomenon which leads to ill conditioning of numerical computations performed to obtain the stabilizing solution $\tilde{\mathbb{X}}$ in (10). On the other hand, in many applications the precise value of some of the small parameters $\varepsilon_k$ involved in the mathematical model of the regulated phenomenon are not known. That is why, in the case of Riccati equations of type (10) with the structure of the coefficients given in (5), is preferable to be done a detailed study of the dependence of the stabilizing solution $\tilde{\mathbb{X}}$ of (10) with respect to the small parameters $\varepsilon_k$, $1 \le k \le 2$.
The goal of this work is to perform such an investigation of the asymptotic behaviors of the stabilizing solution $\tilde{\mathbb{X}}$ of SARE (10) satisfying the sign conditions (11) when the small parameters $\varepsilon_k$ satisfy the condition (3). This study allows us to point out the dominant part of the stabilizing solution $\tilde{\mathbb{X}}$ as well

as the dominant part of the optimal feedback gain $\tilde{\mathbb{F}}$ given in (9). Based on the dominant part of the optimal feedback gain, we shall construct a suboptimal control $u_{app}(t) = F_{app}(\eta_t)x(t)$ which stabilizes the given system (1) and achieves a near optimal value $J(x_0, u_{app})$ of the cost (2). We shall provide an estimation of the deviation of this suboptimal value of the cost from the optimal value $J(x_0, \tilde{u})$.
In the deterministic case, that is $C_{kj}(i) = 0$, $D_k(i) = 0$, $0 \leq k, j \leq 2$, $i \in \mathfrak{N}$, an analogous study was done in [2], [8], [10] and [12] for the case of system with only one fast time scale and in [1] respectively [15] for case with several fast time scales.

## 3.1. THE DERIVATION OF THE REDUCED COUPLED ALGEBRAIC RICCATI EQUATIONS

Following the approach from [1] we shall investigate the asymptotic behavior of the solution $(\mathbb{X}, \mathbb{F})$ of the following Lurie-Yakubovich-Popov type system of stochastic control:

$$\begin{aligned} &\mathbb{B}^T(i)X(i) + \mathbb{D}^T(i)X(i)\mathbb{C}(i) + \mathbb{L}^T(i) = -(R(i) + \mathbb{D}^T(i)X(i)\mathbb{D}(i))\mathbb{F}(i) \\ &\mathbb{A}^T(i)X(i) + X(i)\mathbb{A}(i) + \mathbb{C}^T(i)X(i)\mathbb{C}(i) + \mathbb{M}(i) - \\ &- \mathbb{F}^T(i)(R(i) + \mathbb{D}^T(i)X(i)\mathbb{D}(i))\mathbb{F}(i) + \sum_{j=1}^{N} q_{ij}X(j) = 0. \end{aligned} \tag{12}$$

Proceeding in this way, we obtain simultaneously both the asymptotic structure of the stabilizing solution of SARE (10)-(11) as well as the asymptotic structure of the optimal stabilizing feedback gain (9).
We take

$$X(i) = \begin{pmatrix} X_{00}(i) & \varepsilon_1 X_{01}(i) & \varepsilon_2 X_{02}(i) \\ \varepsilon_1 X_{01}^T(i) & \varepsilon_1 X_{11}(i) & \varepsilon_2 X_{12}(i) \\ \varepsilon_2 X_{02}^T(i) & \varepsilon_2 X_{12}^T(i) & \varepsilon_2 X_{22}(i) \end{pmatrix}$$

and $\mathbb{F}(i) = (F_0(i) \quad F_1(i) \quad F_2(i))$, $X_{kj}(i) \in \mathbb{R}^{n_k \times n_j}$, $X_{jj}(i) = X_{jj}^T(i)$, $F_j(i) \in \mathbb{R}^{m \times n_j}$, $\quad k, j = 0, 1, 2$.
Using (5) we obtain the following partition of the system (12):

$$\begin{aligned} &B_0^T(i)X_{00}(i) + B_1^T(i)X_{01}^T(i) + B_2^T(i)X_{02}^T(i) + D_0^T(i)(X_{00}(i)C_{00}(i) \\ &+\varepsilon_1^{\delta}X_{01}(i)C_{10}(i) + +\varepsilon_2^{\delta}X_{02}(i)C_{20}(i)) + D_1^T(i)(\varepsilon_1^{\delta}X_{01}^T(i)C_{00}(i) + \varepsilon_1^{2\delta-1}X_{11}(i)C_{10}(i) \\ &+(\frac{\varepsilon_2}{\varepsilon_1})^{\delta}\varepsilon_1^{2\delta-1}X_{12}(i)C_{20}(i)) + D_2^T(i)(\varepsilon_2^{\delta}X_{02}^T(i)C_{00}(i) + (\frac{\varepsilon_2}{\varepsilon_1})^{\delta}\varepsilon_1^{2\delta-1}X_{12}^T(i)C_{10}(i) \\ &+\varepsilon_2^{2\delta-1}X_{22}(i)C_{20}(i)) + L_0^T(i) = -(R(i) + \mathbb{D}^T(i)X(i)\mathbb{D}(i))F_0(i) \end{aligned}$$

$$\varepsilon_1 B_0^T(i)X_{01}(i)+B_1^T(i)X_{11}(i)+B_2^T(i)X_{12}^T(i)+D_0^T(i)(X_{00}(i)C_{01}(i)$$
$$+\varepsilon_1^{\delta}X_{01}(i)C_{11}(i)+\varepsilon_2^{\delta}X_{02}(i)C_{21}(i))+D_1^T(i)(\varepsilon_1^{\delta}X_{01}^T(i)C_{01}(i)+\varepsilon_1^{2\delta-1}X_{11}(i)C_{11}(i)$$
$$+(\frac{\varepsilon_2}{\varepsilon_1})^{\delta}\varepsilon_1^{2\delta-1}X_{12}(i)C_{21}(i))+D_2^T(i)(\varepsilon_2^{\delta}X_{02}^T(i)C_{01}(i)+(\frac{\varepsilon_2}{\varepsilon_1})^{\delta}\varepsilon_1^{2\delta-1}X_{12}^T(i)C_{11}(i)$$
$$+\varepsilon_2^{2\delta-1}X_{22}(i)C_{21}(i))+L_1^T(i)=-(R(i)+\mathbb{D}^T(i)X(i)\mathbb{D}(i))F_1(i)$$

$$\varepsilon_2 B_0^T(i)X_{02}(i)+(\frac{\varepsilon_2}{\varepsilon_1})B_1^T(i)X_{12}(i)+B_2^T(i)X_{22}(i)+D_0^T(i)(X_{00}(i)C_{02}(i)$$
$$+\varepsilon_1^{\delta}X_{01}(i)C_{12}(i)+ +\varepsilon_2^{\delta}X_{02}(i)C_{22}(i))+D_1^T(i)(\varepsilon_1^{\delta}X_{01}^T(i)C_{02}(i)+\varepsilon_1^{2\delta-1}X_{11}(i)C_{12}(i)$$
$$+(\frac{\varepsilon_2}{\varepsilon_1})^{\delta}\varepsilon_1^{2\delta-1}X_{12}(i)C_{22}(i))+ +D_2^T(i)(\varepsilon_2^{\delta}X_{02}^T(i)C_{02}(i)+(\frac{\varepsilon_2}{\varepsilon_1})^{\delta}\varepsilon_1^{2\delta-1}X_{12}^T(i)C_{12}(i)$$
$$+\varepsilon_2^{2\delta-1}X_{22}(i)C_{22}(i))+L_2^T(i)=-(R(i)+\mathbb{D}^T(i)X(i)\mathbb{D}(i))F_2(i)$$

$$A_{00}^T(i)X_{00}(i)+A_{10}^T(i)X_{01}^T(i)+A_{20}^T(i)X_{02}^T(i)+X_{00}(i)A_{00}(i)+X_{01}(i)A_{10}(i)$$
$$+X_{02}(i)A_{20}(i)+[\mathbb{C}^T(i)X(i)\mathbb{C}(i)]_{00}-F_0^T(i)(R(i)$$
$$+\mathbb{D}^T(i)X(i)\mathbb{D}(i))F_0(i)+M_{00}(i)+\sum_{j=1}^{N}q_{ij}X_{00}(j)=0$$

$$\varepsilon_1 A_{00}^T(i)X_{01}(i)+A_{10}^T(i)X_{11}(i)+A_{20}^T(i)X_{12}^T(i)+X_{00}(i)A_{01}(i)+X_{01}(i)A_{11}(i)$$
$$+X_{02}(i)A_{21}(i)+[\mathbb{C}^T(i)X(i)\mathbb{C}(i)]_{01}-F_0^T(i)(R(i)$$
$$+\mathbb{D}^T(i)X(i)\mathbb{D}(i))F_1(i)+M_{01}(i)+\varepsilon_1\sum_{j=1}^{N}q_{ij}X_{01}(j)=0$$

$$\varepsilon_2 A_{00}^T(i)X_{02}(i)+\frac{\varepsilon_2}{\varepsilon_1}A_{10}^T(i)X_{12}(i)+A_{20}^T(i)X_{22}(i)+X_{00}(i)A_{02}(i)$$
$$+X_{01}(i)A_{12}(i)+X_{02}(i)A_{22}(i)+[\mathbb{C}^T(i)X(i)\mathbb{C}(i)]_{02}-F_0^T(i)(R(i)$$
$$+\mathbb{D}^T(i)X(i)\mathbb{D}(i))F_2(i)+M_{02}(i)+\varepsilon_2\sum_{j=1}^{N}q_{ij}X_{02}(j)=0$$

$$\varepsilon_1 A_{01}^T(i)X_{01}(i)+A_{11}^T(i)X_{11}(i)+A_{21}^T(i)X_{12}^T(i)+\varepsilon_1 X_{01}^T(i)A_{01}(i)$$
$$+X_{11}(i)A_{11}(i)+X_{12}(i)A_{21}(i)+[\mathbb{C}^T(i)X(i)\mathbb{C}(i)]_{11}-F_1^T(i)(R(i)$$
$$+\mathbb{D}^T(i)X(i)\mathbb{D}(i))F_1(i)+M_{11}(i)+\varepsilon_1\sum_{j=1}^{N}q_{ij}X_{11}(j)=0$$

$$\varepsilon_2 A_{01}^T(i)X_{02}(i)+\frac{\varepsilon_2}{\varepsilon_1}A_{11}^T(i)X_{12}(i)+A_{21}^T(i)X_{22}(i)+\varepsilon_1 X_{01}^T(i)A_{02}(i)+$$
$$+X_{11}(i)A_{12}(i)+X_{12}(i)A_{22}(i)+[\mathbb{C}^T(i)X(i)\mathbb{C}(i)]_{12}- \tag{13}$$
$$-F_1^T(i)(R(i)+\mathbb{D}^T(i)X(i)\mathbb{D}(i))F_2(i)+M_{12}(i)+\varepsilon_2\sum_{j=1}^{N}q_{ij}X_{12}(j)=0$$

$$\varepsilon_2 A_{02}^T(i)X_{02}(i)+\frac{\varepsilon_2}{\varepsilon_1}A_{12}^T(i)X_{12}(i)+A_{22}^T(i)X_{22}(i)+\varepsilon_2 X_{02}^T(i)A_{02}(i)+$$
$$+\frac{\varepsilon_2}{\varepsilon_1}X_{12}^T(i)A_{12}(i)+X_{22}(i)A_{22}(i)+[\mathbb{C}^T(i)X(i)\mathbb{C}(i)]_{22}$$
$$-F_2^T(i)(R(i)+\mathbb{D}^T(i)X(i)\mathbb{D}(i))F_2(i)+M_{22}(i)+\varepsilon_2\sum_{j=1}^N q_{ij}X_{22}(j)=0$$

where $[\mathbb{C}^T(i)X(i)\mathbb{C}(i)]_{kj}$ is the $kj$-block of the matrix $\mathbb{C}^T(i)X(i)\mathbb{C}(i)$, $0\le k$, $j\le 2$.
Setting formally $\varepsilon=0$ in (13) and taking into account (3) and $\delta>\frac{1}{2}$ we obtain:

$$B_0^T(i)X_{00}(i)+B_1^T(i)X_{01}^T(i)+B_2^T(i)X_{02}^T(i)+D_0^T(i)X_{00}(i)C_{00}(i)+L_0^T(i)$$
$$=-(R(i)+D_0^T(i)X_{00}(i)D_0(i))F_0(i)$$

$$B_1^T(i)X_{11}(i)+B_2^T(i)X_{12}^T(i)+D_0^T(i)X_{00}(i)C_{01}(i)+L_1^T(i)$$
$$=-(R(i)+D_0^T(i)X_{00}(i)D_0(i))F_1(i)$$

$$B_2^T(i)X_{22}(i)+D_0^T(i)X_{00}(i)C_{02}(i)+L_2^T(i)=-(R(i)+D_0^T(i)X_{00}(i)D_0(i))F_2(i)$$

$$A_{00}^T(i)X_{00}(i)+A_{10}^T(i)X_{01}^T(i)+A_{20}^T(i)X_{02}^T(i)+X_{00}(i)A_{00}(i)$$
$$+X_{01}(i)A_{10}(i)+X_{02}(i)A_{20}(i)+C_{00}^T(i)X_{00}(i)C_{00}(i)$$
$$-F_0^T(i)(R(i)+D_0^T(i)X_{00}(i)D_0(i))F_0(i)+M_{00}(i)+\sum_{j=1}^N q_{ij}X_{00}(j)=0$$

$$A_{10}^T(i)X_{11}(i)+A_{20}^T(i)X_{12}^T(i)+X_{00}(i)A_{01}(i)+X_{01}(i)A_{11}(i)+X_{02}(i)A_{22}(i)+$$
$$+C_{00}^T(i)X_{00}(i)C_{01}(i)-F_0^T(i)(R(i)+D_0^T(i)X_{00}(i)D_0(i))F_1(i)+M_{01}(i)=0$$

$$A_{20}^T(i)X_{22}(i)+X_{00}(i)A_{02}(i)+X_{01}(i)A_{12}(i)+X_{02}(i)A_{22}(i)$$
$$+C_{00}^T(i)X_{00}(i)C_{02}(i)-F_0^T(i)(R(i)+D_0^T(i)X_{00}(i)D_0(i))F_2(i)+M_{02}(i)=0$$

$$A_{11}^T(i)X_{11}(i)+A_{21}^T(i)X_{12}^T(i)+X_{11}(i)A_{11}(i)+X_{12}(i)X_{21}(i)$$
$$+C_{01}^T(i)X_{00}(i)C_{01}(i)-F_1^T(i)(R(i)+D_0^T(i)X_{00}(i)D_0(i))F_1(i)+M_{11}(i)=0$$

$$A_{21}^T(i)X_{12}(i)+X_{11}(i)A_{12}(i)+X_{12}(i)A_{22}(i)+C_{01}^T(i)X_{00}(i)C_{02}(i)$$
$$-F_1^T(i)(R(i)+D_0^T(i)X_{00}(i)D_0(i))F_2(i)+M_{12}(i)=0$$

$$\begin{aligned} &A_{22}^T(i)X_{22}(i)+X_{22}(i)A_{22}(i)+C_{02}^T(i)X_{00}(i)C_{02}(i)-\\ &-F_2^T(i)(R(i)+D_0^T(i)X_{00}(i)D_0(i))F_2(i)+M_{22}(i)=0. \end{aligned} \tag{14}$$

Assuming that $A_{22}(i)$ are invertible, we may introduce the notations:

$$\begin{aligned} A_{kj}^1(i) &= A_{kj}(i)-A_{k2}(i)A_{22}^{-1}(i)A_{2j}(i) \\ B_j^1(i) &= B_j(i)-A_{j2}(i)A_{22}^{-1}(i)B_2(i), \quad k,j=0,1 \end{aligned} \tag{14}$$

$$\begin{aligned} C_{0k}^1(i) &= C_{0k}(i)-C_{02}(i)A_{22}^{-1}(i)A_{2k}(i) \\ L_k^1(i) &= L_k(i)-A_{2k}^T(i)A_{22}^{-T}(i)L_2(i)- \\ &\quad -(M_{k2}(i)-A_{2k}^T(i)A_{22}^{-T}(i)M_{22}(i))A_{22}^{-1}(i)B_2(i), \quad k=0,1 \end{aligned} \tag{15}$$

$$\begin{aligned} R^1(i) &= R(i)-L_2^T(i)A_{22}^{-1}(i)B_2(i)- \\ &\quad -B_2^T(i)A_{22}^{-T}(i)L_2(i)+B_2^T(i)A_{22}^{-T}(i)M_{22}(i)A_{22}^{-1}(i)B_2(i) \\ D_0^1(i) &= D_0(i)-C_{02}(i)A_{22}^{-1}(i)B_2(i) \end{aligned} \tag{16}$$

and

$$\begin{pmatrix} M_{00}^1(i) & M_{01}^1(i) \\ (M_{01}^1)^T(i) & M_{11}^1(i) \end{pmatrix} = \begin{pmatrix} I_{n_0} & 0 \\ 0 & I_{n_1} \\ -A_{22}^{-1}(i)A_{20}(i) & -A_{22}^{-1}(i)A_{21}(i) \end{pmatrix}^T$$
$$\times \begin{pmatrix} M_{00}(i) & M_{01}(i) & M_{02}(i) \\ M_{01}^T(i) & M_{11}(i) & M_{12}(i) \\ M_{02}^T(i) & M_{12}^T(i) & M_{22}(i) \end{pmatrix} \times \begin{pmatrix} I_{n_0} & 0 \\ 0 & I_{n_1} \\ -A_{22}^{-1}(i)A_{20}(i) & -A_{22}^{-1}(i)A_{21}(i) \end{pmatrix} \tag{17}$$

The next result allows us to reduce the number of equations and the number of unknowns of the systems (14).

**Lemma 3.1.** *If $A_{22}(i)$ are invertible, the following hold:*

*(i) If $(X_{00}(i), X_{01}(i), X_{11}(i), X_{02}(i), X_{12}(i), X_{22}(i), F_0(i), F_1(i), F_2(i)), i \in \mathfrak{N}$ is a solution of the system (14) with the property that the matrices $A_{22}(i)+B_2(i)F_2(i)$ are invertible, then*

$$(X_{00}(i), X_{01}(i), X_{11}(i), X_{22}(i), F_0^1(i), F_1^1(i), F_2(i))$$

*is a solution of the following system:*

$$\begin{aligned} &(B_0^1)^T(i)X_{00}(i)+(B_1^1)^T(i)X_{01}^T(i)+(D_0^1)^T(i)X_{00}(i)C_{00}^1(i)+(L_0^1)^T(i) = \\ &\qquad -(R^1(i)+(D_0^1)^T(i)X_{00}(i)D_0^1(i))F_0^1(i) \end{aligned}$$

$$(B_1^1)^T(i)X_{11}(i)+(D_0^1)^T(i)X_{00}(i)C_{01}^1(i)+(L_1^1)^T(i)= \\ -(R^1(i)+(D_0^1)^T(i)X_{00}(i)D_0^1(i))F_1^1(i)$$

$$B_2^T(i)X_{22}(i)+D_0^T(i)X_{00}(i)C_{02}(i)+L_2^T(i)= \\ -(R(i)+D_0^T(i)X_{00}(i)D_0(i))F_2(i)$$

$$(A_{00}^1)^T(i)X_{00}(i)+(A_{10}^1)^T(i)X_{01}^T(i)+X_{00}(i)A_{00}^1(i)+X_{01}(i)A_{10}^1(i)+ \\ +(C_{00}^1)^T(i)X_{00}(i)C_{00}^1(i)-(F_0^1)^T(i)(R^1(i) \\ +(D_0^1)^T(i)X_{00}(i)D_0^1(i))F_0^1(i)+M_{00}^1(i)+\sum_{j=1}^N q_{ij}X_{00}(j)=0 \tag{19}$$

$$(A_{10}^1)^T(i)X_{11}(i)+X_{00}(i)A_{01}^1(i)+X_{01}(i)A_{11}^1(i)+(C_{00}^1)^T(i)X_{00}(i)C_{01}^1(i) \\ -(F_0^1)^T(i)(R^1(i)+(D_0^1)^T(i)X_{00}(i)D_0^1(i))F_1^1(i)+M_{01}^1(i)=0$$

$$(A_{11}^1)^T(i)X_{11}(i)+X_{11}(i)A_{11}^1(i)+(C_{01}^1)^T(i)X_{00}(i)C_{01}^1(i) \\ -(F_1^1)^T(i)(R^1(i)+(D_0^1)^T(i)X_{00}(i)D_0^1(i))F_1^1(i)+M_{11}^1(i)=0$$

$$A_{22}^T(i)X_{22}(i)+X_{22}(i)A_{22}(i)+C_{02}^T(i)X_{00}(i)C_{02}(i) \\ -F_2^T(i)(R(i)+D_0^T(i)X_{00}(i)D_0(i))F_2(i)+M_{22}(i)=0.$$

*where*

$$F_j^1(i) =(I_m + F_2(i)A_{22}^{-1}(i)B_2(i))^{-1}(F_j(i) - F_2(i)A_{22}^{-1}(i)A_{2j}(i)), \\ j=0,1 \tag{20}$$

*(ii) If* $(X_{00}(i), X_{01}(i), X_{11}(i), X_{22}(i), F_0^1(i), F_1^1(i), F_2(i)), i \in \mathfrak{N}$ *is a solution of the system (19) with the property that* $A_{22}(i) + B_2(i)F_2(i)$ *are invertible, then*

$$(X_{00}(i), X_{01}(i), X_{11}(i), X_{02}(i), X_{12}(i), X_{22}(i), F_0(i), F_1(i), F_2(i)), i \in \mathfrak{N}$$

*is a solution of the system (14) if*

$$F_j(i) =(I_m + F_2(i)A_{22}^{-1}(i)B_2(i))F_j^1(i) + F_2(i)A_{22}^{-1}(i)A_{2j}(i), \\ j=0,1 \tag{21}$$

*and*

$$\begin{aligned}X_{02}(i) = &- [A_{20}^T(i)X_{22}(i) + X_{00}(i)A_{02}(i) + X_{01}(i)A_{11}(i)\\ &+ C_{00}^T(i)X_{00}(i)C_{02}(i) - F_0^T(i)(R(i)\\ &+ D_0^T(i)X_{00}(i)D_0(i))F_2(i) + M_{02}(i)]A_{22}^{-1}(i)\end{aligned} \tag{22}$$

$$\begin{aligned}X_{12}(i) = &-[A_{21}^T(i)X_{22}(i) + X_{11}(i)A_{12}(i) + C_{01}^T(i)X_{00}(i)C_{02}(i)\\ &- F_1^T(i)(R(i) + D_0^T(i)X_{00}(i)D_0(i))F_2(i) + M_{12}(i)]A_{22}^{-1}(i).\end{aligned} \tag{23}$$

*Proof.* Proof is done by direct calculation. The details are omitted. ∎

We assume now that $A_{11}^1(i)$ are invertible and introduce the new notations:

$$\begin{array}{rcl} A_{00}^0(i) &=& A_{00}^1(i) - A_{01}^1(i)(A_{11}^1)^{-1}(i)A_{10}^1(i)\\ C_{00}^0(i) &=& C_{00}^1(i) - C_{01}^1(i)(A_{11}^1)^{-1}(i)A_{10}^1(i)\\ B_0^0(i) &=& B_0^1(i) - A_{01}^1(i)(A_{11}^1)^{-1}(i)B_1^1(i)\\ D_0^0(i) &=& D_0^1(i) - C_{01}^1(i)(A_{11}^1)^{-1}(i)B_1^1(i)\end{array} \tag{23}$$

$$\begin{array}{rcl} L_0^0(i) &=& L_0^1(i) - (A_{10}^1)^T(i)(A_{11}^1)^{-T}(i)L_1^1(i) - \\ && -(M_{01}^1(i) - (A_{10}^1)^T(i)(A_{11}^1)^{-T}(i)M_{11}^1(i))(A_{11}^1)^{-1}(i)B_1^1(i) \quad (24)\\ R^0(i) &=& R^1(i) - (B_1^1)^T(i)(A_{11}^1)^{-T}(i)L_1^1(i) - (L_1^1)^T(i)(A_{11}^1)^{-1}(i)B_1^1(i)\\ && +(B_1^1)^T(i)(A_{11}^1)^{-T}(i)M_{11}^1(i)(A_{11}^1)^{-1}(i)B_1^1(i)\\ M^0(i) &=& \begin{pmatrix} I_{n_0} \\ -(A_{11}^1)^{-1}(i)A_{10}^1(i)\end{pmatrix}^T \begin{pmatrix} M_{00}^1(i) & M_{01}^1(i)\\ (M_{01}^1)^T(i) & M_{11}^1(i)\end{pmatrix} \times\\ && \times \begin{pmatrix} I_{n_0} \\ -(A_{11}^1)^{-1}(i)A_{10}^1(i)\end{pmatrix}.\end{array}$$

The next result allows us to reduce the number of equations and unknowns of the system (19).

**Lemma 3.2.** *Assume that $A_{11}^1(i)$ are invertible, then the following hold:*

*(i) If $(X_{00}(i), X_{01}(i), X_{11}(i), X_{22}(i), F_0^1(i), F_1^1(i), F_2(i)), i \in \mathfrak{N}$ is a solution of the system (19) with the property that $A_{11}^1(i) + B_1^1(i)F_1^1(i)$ are invertible, then $(X_{00}(i), X_{11}(i), X_{22}(i), F_0^0(i), F_1^1(i), F_2(i))$ is a solution of the*

*system:*

$$
\begin{aligned}
&(B_0^0)^T(i)X_{00}(i)+(D_0^0)^T(i)X_{00}(i)C_{00}^0(i)+L_0^0(i)\\
&=-(R^0(i)+(D_0^0)^T(i)X_{00}(i)D_0^0(i))F_0^0(i)\\
&(B_1^1)^T(i)X_{11}(i)+(D_0^1)^T(i)X_{00}(i)C_{01}^1(i)+(L_1^1)^T(i)\\
&=-(R^1(i)+(D_0^1)^T(i)X_{00}(i)D_0^1(i))F_1^1(i)\\
&B_2^T(i)X_{22}(i)+D_0^T(i)X_{00}(i)C_{02}(i)+L_2^T(i)\\
&=-(R(i)+D_0^T(i)X_{00}(i)D_0(i))F_2(i)\\
&(A_{00}^0)^T(i)X_{00}(i)+X_{00}(i)A_{00}^0(i)+(C_{00}^0)^T(i)X_{00}(i)C_{00}^0(i)-(F_0^0)^T(i)(R^0(i)\\
&+(D_0^0)^T(i)X_{00}(i)D_0^0(i))F_0^0(i)+M^0(i)+\sum_{j=1}^{N}q_{ij}X_{00}(j)=0\\
&(A_{11}^1)^T(i)X_{11}(i)+X_{11}(i)A_{11^1}(i)+(C_{01}^1)^T(i)X_{00}(i)C_{01}^1(i)\\
&-(F_1^1)^T(i)(R^1(i)+(D_0^1)^T(i)X_{00}(i)D_0^1(i))F_1^1(i)+M_{11}^1(i)=0 \qquad (26)\\
&A_{22}^T(i)X_{22}(i)+X_{22}(i)A_{22}(i)+(C_{02})^T(i)X_{00}(i)C_{02}(i)-\\
&-F_2^T(i)(R(i)+D_0^T(i)X_{00}(i)D_0(i))F_2(i)+M_{22}(i)=0
\end{aligned}
$$

*where*

$$F_0^0(i)=(I_m+F_1^1(i)(A_{11}^1)^{-1}(i)B_1^1(i))^{-1}(F_0^1(i)-F_1^1(i)(A_{11}^1)^{-1}(i)A_{10}^1(i)). \qquad (26)$$

*(ii) If* $(X_{00}(i),X_{11}(i),X_{22}(i),F_0^0(i),F_1^1(i),F_2(i)),i\in\mathfrak{N}$ *is a solution of the system (26) with the property that*

$$A_{11}^1(i)+B_1^1(i)F_1^1(i)$$

*are invertible matrices, then*

$$(X_{10}(i),X_{01}(i),X_{11}(i),X_{22}(i),F_0^1(i),F_1^1(i),F_2(i)),i\in\mathfrak{N}$$

*is a solution of the system (19) if*

$$F_0^1(i)=(I_m+F_1^1(i)(A_{11}^1)^{-1}(i)B_1^1(i))F_0^0(i)+F_1^1(i)(A_{11}^1)^{-1}(i)A_{10}^1(i) \qquad (27)$$

*and*

$$
\begin{aligned}
&X_{01}(i)=-[(A_{10}^1)^T(i)X_{11}(i)+X_{00}(i)A_{01}^1(i)+(C_{00}^1)^T(i)X_{00}(i)C_{01}^1(i)-\\
&(F_0^1)^T(i)(R^1(i)+(D_0^1)^T(i)X_{00}(i)D_0^1(i))F_1^1(i)+M_{01}^1(i)](A_{11}^1)^{-1}(i). \qquad (29)
\end{aligned}
$$

*Proof.* The proof may be done by a laborious calculation. ∎

Assuming that $X_{00}(i)$ are such that the matrices

$$R^0(i)+(D_0^0)^T(i)X_{00}(i)D_0^0(i),\ \ R^1(i)+(D_0^1)^T(i)X_{00}(i)D_0^1(i),\ \ R(i)+D_0^T(i)X_{00}(i)D_0(i)$$

are invertible we may remove the unknowns $F_0^0(i)$, $F_1^1(i)$, $F_2(i)$ from (26) and obtain the following system of nonlinear equations with the unknowns $X_{00}(i)$, $X_{11}(i)$, $X_{22}(i)$:

$$\begin{aligned}
&(A_{00}^0)^T(i)X_{00}(i)+X_{00}(i)A_{00}^0(i)+(C_{00}^0)^T(i)X_{00}(i)C_{00}^0(i)-\\
&-(X_{00}(i)B_0^0(i)+(C_{00}^0)^T(i)X_{00}(i)D_0^0(i)+L_0^0(i))(R^0(i)\\
&+(D_0^0)^T(i)X_{00}(i)D_0^0(i))^{-1}\times((B_0^0)^T(i)X_{00}(i)\\
&+(D_0^0)^T(i)X_{00}(i)C_{00}^0(i)+(L_0^0)^T)(i)+M^0(i)+\sum_{j=1}^{N}q_{ij}X_{00}(j)=0
\end{aligned}$$

$$\begin{aligned}
&(A_{11}^1)^T(i)X_{11}(i)+X_{11}(i)A_{11}^1(i)+(C_{01}^1)^T(i)X_{00}(i)C_{01}^1(i)-(X_{11}(i)B_1^1(i)\\
&+(C_{01}^1)^T(i)X_{00}(i)D_0^1(i)+L_1^1(i))(R^1(i)+(D_0^1)^T(i)X_{00}(i)D_0^1(i))^{-1}\\
&\times((B_1^1)^T(i)X_{11}(i)+(D_0^1)^T(i)X_{00}(i)C_{01}^1(i)+(L_1^1)^T(i))+M_{11}^1(i)=0 \qquad (30)
\end{aligned}$$

$$\begin{aligned}
&A_{22}^T(i)X_{22}(i)+X_{22}(i)A_{22}(i)+C_{02}^T(i)X_{00}(i)C_{02}(i)-(X_{22}(i)B_2(i)\\
&+C_{02}^T(i)X_{00}(i)D_0(i)+L_2(i))(R(i)+D_0^T(i)X_{00}(i)D_0(i))^{-1}\times(B_2^T(i)X_{22}(i)\\
&+D_0^T(i)X_{00}(i)C_{02}(i)+L_2^T(i))+M_{22}(i)=0.
\end{aligned}$$

In the special case

$$C_{jk}(i)=0,\quad D_k(i)=0,\quad 0\le j,k\le 2,\ \ i=1$$

the system of nonlinear equations (30) reduces to three uncoupled algebraic Riccati equations arising in the investigations of the asymptotic structure of the stabilizing solution of an algebraic Riccati equation associated to a linear quadratic optimization problem for a deterministic controlled system with several time scales (see e.g. [1]).

In the stochastic context considered on this work, the system of nonlinear equation (30) will play the same role which, in the deterministic case, is played by the algebraic Riccati equations of lower dimension obtained neglecting the small parameters. That is why, in the following, the system (30) will be called **the system of coupled reduced algebraic Riccati equations** (SCRARE).

In the next subsection we shall introduce the concept of stabilizing solution of the system (30) and we shall provide a set of necessary and sufficient conditions for the existence of the stabilizing solution.

## 3.2. STABILIZING SOLUTION OF SCRARE

Let $\mathcal{S}_{n_k} \subset \mathbb{R}^{n_k \times n_k}$ be the linear space of symmetric matrices of size $n_k \times n_k$, $0 \leq k \leq 2$. We set

$$\mathfrak{X} = (\mathcal{S}_{n_0} \times \mathcal{S}_{n_1} \times \mathcal{S}_{n_2}) \times (\mathcal{S}_{n_0} \times \mathcal{S}_{n_1} \times \mathcal{S}_{n_2}) \times \cdots \times (\mathcal{S}_{n_0} \times \mathcal{S}_{n_1} \times \mathcal{S}_{n_2}).$$

Hence, $\mathbf{X} \in \mathfrak{X}$ if and only if

$$\mathbf{X} = \left((X_0(1), X_1(1), X_2(1)), \cdots, (X_0(N), X_1(N), X_2(N))\right).$$

$\mathfrak{X}$ is a real Hilbert space with respect to the inner product

$$\langle \mathbf{X}, \mathbf{Y} \rangle = \sum_{i=1}^{N} \sum_{j=0}^{2} Tr[X_j(i) Y_j(i)] \tag{30}$$

for all $\mathbf{X}, \mathbf{Y} \in \mathfrak{X}$. On $\mathfrak{X}$ we consider the ordering relation "$\geq$" induced by the convex closed cone with not empty interior

$$\begin{aligned}\mathfrak{X}_+ = &\{\mathbf{X} \in \mathfrak{X} | \mathbf{X} = \left((X_0(1), X_1(1), X_2(1)), \cdots, (X_0(N), X_1(N), X_2(N))\right), \\ &X_j(i) \geq 0, j = 0, 1, 2, \quad i \in \mathfrak{N}\}.\end{aligned}$$

The system of nonlinear equations (30) can be regarded as a generalized algebraic Riccati equation on $\mathfrak{X}$ of the form:

$$\begin{aligned}\mathbf{A}^T\mathbf{X} + \mathbf{X}\mathbf{A} + \mathbf{\Pi}_1[\mathbf{X}] + \mathbf{M} - \\ (\mathbf{X}\mathbf{B} + \mathbf{\Pi}_2[\mathbf{X}] + \mathbf{L})(\mathbf{R} + \mathbf{\Pi}_3[\mathbf{X}])^{-1}(\mathbf{X}\mathbf{B} + \mathbf{\Pi}_2[\mathbf{X}] + \mathbf{L})^T &= 0 \end{aligned} \tag{31}$$

where

$$\begin{aligned}\mathbf{A} = \Big( &(A_{00}^0(1) + \frac{1}{2} q_{11} I_{n_0}, A_{11}^1(1), A_{22}(1)), \\ &\cdots, (A_{00}^0(N) + \frac{1}{2} q_{NN} I_{n_0}, A_{11}^1(N), A_{22}(N)) \Big)\end{aligned}$$

$$\mathbf{B} = \left((B_0^0(1), B_1^1(1), B_2(1)), \cdots, (B_0^0(N), B_1^1(N), B_2(N))\right)$$

$$\mathbf{M} = \left((M^0(1), M_{11}^1(1), M_{22}(1)), \cdots, (M^0(N), M_{11}^1(N), M_{22}(N))\right)$$

$$\mathbf{R} = \left((R^0(1), R^1(1), R(1)), \cdots, (R^0(N), R^1(N), R(N))\right)$$

$$\begin{aligned}\mathbf{\Pi}_1[\mathbf{X}](i) = &\big((C_0^0)^T(i) X_0(i) C_0^0(i) + \sum_{j \neq i, j=1}^{N} q_{ij} X_0(j), \\ &C_{01}^{1T}(i) X_0(i) C_{01}^1(i), C_{02}^T(i) X_0(i) C_{02}(i)\big)\end{aligned}$$

$$\mathbf{\Pi}_2[\mathbf{X}](i) = \big((C_{00}^0)^T(i)X_0(i)D_0^0(i), (C_{01}^1)^T(i)X_0(i)D_0^1(i), C_{02}^T(i)X_0(i)D_0(i)\big)$$
$$\mathbf{\Pi}_3[\mathbf{X}](i) = \big((D_0^0)^T(i)X_0(i)D_0^0(i), (D_0^1)^T(i)X_0(i)D_0^1(i), D_0^T(i)X_0(i)D_0(i)\big)$$

$1 \leq i \leq N$ and for all $\mathbf{X} \in \mathfrak{X}$.
We set $\mathbf{\Pi}[\mathbf{X}](i) = \begin{pmatrix} \mathbf{\Pi}_1[\mathbf{X}](i) & \mathbf{\Pi}_2[\mathbf{X}](i) \\ \mathbf{\Pi}_2^T[\mathbf{X}](i) & \mathbf{\Pi}_3[\mathbf{X}](i) \end{pmatrix}$. To the triple $(\mathbf{A}, \mathbf{B}, \mathbf{\Pi})$ and the feedback gain $\mathbf{F} = ((F_0(1), F_1(1), F_2(1)), \cdots, (F_0(N), F_1(N), F_2(N)))$ with $F_j(i) \in \mathbb{R}^{m \times n_j}$, $j = 0, 1, 2, i \in \mathfrak{N}$ we associate the linear operator $\mathbb{L}_{\mathbf{F}} : \mathfrak{X} \to \mathfrak{X}$ by

$$\mathbb{L}_{\mathbf{F}}[\mathbf{X}](i) = (\mathbb{L}_{\mathbf{F}0}[\mathbf{X}](i), \mathbb{L}_{\mathbf{F}1}[\mathbf{X}](i), \mathbb{L}_{\mathbf{F}2}[\mathbf{X}](i))$$

with

$$\begin{aligned}
\mathbb{L}_{\mathbf{F}0}[\mathbf{X}](i) =& (A_{00}^0(i)+B_0^0(i)F_0(i))^T X_0(i) + X_0(i)(A_{00}^0(i)+B_0^0(i)F_0(i)) \\
&+ (C_{00}^0(i)+D_0^0(i)F_0(i))^T X_0(i)(C_{00}^0(i) \\
&+ D_0^0(i)F_0(i)) + \sum_{j=1}^N q_{ij}X_0(j) \\
\mathbb{L}_{\mathbf{F}1}[\mathbf{X}](i) =& (A_{11}^1(i)+B_1^1(i)F_1(i))^T X_1(i) + X_1(i)(A_{11}^1(i)+B_1^1(i)F_1(i)) \\
&+ (C_{01}^1(i)+D_0^1(i)F_1(i))^T X_0(i)(C_{01}^1(i)+D_0(i)F_1(i)) \qquad (33) \\
\mathbb{L}_{\mathbf{F}2}[\mathbf{X}](i) =& (A_{22}(i)+B_2(i)F_2(i))^T X_2(i) + X_2(i)(A_{22}(i)+B_2(i)F_2(i)) \\
&+ (C_{02}(i)+D_0(i)F_2(i))^T X_0(i)(C_{02}(i)+D_0(i)F_2(i))
\end{aligned}$$

for all $\mathbf{X} \in \mathfrak{X}, i \in \mathfrak{N}$.
Several properties of the operator of type (33) are summarized in the following proposition:

**Propoziia 3.1.** *(i) For each feedback gain the corresponding operator $\mathbb{L}_{\mathbf{F}}$ generates a positive evolution on the space $\mathfrak{X}$, that is:*

$$e^{\mathbb{L}_{\mathbf{F}}t}\mathfrak{X}_+ \subset \mathfrak{X}_+$$

*for all $t \geq 0$.*

*(ii) The spectrum of the operator $\mathbb{L}_{\mathbf{F}}$ is in the half plane $\mathbb{C}_- = \{\lambda \in \mathbb{C}, Re\lambda < 0\}$ if and only if, there exists $\hat{\mathbf{X}} \in Int\mathfrak{X}_+$ such that $\mathbb{L}_{\mathbf{F}}[\hat{\mathbf{X}}] < 0$.*

Now we are in position to introduce the concept of stabilizing solution of SCRARE (30).

**Definition 3.1.** *A solution*

$$\tilde{\mathbf{X}} = \Big((\tilde{X}_0(1), \tilde{X}_1(1), \tilde{X}_2(1)), \cdots, (\tilde{X}_0(N), \tilde{X}_1(N), \tilde{X}_2(N))\Big)$$

*of (30) is named "stabilizing solution" if the spectrum of the linear operator* $\mathbb{L}_{\tilde{\mathbf{F}}}$ *is located in the half plane* $\mathbb{C}_-$, $\mathbb{L}_{\tilde{\mathbf{F}}}$ *being the linear operator of type (33) defined for* $\tilde{\mathbf{F}} = \Big((\tilde{F}_0(1), \tilde{F}_1(1), \tilde{F}_2(1)), \cdots, (\tilde{F}_0(N), \tilde{F}_1(N), \tilde{F}_2(N))\Big)$ *where*

$$\begin{aligned}
\tilde{F}_0(i) &= -(R^0(i)+(D_0^0)^T(i)\tilde{X}_0(i)D_0^0(i))^{-1} \times \\
&\quad ((B_0^0)^T(i)\tilde{X}_0(i)+(D_0^0)^T(i)\tilde{X}_0(i)C_{00}^0(i)+(L_0^0)^T(i)) \\
\tilde{F}_1(i) &= -(R^1(i)+(D_0^1)^T(i)\tilde{X}_0(i)D_0^1(i))^{-1} \times \\
&\quad ((B_1^1)^T(i)\tilde{X}_1(i)+(D_0^1)^T(i)\tilde{X}_0(i)C_{01}^1(i)+(L_1^1)^T(i)) \\
\tilde{F}_2(i) &= -(R(i)+D_0^T(i)\tilde{X}_0(i)D_0(i))^{-1} \times \\
&\quad (B_2^T(i)\tilde{X}_2(i)+D_0^T(i)\tilde{X}_0(i)C_{02}(i)+L_2^T(i))
\end{aligned} \tag{33}$$

**Definition 3.2.** *We say that the triple* $(\mathbf{A}, \mathbf{B}, \mathbf{\Pi})$ *is stabilizable if there exists a feedback gain* $\mathbf{F} = ((F_0(1), F_1(1), F_2(1)), \cdots, (F_0(N), F_1(N), F_2(N)))$ *with the property that the spectrum of the corresponding linear operator* $\mathbb{L}_{\mathbf{F}}$ *is located in the half plane* $\mathbb{C}_-$.

Necessary and sufficient conditions for the stabilizability of the triple $(\mathbf{A}, \mathbf{B}, \mathbf{\Pi})$ can be derived employing part (ii) of Proposition 3.1 combined with Schur complement technique.
The next result provides a set of conditions equivalent to the existence of the stabilizing solution $\tilde{\mathbf{X}}$ of SCRARE (30) satisfying the sign conditions:

$$\begin{aligned}
&R^0(i)+(D_0^0)^T(i)\tilde{X}_0(i)D_0^0(i) > 0 \\
&R^1(i)+(D_0^1)^T(i)\tilde{X}_0(i)D_0^1(i) > 0 \\
&R(i) + D_0^T(i)\tilde{X}_0(i)D_0(i) > 0, \qquad i \in \mathfrak{N}.
\end{aligned} \tag{34}$$

**Theorem 3.1.** *The following are equivalent:*

(i) *the SCRARE (30) has a unique stabilizing solution*

$$\tilde{\mathbf{X}} = \Big((\tilde{X}_0(1), \tilde{X}_1(1), \tilde{X}_2(1)), \cdots, (\tilde{X}_0(N), \tilde{X}_1(N), \tilde{X}_2(N))\Big)$$

*satisfying the sign conditions (35);*

(ii) *the triple* $(\mathbf{A}, \mathbf{B}, \mathbf{\Pi})$ *is stabilizable and there exists*

$$\mathbf{Y} = ((Y_0(1), Y_1(1), Y_2(1)), \cdots, (Y_0(N), Y_1(N), Y_2(N))) \in \mathfrak{X}$$

*solving the following system of LMIs:*

$$\begin{pmatrix} \Theta_0(Y_0,i) & Y_0(i)B_0^0(i)+C_{00}^{0T}(i)Y_0(i)D_0^0(i)+L_0^0(i) \\ B_0^{0T}(i)Y_0(i)+D_0^{0T}(i)Y_0(i)C_{00}^0(i)+L_0^{0T}(i) & R^0(i)+D_0^{0T}(i)Y_0(i)D_0^0(i) \end{pmatrix} > 0$$

$$\begin{pmatrix} \Theta_1(Y_0,Y_1,i) & Y_1(i)B_1^1(i)+C_{01}^{1T}(i)Y_0(i)D_0^1(i)+L_1^1(i) \\ B_1^{1T}(i)Y_1(i)+D_0^{1T}(i)Y_0(i)C_{01}^1(i)+L_1^{1T}(i) & R^1(i)+D_0^{1T}(i)Y_0(i)D_0^1(i) \end{pmatrix} > 0$$

$$\begin{pmatrix} \Theta_2(Y_0,Y_2,i) & Y_2(i)B_2(i)+C_{02}^T(i)Y_0(i)D_0(i)+L_2(i) \\ B_2^T(i)Y_2(i)+D_0^T(i)Y_0(i)C_{02}(i)+L_2^T(i) & R(i)+D_0^T(i)Y_0(i)D_0(i) \end{pmatrix} > 0.$$

*where*

$$\begin{aligned}\Theta_0(Y_0,i) =&(A_0^0)^T(i)Y_0(i) + Y_0(i)A_0^0(i) + (C_{00}^0)^T(i)Y_0(i)C_{00}^0(i) \\ &+ M^0(i) + \sum_{j=1}^N q_{ij}Y_0(j)\end{aligned}$$

$$\Theta_1(Y_0,Y_1,i) = (A_{11}^1)^T(i)Y_1(i) + Y_1(i)A_{11}^1(i) + (C_{01}^1)^T(i)Y_0(i)C_{01}^1(i) + M_{11}^1(i)$$
$$\Theta_2(Y_0,Y_2,i) = A_{22}^T(i)Y_2(i) + Y_2(i)A_{22}(i) + C_{02}^T(i)Y_0(i)C_{02}(i) + M_{22}(i)$$

## 3.3. THE ASYMPTOTIC STRUCTURE OF THE STABILIZING SOLUTION OF SARE

Let us assume that conditions from Theorem 3.1 (ii) are fulfilled. Then SCRARE (30) has a unique stabilizing solution

$$\tilde{\mathbf{X}} = \Big((\tilde{X}_0(1),\tilde{X}_1(1),\tilde{X}_2(1)),\cdots,(\tilde{X}_0(N),\tilde{X}_1(N),\tilde{X}_2(N))\Big)$$

satisfying the sign conditions (35). Let $(\tilde{F}_0(i),\tilde{F}_1(i),\tilde{F}_2(i))$ be the corresponding feedback gains defined in (34). Employing Proposition 3.1 (ii) in the case of operator $\mathbb{L}_{\tilde{\mathbf{F}}}$ we may infer that the matrices $A_{jj}^j(i)+B_j^j(i)\tilde{F}_j(i)$ are Hurwitz matrices $0\le j\le 2, i\in\mathfrak{N}$, with $A_{22}^2(i)\triangleq A_{22}(i)$ and $B_2^2(i)\triangleq B_2(i)$. Taking $\tilde{F}_1^1(i)\triangleq\tilde{F}_1(i)$ we compute $\tilde{F}_0^1(i)$ by

$$\tilde{F}_0^1(i) = (I_m+\tilde{F}_1^1(i)(A_{11}^1)^{-1}(i)B_1^1(i))\tilde{F}_0(i)+\tilde{F}_1^1(i)(A_{11}^1)^{-1}(i)A_{10}^1(i). \tag{35}$$

Further, we compute $\tilde{\tilde{F}}_j(i)$ by

$$\tilde{\tilde{F}}_j(i) = (I_m+\tilde{F}_2(i)A_{22}^{-1}(i)B_2(i))\tilde{F}_j^1(i)+\tilde{F}_2(i)A_{22}^{-1}(i)A_{2j}(i),\ \ j=0,1 \tag{36}$$

and

$$\tilde{\tilde{F}}_2(i)\triangleq\tilde{F}_2(i). \tag{37}$$

We compute $\tilde{X}_{01}(i)$ by

$$\begin{aligned}\tilde{X}_{01}(i) =&-[(A_{10}^1)^T(i)\tilde{X}_1(i)+\tilde{X}_0(i)A_{01}^1(i)+(C_{00}^1)^T(i)\tilde{X}_0(i)C_{01}^1(i) \\ &-(\tilde{F}_0^1)^T(i)(R^1(i)+(D_0^1)^T(i)\tilde{X}_0(i)D_0^1(i))\tilde{F}_1^1(i)+M_{01}^1(i)](A_{11}^1)^{-1}(i).\end{aligned} \tag{38}$$

Then, we compute $\tilde{X}_{02}(i)$, $\tilde{X}_{12}(i)$ by:

$$\begin{aligned}\tilde{X}_{02}(i) =& -[A_{20}^T(i)\tilde{X}_2(i)+\tilde{X}_0(i)A_{02}(i)+\tilde{X}_{01}(i)A_{12}(i)+C_{00}^T(i)\tilde{X}_0(i)C_{02}(i)\\ &-\tilde{\tilde{F}}_0^T(i)(R(i)+D_0^T(i)\tilde{X}_0(i)D_0(i))\tilde{\tilde{F}}_2(i)+M_{02}(i)]A_{22}^{-1}(i)\end{aligned} \tag{40}$$

$$\begin{aligned}\tilde{X}_{12}(i) =& -[A_{21}^T(i)\tilde{X}_2(i)+\tilde{X}_1(i)A_{12}(i)+C_{01}^T(i)\tilde{X}_1(i)A_{12}(i)+C_{01}^T(i)\tilde{X}_0(i)C_{02}(i)\\ &-\tilde{\tilde{F}}_1^T(i)(R(i)+D_0^T(i)\tilde{X}_0(i)D_0(i))\tilde{\tilde{F}}_2(i)+M_{12}(i)]A_{22}^{-1}(i).\end{aligned} \tag{41}$$

Applying Lemma 3.1 (ii), Lemma 3.2 (ii) together with (36)-(41) we obtain that

$$(\tilde{X}_0(i),\tilde{X}_{01}(i),\tilde{X}_1(i),\tilde{X}_{02}(i),\tilde{X}_{12}(i),\tilde{X}_2(i),\tilde{\tilde{F}}_0(i),\tilde{\tilde{F}}_1(i),\tilde{\tilde{F}}_2(i)), i\in\mathfrak{N}$$

is a solution of the system (14) obtained starting from the stabilizing solution of SCRARE (30).
In order to obtain the existence of the stabilizing solution of SARE (10) satisfying the sign condition (11), we shall use the implicit function theorem applied to the system (13). To this end, we regard (13) as an equation of type

$$\Phi(\mathbb{W},\nu)=0 \tag{41}$$

on the finite dimensional Banach space

$$\begin{aligned}\mathcal{W} =& \mathcal{S}_{n_0}^N\times(\mathbb{R}^{n_0\times n_1})^N\times\mathcal{S}_{n_1}^N\times(\mathbb{R}^{n_0\times n_1})^N\times(\mathbb{R}^{n_1\times n_2})^N\\ &\times\mathcal{S}_{n_2}^N\times(\mathbb{R}^{m\times n_0})^N\times(\mathbb{R}^{m\times n_1})^N\times(\mathbb{R}^{m\times n_2})^N.\end{aligned}$$

In (42) $\nu$ stands for $(\varepsilon_1,\varepsilon_2,\frac{\varepsilon_2}{\varepsilon_1},\varepsilon_1^\delta,\varepsilon_1^{2\delta-1},\varepsilon_2^\delta,\varepsilon_2^{2\delta-1},(\frac{\varepsilon_2}{\varepsilon_1})^\delta)$. One shows that the assumptions of the implicit function theorem are fulfilled for equation (42) around to the special solution $\tilde{\mathbb{W}}=(W(1),W(2),\cdots,W(n))$ with

$$W(i)=(\tilde{X}_0(i),\tilde{X}_{01}(i),\tilde{X}_1(i),\tilde{X}_{02}(i),\tilde{X}_{12}(i),\tilde{X}_2(i),\tilde{\tilde{F}}_0(i),\tilde{\tilde{F}}_1(i),\tilde{\tilde{F}}_2(i))$$

and $\nu=0$.
Thus we obtain the main results of this work:

**Theorem 3.2.** *Assume:*

*(a) the matrices $A_{22}(i)$ and $A_{11}^1(i)$ are invertible;*

*(b) conditions from (ii) of Theorem 3.1 are fulfilled.*

*Then the following hold: there exists $\varepsilon_k^*>0$, $\rho^*>0$, with the property that for any $0<\varepsilon_k<\varepsilon_k^*$, $k=1,2$ and $0<\varepsilon_2/\varepsilon_1<\rho^*$ the SARE (10) has a stabilizing solution $\tilde{\mathbf{X}}(\varepsilon_1,\varepsilon_2)=(\tilde{X}(\varepsilon_1,\varepsilon_2,1),\cdots,\tilde{X}(\varepsilon_1,\varepsilon_2,N))$ satisfying the* *sign condition (11). Furthermore the solution* $\tilde{\mathbf{X}}(\varepsilon_1, \varepsilon_2)$ *has the asymptotic structure:*

$$\tilde{X}(\varepsilon_1, \varepsilon_2, i) = \begin{pmatrix} \tilde{X}_0(i)+O(\nu) & \varepsilon_1(\tilde{X}_{01}(i)+O(\nu)) & \varepsilon_2(\tilde{X}_{02}(i)+O(\nu)) \\ \varepsilon_1(\tilde{X}_{01}^T(i)+O(\nu)) & \varepsilon_1(\tilde{X}_1(i)+O(\nu)) & \varepsilon_2(\tilde{X}_{12}(i)+O(\nu)) \\ \varepsilon_2(\tilde{X}_{02}^T(i)+O(\nu)) & \varepsilon_2(\tilde{X}_{12}^T(i)+O(\nu)) & \varepsilon_2(\tilde{X}_2(i)+O(\nu)) \end{pmatrix}$$

*and the corresponding feedback gain (9) of the optimal control has the asymptotic structure:*

$$\tilde{F}(\varepsilon_1, \varepsilon_2, i) = \begin{pmatrix} \tilde{\tilde{F}}_0(i) & \tilde{\tilde{F}}_1(i) & \tilde{\tilde{F}}_2(i) \end{pmatrix} + O(\nu). \tag{42}$$

## 3.4. NEAR OPTIMAL LQ REGULATOR

The asymptotic structure (43) of the optimal feedback allows us to design a near optimal control whose gain matrices do not depend upon small parameters $\varepsilon_k$, $k = 1, 2$.

**Theorem 3.3.** *Assume that the assumptions of Theorem 3.2 are fulfilled. Consider the control*

$$u_{app}(t) = \tilde{\tilde{F}}_0(\eta_t)x_0(t) + \tilde{\tilde{F}}_1(\eta_t)x_1(t) + \tilde{\tilde{F}}_2(\eta_t)x_2(t) \tag{43}$$

*whose gain matrices* $\tilde{\tilde{F}}_j$, $0 \le j \le 2$ *are computed via (36)- (38) based on the stabilizing feedback gains* $(\tilde{F}_0(i), \tilde{F}_1(i), \tilde{F}_2(i))$ *defined in (34) corresponding to the stabilizing solution* $\tilde{\mathbf{X}}$ *of SCRARE (30). Under the considered assumptions the control (44) stabilizes the system (1) for* $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ *small enough. An upper bound of the loss of performance is given by:*

$$0 \le J(x_0, u_{app}) - J(x_0, \tilde{u}) \le \gamma \|\nu\|^2 |x_0|^2.$$

**Remark 3.1.** *The gain matrices of the control (44) do not depend upon the small parameters* $\varepsilon_1$, $\varepsilon_2$ *and* $\frac{\varepsilon_2}{\varepsilon_1}$. *They may be computed using (34), (36)-(38), based on the stabilizing solution of SCRARE (30), satisfying the sign conditions (35).*

## 4. CONCLUSION

We have addressed in this note an LQ-type control problem for a class of Itô differential equations with two fast time scales and Markovian jumping of there parameters. Our goal was to perform a detailed investigation of the $\varepsilon$-dependence of the stabilizing solution of the Riccati equation involved in the construction of the optimal control of the considered LQ problem. The asymptotic structure of the stabilizing solution of the algebraic Riccati equation associated to the considered LQ control problem was obtained applying the implicit functions theorem. Based on the dominant part of the gain matrices of the optimal control we have constructed a near optimal $\varepsilon$-independent control law.

## References


[1] V. Dragan, A. Halanay, *Suboptimal Stabilization of Linear Systems with Several Time Scales,* Int. J. of Control, **36**(1982), 109-126.

[2] V. Dragan V, A. Halanay , *Stabilization of Linear Systems*, Birkhauser: Boston, 1999.

[3] V. Dragan and T. Morozan, *Exponential stability for a class of linear time-varying singularly perturbed stochastic systems*, Dynamics of Continuous, Discrete and Impulsive Systems, Series B: Applications & Algorithms, **9** (2002), 213-231.

[4] V. Dragan, T. Morozan and P. Shi, *Asymptotic Properties of Input-Output Operators Norm Associated with Singularly Perturbed Systems with Multiplicative White Noise*, SIAM Journal on Control and Optimization, **41** (2002), 141-163.

[5] V. Dragan, H. Mukaidani and P. Shi, *The Linear Quadratic Regulator Problem for a Class of Controlled Systems Modeled by Singularly Perturbed Itô Differential Equations*, SIAM Journal on Control and Optimization, **50** (2012), 448-470.

[6] V. Dragan, H. Mukaidani, *Optimal control for a singularly perturbed linear stochastic system with multiplicative white noise perturbations and Markovian jumping*, Optim. Control Appl. Meth., **38** (2017), 205-228.

[7] V. Dragan, T. Morozan, A.M. Stoica, *Mathematical methods in robust control of linear stochastic systems*, Springer New-York, 2013, second edition.

[8] Z. Gajic, M. T. Lim, *Optimal Control of Singularly Perturbed Linear Systems and Applications*, Marcel Dekker Inc.:New York, 2001.

[9] E. Ya. Gorelova, *Stability of singularly perturbed stochastic systems* Automation and Remote Control, **58** (1997), 112–121.

[10] P. V. Kokotovic, H. Khalil, J. Oreilly, *Singular Perturbations Methods in Control: Analysis and Design*, Academic Press: London, 1986.

[11] X. Li, X.Y. Zhou, M.A.Rami, *Indefinite Stochastic Linear Quadratic Control with Markovian Jumps in Infinite Time Horizon,* J. Global Optim., **27**(2003), 149 - 175.

[12] D. S. Naidu, *Singular Perturbation Methodology in Control Systems*, Peter Peregrinus Limited: Stevenage Herts, 1988.

[13] D. S. Naidu, M. S. Krishnarayalu, *Singular perturbation method for initial value problems in two-parameter discrete control systems*, Internat. J. Systems Sci., **18** (1987), 2197-2208

[14] Z. Pan and T. Basar, *Model Simplification and Optimal Control of Stochastic Singularly Perturbed Systems under Exponentiated Quadratic Cost*, Proceedings of the 33rd Conference on Decision and Control, Lake Buena Vista, FL, 1994, 1700-1705.

[15] Z. Pan, T. Basar, *Multi-time scale zero-sum differential games with perfect states measurements*, Dynamics and Control, **5**(1995), 7-29.

[16] L. Rybarska-Rusinek and L. Socha, *String Stability of Singularly Perturbed Stochastic Systems*, Stochastic Analysis and Applications, **25** (2007), 719–737.

[17] L. Socha, *Exponential stability of singularly perturbed stochastic systems*, IEEE Transactions on Automatic Control, **45** (2000), 576-580.