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ON THE REGULARIZATION OF SOME SINGULAR INTEGRAL OPERATORS IN THE SPACES WITH WEIGHTS

Petru Moloşnic, Vasile Nyaga
State Agrarian University of Moldova, Chişinău, Republic of Moldova
Tiraspol State University, Chişinău, Republic of Moldova
vasileneagu45@gmail.com

Abstract A method for regularization of singular integral equations containing an involution operator, in particular a complex conjugation operator or a Carleman type translation is studied. The proposed method allows to determine the resolvability conditions and the Noetherian equations for such equations, as well as the formula for calculating the indices of these equations.

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1. INTRODUCTION

Let us consider the singular integral equation

$$\begin{aligned}(A\varphi)(t) = a(t)\varphi(t) + b(t)(S\varphi)(t) + c(t)(V\varphi)(t) + \\ + d(t)(SV\varphi)(t) + (T\varphi)(t) = f(t)\end{aligned} \tag{1}$$

where $a(t)$, $b(t)$, $c(t)$, $d(t)$, $f(t)$ are given functions on a curve Γ; φ is an unknown function; S is the operator of singular integration,

$$(S\varphi)(t) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}d\tau \quad (t \in \Gamma),$$

$(W\varphi)(t) = \overline{\varphi(t)}$ (or $(V\varphi)(t) = \varphi(\omega(t))$ and T is a compact operator in the same space, where φ is contained. An operator

$$A = aI + bS + (cI + dS)V + T \tag{2}$$

is called a general singular operator with shift and the operator

$$A = aI + bS + (cI + dS)V \tag{3}$$

is called a characteristic singular integral operator.

By constructing the theory of integral equations (1) whis $\varphi \in L_p(\Gamma, \rho)$ (see [1]–[3]) the following properties of integral operators we will be used:

1) if $\varphi \in L_p(\Gamma, \rho)$, then $(S\varphi)(t)$ is almost everywhere finite;

2) the operators S and V are bounded in the space $L_p(\Gamma, \rho)$;

3) if $a(t)$ is a continuous function on Γ then the operator $aS - SaI$ is compact in the space $L_p(\Gamma, \rho)$;

4) the operator $VS + \varepsilon SV$ is compact in the space $L_p(\Gamma, \rho)$, where $\varepsilon = 1$ or $\varepsilon = -1$.

It is well known (see [1]–[3]) that the operator S has the properties 1), 2) and 3) if Γ is a Lyapunov curve. Naturally the problem arises to justify these properties for a more large class of curves than Lyapunov. In particular, to establish if the properties 1)–4) hold in the case of a smooth curve Γ. Unfortunately, this problem is still open. There are found different classes of nonsmooth curves for which the properties 1)–3) hold (see [4]). In [5]–[7] it was established that if Γ is a piecewise smooth curve, then property 4) is not realized.

By constructing the theory of singular equations (1) we shall require from Γ to provide some properties of the operators S and V. In this paper we realize this task in solving of the problem of regularization for operator (2), important from our point of view. Note that the Noether criteria for the operator (2) are established in [2]. We shall use these results proving the necessity of the existence of regularization for (2).

2. CONTINUITY OF THE OPERATOR A

Let Γ be a simple Lyapunov oriented contour on the complex plane and $\alpha : \Gamma \to \Gamma$ be a homeomorphic map of contour Γ on itself satisfying the conditions:

i) $\alpha(\alpha(t)) \equiv t \ \ (\alpha(t) \not\equiv t)$;

ii) there exists $\alpha'(t) \neq 0$ and $\alpha'(t) \in H_\mu(\Gamma)$.

Under these conditions the following theorem holds

Theorem 2.1. *The operator A defined by equality* (2) *is bounded in the space $L_p(\Gamma, \rho)$, where*

$$\rho(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k}, \ \ t_k \in \Gamma, \ -1 < \beta_k < p - 1, \ k = \overline{1, n}. \tag{4}$$

Proof. The boundedness of the operator $(V\varphi)(t) = \overline{\varphi(t)}$ is evident and the boundedness of the operator S was proved in [1]. Let us prove the boundedness of the operator $(V\varphi)(t) = \varphi(\alpha(t))$. We have

$$|V\varphi|^p = \int_\Gamma |\varphi(\alpha(t))|^p \prod_{k=1}^{n} |t - t_k|^{\beta_k} \, |dt| =$$

$$= \int_\Gamma |\varphi(t)|^p \prod_{k=1}^n |\alpha(t) - t_k|^{\beta_k} \left|\alpha'(t)\right| |dt| \leq$$

$$\leq \max_t \left|\alpha'(t)\right| \int_\Gamma |\varphi(t)|^p \prod_{k=1}^n \left|\frac{\alpha(t) - t_k}{t - t_k}\right|^{\beta_k} \prod_{k=1}^n |t - t_k|^{\beta_k} |dt| . \tag{5}$$

Since α' satisfies the Hölder condition, the function $\prod_{k=1}^n \left|\frac{\alpha(t)-t_k}{t-t_k}\right|^{\beta_k}$ is bounded on Γ. Hence

$$|(V\varphi(t))|^p \leq \max_t \left|\alpha'(t)\right| \sup_t \prod_{k=1}^n \left|\frac{\alpha(t) - t_k}{t - t_k}\right|^{\beta_k} \|\varphi\|^p .$$

The boundedness of the operator $(V\varphi)(t) = \varphi(\alpha(t))$ is proved. Theorem 2.1 is proved. ∎

Let us consider the case of an unbounded contour. An unbounded contour Γ is called admissible, if by mapping $t \to (z - z_0)^{-1}$ $(z_0 \notin \Gamma)$ it passes in a simple Lyapunov contour $\widetilde{\Gamma}$. We say that a function $\alpha : \Gamma \to \Gamma$ belongs to a class $V(\Gamma)$ if it satisfies the conditions:

$i')$ $\alpha(\alpha(t)) \equiv t$;

$ii')$ there exists $\alpha'(t)$ $(\alpha'(t) \neq 0)$ and a function $\left[\alpha'(t)(t - z_0)^2\right] / (\alpha(t) - z_0)^2$ belongs to $H_\mu(\Gamma)$.

Here it is also assumed that there exist the finite limits

$$\lim_{t\to\infty} \left[\alpha'(t)(t - z_0)^2\right] / (\alpha(t) - z_0)^2 \ (t \in \Gamma)$$

and

$$\lim_{t\to\alpha(\infty)} \left[\alpha'(t)(t - z_0)^2\right] / (\alpha(t) - z_0)^2 \ (t \in \Gamma).$$

In the case of an unbounded contour we can take as the operator of shift the following operator

$$(V\varphi)(t) = \frac{\alpha(t) - z_0}{t - z_0} \varphi(\alpha(t)). \tag{6}$$

We examine the following example. Assume that $\Gamma = [-\infty, +\infty]$, $\alpha : \Gamma \to \Gamma$, $\alpha(t) = \frac{1}{t}$ and let $z_0 = -i$, then $\frac{\alpha'(t)(t-z_0)^2}{(\alpha(t)-z_0)^2} = \left(\frac{t+i}{t-i}\right)^2$ and $(V\varphi)(t) = \frac{i}{t}\left(\frac{t+i}{t-i}\right)\varphi(\frac{1}{t})$. Obviously $\alpha(t) \in V(\Gamma)$.

Note that the operator V defined by equality (6) and S satisfies the conditions 1)–4).

Theorem 2.2. *Let Γ be an admissible contour and $\alpha(t_0) = t_0$ $(t_0 \neq \infty)$. Then the operator A is bounded in the spaces $L_p(\Gamma, |t - z_0|^{p-2})$ and $L_p(\Gamma, |t - t_0|^{\frac{p}{2}-1})$.*

The proof of this theorem follows easily from the results of the paper [6].

3. COMPACTNESS OF THE OPERATOR $K_\omega = S\omega I - \omega S$

3.1. THE CASE OF A BOUNDED CONTOUR

Theorem 3.1. *Let $\omega \in C(\Gamma)$. Then the operator $K_\omega = S\omega I - \omega S$ is compact in the space $L_p(\Gamma, \rho)$.*

Proof. In the case when $\omega(t)$ is a polynomial or a rational function the validity of theorem is evident by finite dimentionality of the operator K_ω in the space $L_p(\Gamma, \rho)$. If $\omega(t)$ is an arbitrary continuous function on Γ then we can find a sequence $\{r_n(t)\}$ of polynomials (if Γ is an open curve) or of rational functions (if Γ is a closed curve) uniformly convergent on Γ to $\omega(t)$.

It is easy to see that if $\omega(t)$ is a bounded measurable function on Γ, then the operator $S\omega I - \omega S$ is bounded in $L_p(\Gamma, \rho)$. Therefore, we have

$$|K_\omega - K_{r_n}| \to 0, \ n \to \infty. \tag{7}$$

From (7) it follows the validity of theorem. ∎

Theorem 3.1 was proved for the first time by Mihlin in the case of space $L_2(\Gamma)$. When Γ is a Lyapunov curve it was studied by Hvedelidze in [1].

On interest is also the investigation of the following problem: does the prove of Theorem 3.1 hold true if the function is piecewise continuous on Γ. The answer is negative. We shall find a concrete piecewise continuous function ω for which the operator K_ω will be not compact in the space $L_p(\Gamma)$.

Let Γ_{ab} be an open curve, where a and b are its endpoints. Let c be some fixed interior point on Γ_{ab}. Consider the function

$$\omega(t) = \begin{cases} 0, & if\, t \in \Gamma_{ac} \\ 1, & if\, t \in \Gamma_{cb}, \end{cases} \tag{8}$$

where $\Gamma_{ab} = \Gamma_{ac} \bigcup \Gamma_{cb}$.

Consider in $L_p(\Gamma_{ab})$ the set of functions $\{\varphi_n(t)\}$, $n \in N$, where

$$\varphi_n(t) = \begin{cases} 0, & if\, t \in \Gamma_{ac} \bigcup \Gamma_{c_n b} \\ n^{1/p}, & if\, t \in \Gamma_{cc_n}, \end{cases} \tag{9}$$

and c_n is a point on Γ_{cb} such that the length of the curve $\Gamma_{cc_n}$is equal to $\frac{1}{n}$. In this case

$$\|\varphi_n\|^p_{L_p(\Gamma)} = \int_{\Gamma_{ab}} |\varphi_n(t)|^p \, |dt| = n \int_{\Gamma_{cc_n}} |dt| = 1.$$

Let us find the image of a set $\{\varphi_n(t)\}$ by map K_ω :

$$(K_\omega \varphi_n)(t) = \int_{\Gamma_{cc_n}} \frac{\omega(\tau) - \omega(t)}{\tau - t} \varphi_n(\tau) d\tau = n^{1/p} \chi(\Gamma_{ac}) Ln \frac{c_n - t}{c - t},$$

where $\chi(\Gamma_{ac})$ is a characteristic function of a set Γ_{ac}.

Let λ be the length of the curve Γ_{ac} and $\Phi_n(t) = (K_\omega \varphi_n)(t)$. Then for every $\varepsilon > 0$ we have

$$\int_{\Gamma_{ab}} |\Phi_n(t(s+\varepsilon)) - \Phi_n(t(s))|^p \, ds \geq \int_{\lambda-\varepsilon/2}^{\lambda} |\Phi_n(t(s+\varepsilon)) - \Phi_n(t(s))|^p \, ds =$$

$$\int_{\lambda-\varepsilon/2}^{\lambda} |\Phi_n(t(s))|^p \, ds \geq n \int_{\lambda-\varepsilon/2}^{\lambda} \left| \ln \left| \frac{c_n - t}{c - t} \right| \right|^p ds \tag{10}$$

Denote the length of the curves Γ_{tc_n} and Γ_{tc} by $\lambda_{tc_n}$and λ_{tc}, respectively. Then taking into account that Γ_{ab} is a piecewise Lyapunov curve without cusps, we have

$$\left| \frac{c_n - t}{c - t} \right| \geq m(1 + \frac{\lambda_{cc_n}}{\lambda_{tc}}) \geq m(1 + \frac{2}{n\varepsilon}).$$

Put now $\varepsilon = \frac{2m}{n}$, then

$$\left| \frac{c_n - t}{c - t} \right| \geq 1 + m.$$

From relation (10) we get

$$\int_{\Gamma_{ab}} |\Phi_n(t(s+\varepsilon)) - \Phi_n(t(s))|^p \, ds \geq m \, |\ln(1+m)|^p \, .$$

Hence by Riesz's theorem on compactness of functions in the space L_p, the set of functions $\Phi_n(t) = (K_\omega \varphi_n)(t)$ is not compact in $L_p(\Gamma_{ab})$ and therefore the operator K_ω is not compact in $L_p(\Gamma_{ab})$. Thus if the function ω has the points of discontinuity, then the operator K_ω is not compact. This statement remains true in the space $L_p(\Gamma, \rho)$. The proof can be done similarly.

In the case of a Lyapunov curve by Gohberg and Krupnik in [3] it was proved the following theorem.

Theorem 3.2. *If $\omega(t)$ is a piecewise continuous function on Γ, then operator $K_\omega = S\omega I - \omega S$ is compact in $L_p(\Gamma, \rho)$ if and only if $\omega(t)$ is a continuous function.*

We shall show that Theorem 3.2 can be proved rather simple for more general curve such that $\forall t_1, t_2 \in \Gamma$, $|t_1 - t_2| \geq m\, l(t_1, t_2)$, where $m = const$, $l(t_1, t_2)$ is the length of the arc (the smallest if Γ is closed connecting the points t_1 and t_2), if we take into consideration the above constructed example. It is sufficient to show that if K_a is compact in $L_p(\Gamma)$, then a function $a \in C(\Gamma)$. Suppose the contrary that K_a is compact in $L_p(\Gamma)$ and the function a has one point of discontinuity $c \in \Gamma$.

Let $\sigma = a(c+0) - a(c-0)$ and consider the function $a_1(t) = a(t) - \sigma\omega(t)$, where $\omega(t)$ is defined by equality (8). It is evident that $a_1 \in C(\Gamma)$ and therefore by Theorem 3.1, the operator $K_{a_1}$is compact in $L_p(\Gamma)$.

Now from the obvious equality

$$K_a = \sigma K_\omega + K_{a_1} \quad \Rightarrow \quad K_\omega = \frac{1}{\sigma}(K_a - K_{a_1})$$

it follows that the operator K_ω is compact and we have contradiction with the above obtained result.

Let us study now the operator K_a when it acts from one functional Lebesgue space into another. Denote by $CP(\Gamma, t_1, t_2, \ldots, t_n)$ the set of all continuous functions on Γ with the exception of points $t_2, \ldots, t_n$ at which there exist $a(t_k+0)$ and $a(t_k-0)$.

Theorem 3.3. *Let $a \in CP(\Gamma, t_1, t_2, \ldots, t_n)$, $b \in C(\Gamma)$ and $b(t_k) = 0$, $k = \overline{1,n}$, then the operator bK_a is compact in the space $L_p(\Gamma)$.*

Proof. Taking into account the identity

$$b(t)\,[a(\tau) - a(t)] = [a(\tau)b(\tau) - a(t)b(t)] - [b(\tau) - b(t)]\,a(\tau)\,,$$

we get

$$bK_a = K_{ab} - K_b aI. \tag{11}$$

From Theorem 3.1 and equality (11) by taking into account that the functions ab and b are continuous on Γ it follows the validity of Theorem 3.3. ∎

Remark 3.1. *It is clear that if S is bounded in some space $L_p(\Gamma, \rho)$, then in Theorem 3.3 we can replace $L_p(\Gamma)$ with the space $L_p(\Gamma, \rho)$.*

Consequence 3.1. *Let $a \in CP(\Gamma, t_1, t_2, \ldots, t_n)$, $b \in C(\Gamma)$ and $b(t_k) = 0, k = 1, 2, \ldots, n$, then the operator $K_a \in L(L_p(\Gamma), L_p(\Gamma, b))$ is compact.*

The validity of Consequence 3.1 follows from equality $K_a = b^{-1}bK_a$ taking into account Theorem 3.3 and relation $b^{-1}I \in L(L_p(\Gamma), L_p(\Gamma, b))$.

Theorem 3.4. *Let $a \in CP(\Gamma, t_1, t_2, \ldots, t_n)$, then $\forall p_1 \in (1, p)$ the operator $K_a \in L(L_p(\Gamma), L_{p_1}(\Gamma))$ is compact.*

Proof. Put

$$h(t) = \prod_{k=1}^{n} |t - t_k|^{\gamma_k}\,, \tag{12}$$

where $\gamma_1, \gamma_2, \ldots, \gamma_n$ are positive numbers satisfying the condition

$$\max(\gamma_1, \gamma_2, \ldots, \gamma_n) < \frac{p - p_1}{pp_1}. \tag{13}$$

Denote $r = \frac{p}{p_1}$ and apply the Hölder inequality. We obtain

$$\|\varphi\|^{p_1}_{L_{p_1}} = \int_\Gamma |h(t)\varphi(t)|^{p_1} |h(t)|^{-p_1} |dt| \le c^{p_1} \|\varphi\|^{p_1}_{L_p(\Gamma,h)},$$

where

$$c^{p_1} = (\int_\Gamma |h(t)|^{-p} |dt|)^{1/r'} \qquad (r' = r/(r-1).$$

From relation (13) and $r = \frac{p}{p_1}$ we conclude that $c^{p_1} < +\infty$. Hence

$$\|\varphi\|_{L_{p_1}(\Gamma)} \le c \|\varphi\|_{L_p(\Gamma,h)} \quad \forall \varphi \in L_p(\Gamma, h). \tag{14}$$

The assertion of Theorem 3.4 follows from the consequence to Theorem 3.3 and the inequality (14). ∎

Theorem 3.5. *Let $ha \in CP(\Gamma, t_1, t_2, \dots, t_n)$, where h is defined by equality* (12)*. Then operators $hK_a \in L(L_p(\Gamma), L_p(\Gamma, \rho))$, $K_a \in L(L_p(\Gamma), L_p(\Gamma, \rho^2))$ are compact, where*

$$\rho(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k}, 0 < \beta_k < \frac{1}{q}, \; q = \frac{p}{p-1}, \; k = \overline{1, n}. \tag{15}$$

Proof. Denote

$$a_0(t) = h(t)a(t). \tag{16}$$

In virtue of equality (11), we have

$$hK_a = K_{a_0} - K_h aI. \tag{17}$$

Assume that $a_0 \in C(\Gamma)$. In this case the validity of theorem follows from Theorems 3.1, 3.4 and relation

$$aI, h^{-1} \in L(L_p(\Gamma), L_p(\Gamma, \rho)) \cap L(L_p(\Gamma, \rho), L_p(\Gamma, \rho^2)), K_a = hh^{-1}K_a.$$

Suppose that $a_0 \in CP(\Gamma, t_1, t_2, \dots, t_n)$. In this case we can represent the numbers γ_k in the form of a sum of two positive numbers $\gamma_k = \gamma'_k + \gamma''_k$, $k = 1, 2, \dots, n$. Then $hI = h'Ih''I$, where

$$h'(t) = \prod_{k=1}^{n} |t - t_k|^{\gamma'_k}, h''(t) = \prod_{k=1}^{n} |t - t_k|^{\gamma''_k}.$$

Taking into account the equality (11) we obtain

$$hK_a = h'h''K_a = h'K_{ah''} - K_{h''}aI.$$

By analogy with the last case, the assertion of Theorem 3.5 follows from the last equality. ∎

Theorem 3.6. *If $ha \in CP(\Gamma, t_1, t_2, ..., t_n)$, where h is defined by equality (12), then $\forall p_1 \in (1, p)$ the operator $K_a \in L(L_p(\Gamma), L_{p_1}(\Gamma))$ is compact.*

Proof. By the last theorem it is sufficient to prove that

$$\|\varphi\|_{L_{p_1}(\Gamma)} \leq c \|\varphi\|_{L_p(\Gamma,\rho^2)} . \tag{18}$$

The numbers β_k in the definition of a function $\rho(t)$ are bounded in the following way

$$\max(\beta_1, \beta_2, ..., \beta_n) < \frac{p - p_1}{2pp_1}. \tag{19}$$

Let $r = \frac{p}{p_1}$. By Hölder inequality, we get

$$\|\varphi\|^{p_1}_{L_{p_1}} = \int_\Gamma \left|\rho^2(t)\varphi(t)\right|^{p_1} \rho^{-2p_1}(t) \, |dt| \leq$$

$$(\int_\Gamma \left|\rho^2(t)\varphi(t)\right|^{rp_1})^{1/r} |dt| \, (\int_\Gamma \left|\rho^{-2p_1 r'}(t)\right|)^{rp_1} |dt|)^{1/r'} = c^{p_1} \|\varphi\|^{p_1}_{L_p(\Gamma,\rho^2)} ,$$

where

$$c^{p_1} = (\int_\Gamma \left|\rho^{-2p_1 r'}(t)\right|)^{rp_1} |dt|)^{1/r'}. \tag{20}$$

The condition (19) provide the finiteness of the integral (20). Hence, it holds the inequality (18). ∎

3.2. THE CASE OF AN UNBOUNDED CONTOUR

Let Γ be an admissible unbounded contour. Without loss of generality we can assume that a point $z = 0 \notin \Gamma$. Denote by $C(\overline{\Gamma})$ the set of continuous functions ($f(\infty - 0) = f(\infty + 0)$) on $\overline{\Gamma}$ and by $ha \in CP(\overline{\Gamma}, z_1, z_2, \ldots, z_n, \infty)$ the set of functions continuous to the left on Γ, having a finite number of points $z_1, z_2, \ldots, z_n$ of discontinuity of the first kind on Γ and finite limits $a(\infty - 0)$ and $a(\infty + 0)$. By $L_p(\Gamma, \rho)$ we denote the space L_p on Γ with the weight

$$\rho(t) = |z|^\beta \prod_{k=1}^{n} |z - z_k|^{\beta_k} , \tag{21}$$

where

$$-1 < \beta_k < p - 1, -1 < \beta + \sum_{k=1}^{n} \beta_k < p - 1. \tag{22}$$

Theorem 3.7. *If numbers β, β_k $(k=\overline{1,n})$ satisfy conditions* (22)*, then S is bounded in the space $L_p(\Gamma,\rho)$.*

Proof. Let Γ_0 be an image of the curve Γ by mapping $t=z^{-1}$. Assume $t_k = z_k^{-1}$ $(k=\overline{1,n})$, $z_0=0$, $\beta_0 = p-2-\beta-\sum_{k=1}^{n}\beta_k$ and $\rho_0(t)=\prod_{k=0}^{n}|t-t_k|^{\beta_k}$. It is easy to verify that operator B defined by equality

$$(B\varphi)(t)=\frac{1}{t}\varphi(\frac{1}{t}) \quad (t\in\Gamma_0) \tag{23}$$

in [8] is linear bounded by inverse operator acting from $L_p(\Gamma,\rho)$ into $L_p(\Gamma_0,\rho_0)$. From condition (22) it follows that $-1<\beta_k<p-1$ $(k=0,1,2,...,n)$. By [1] the operator

$$(S_0\varphi)(t)=\frac{1}{\pi\, i}\int_{\Gamma_0}\frac{\psi(\tau)}{\tau-t}d\tau$$

is bounded in the space $L_p(\Gamma_0,\rho_0)$. It can be verified directly that

$$B^{-1}S_0B=S. \tag{24}$$

From here it follows the boundedness of the operator S in the space $L_p(\Gamma,\rho)$. ∎

Remark 3.2. *Theorems similar to the Theorems 3.2–3.6 can be proved using equality* (24)*. Indeed, from conditions $a\in CP(\overline{\Gamma},z_1,z_2,\dots,z_n,\infty)$ it follows that $\tilde{a}(t)=a(\frac{1}{t})\in CP(\Gamma_0,t_1,t_2,\dots,t_n,0)$ and the equality* (24) *for the operator K_a gives*

$$B^{-1}K_aB=B_{\tilde{a}}. \tag{25}$$

From this equality the assertions similar to Theorems 3.2–3.6 follow at once.

4. REGULARIZATION OF THE GENERAL SINGULAR OPERATOR IN THE CASE OF CONTINUOUS COEFFICIENTS

Let $A\in L(B)$, where B is a Banach space. An operator $M\in L(B)$ is said to be regularizing for A in a space B if the operators $AM-I$ and $MA-I$ are compact in B.

Later on we shall denote by T the compact operators in $L_p(\Gamma)$. Moreover, we shall assume that Γ is a closed Lyapunov curve. It is known that operator $A\in L(B)$ is Fredholm if and only if it admits regularization.

Let $A\in L(B)$ be some Fredholm operator. If the regularizing operator M for A is known, then the solution of the equation

$$Ax=y \tag{26}$$

can be reduced to the solution of the equation

$$MAx=My \tag{27}$$

in which the operator $MA-I$ is compact. Many methods known for conversion of the operator $I+T$ can be applied to equations (26). It is clear that a special interest represents the case when equations (26) and (27) are equivalent for every vector y. This can occur if only if $KerM = \{0\}$. Indeed, if $MAx = 0$, then $Ax = z$, where $z \in KerM$.

Assume that equations (26) and (27) are equivalent, then either $KerM = \{0\}$ or $\dim KerM > 0$ and $KerM \cap ImA = \{0\}$. The last equality is impossible. Since in this case the equations $Ax = z$ $(z \in KerM)$ and $MAz = Mz = 0$ are not equivalent. Conversely, if $KerM = \{0\}$, then evidently, the equations (26) and (27) are equivalent.

We say that operator A admits an equivalent regularization if it has a regularizing operator M for which equations (26) and (27) are equivalent for every $y \in B$. In this case the operator M is called an equivalent regularizing operator for A.

From what has been said above it follows that operator M is an equivalent regularizing operator for A if and only if it is regularizing operator for A and reversible from the left.

Theorem 4.1. *Let $A \in L(B)$ admit the regularization. In order the regularization to be equivalent it is necessary and sufficient that*

$$IndA \geq 0. \tag{28}$$

Indeed, if M is an equivalent regularizing operator for A, then it is reversible from the left and $IndM \leq 0$. Since $IndMA = IndA + IndM = 0$, then $IndA \geq 0$. Conversely, let $IndA \geq 0$ and M_1 be a regularizing operator for A. Then M_1 is Fredholm operator and $IndA + IndM = 0$. Therefore $IndM_1 \leq 0$. Then the operator M_1 can be represented [9] in the form $M_1 = M+T$, where M is reversible from the left. Evidently, the operator M is equivalent regularizing for A. Theorem 4.1 is proved.

Consider now the case when the Fredholm operator A does not admit an equivalent regularization, i.e. the condition

$$IndA < 0 \tag{29}$$

holds. Let the operator M_1 be regularizing for A. Since $IndM_1 > 0$, then the operator M_1 can be represented [9] in the form $M_1 = M + T$, where M is reversible from the right. The operator M is also regularizing for A, where all the solutions of the equation

$$Ax = y \quad (y \in ImA) \tag{30}$$

can be obtained by formula $x = Mz$, where z describe all solutions of the equation $AMz = y$.

Let us consider now a general operator $A = A_0 + T$, where

$$A_0 = aI + bS + (cI + dS)V \tag{31}$$

is a characteristic singular operator with conjugation or with shift.

Later on, it is convenient to write the operator A_0 in the form

$$A_0 = (\alpha P + \beta Q) + (\gamma P + \delta Q)V, \tag{32}$$

where $P = (I + S)/2,\ Q = (I + S)/2,\ \alpha = a + b,\ \beta = a - b,\ \gamma = c + d$ and $\delta = c - d$.

Theorem 4.2. *Let $(V\varphi)(t) = \overline{\varphi(t)}$. If the functions $\alpha(t),\ \beta(t),\ \gamma(t),\ \delta(t)$ are continuous on Γ and*

$$\Delta(t) = \alpha(t)\overline{\beta(t)} - \gamma(t)\overline{\delta(t)} \neq 0,\ \ \forall t \in \Gamma, \tag{33}$$

then operator

$$M = \frac{\overline{\beta}}{\Delta}P + \frac{\overline{\alpha}}{\overline{\Delta}}Q - (\frac{\gamma}{\Delta}P + \frac{\delta}{\overline{\Delta}}Q)$$

is a regularization for the operator A in the space $L_p(\Gamma)$.

Proof. We recall that if Γ is a closed curve, then $S^2 = I$ and, hence,

$$P^2 = P,\ \ Q^2 = Q,\ \ PQ = QP = 0. \tag{34}$$

Moreover, from [2] it follows that

$$VPV = Q + T_1,\ \ VQV = P + T_2\ \ \ (T_1, T_2 \in T). \tag{35}$$

Note that

$$V^2 = I,\ \ VaV = \overline{a}I, \tag{36}$$

and from the last section we have

$$PaI - aP = \frac{I + S}{2}aI - a\frac{I + S}{2} = K_{a/2}, \tag{37}$$

$$QaI - aQ = \frac{I - S}{2}aI - a\frac{I - S}{2} = -K_{a/2} \tag{38}$$

If $a \in C(\Gamma)$, then $K_{a/2}$ is compact in $L_p(\Gamma)$. We have

$$MA_0 = \left[\frac{\overline{\beta}}{\Delta}P + \frac{\overline{\alpha}}{\overline{\Delta}}Q - (\frac{\gamma}{\Delta}P + \frac{\delta}{\overline{\Delta}}Q)\right][(\alpha P + \beta Q) + (\gamma P + \delta Q)V] =$$

$$= \frac{\overline{\beta}}{\Delta}(\alpha P + \gamma PV + T_3) + \frac{\overline{\alpha}}{\overline{\Delta}}(\beta Q + \delta QV + T_4) - \frac{\gamma}{\Delta}(\overline{\beta}PV + \overline{\delta}P + T_5) -$$

$$-\frac{\delta}{\overline{\Delta}}(\overline{\alpha}QV+\overline{\gamma}Q+T_6)=\frac{\alpha\overline{\beta}-\gamma\overline{\delta}}{\Delta}P+\frac{\overline{\alpha}\beta-\gamma\overline{\delta}}{\overline{\Delta}}Q+T_7=I+T_7. \tag{39}$$

Similarly we find that

$$A_0M=I+T_8 \quad (T_j\in T). \tag{40}$$

∎

Remark 4.1. *For arbitrary functions α, β, γ, $\delta\in CP(\Gamma,t_1,t_2,\ldots,t_n)$, satisfying condition* (33)*, the operator M is not a regularization operator for A in the space $L_p(\Gamma)$.*

Indeed, let $\gamma(t)=\delta(t)=0$ and $\alpha(t)=-\beta(t)$ have one point of discontinuity of the first kind. Then $A_0=\alpha S$, $\Delta=-|\alpha|^2$ and $M=\alpha^{-1}S$. From the relation $MA_0=\alpha^{-1}S\alpha S=I+T_7$ we get that $K_\alpha S\alpha I-\alpha S=T_9(T_9=S\alpha T_7)$ is compact, contrary to Theorem 3.2.

Let $(V\varphi)(t)=\varphi(\omega(t))$, where $\omega:\Gamma\to\Gamma$, $\omega(\omega(t))=t$ and $0\neq\omega'(t)\in H_\mu(\Gamma)$. In this case the following relation holds (see [2])

$$VSV=\varepsilon S+T, \tag{41}$$

where T is compact in $L_p(\Gamma)$ and $\varepsilon=1$ or $\varepsilon=-1$ in dependence of if ω preserves or changes the orientation on Γ.

Consider the operator

$$A_0=\alpha P+\beta Q+(\gamma P+\delta Q)V. \tag{42}$$

Theorem 4.3. *Let ω keeps the orientation on Γ. If the function $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\delta(t)$ are continuous on Γ and*

$$\begin{cases} \Delta_1(t)=\gamma(t)\tilde{\gamma}(t)+\alpha(t)\tilde{\alpha}(t)\neq 0, & \forall t\in\Gamma,\\ \Delta_2(t)=\delta(t)\tilde{\delta}(t)+\beta(t)\tilde{\beta}(t)\neq 0, & \forall t\in\Gamma,\end{cases} \tag{43}$$

where $\tilde{f}(t)=f(\omega(t))$, then the operator

$$M=\frac{\tilde{\alpha}}{\Delta_1}P+\frac{\tilde{\beta}}{\Delta_2}Q-(\frac{\gamma}{\Delta_1}P+\frac{\delta}{\Delta_2}Q)V$$

is a regularization for the operator A_0 in the space $L_p(\Gamma)$.

Theorem 4.4. *Let ω change the orientation on Γ. If the functions $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\delta(t)$ are continuous on Γ and*

$$\Delta_3(t)=\alpha(t)\tilde{\beta}(t)-\gamma(t)\tilde{\delta}(t)\neq 0, \quad \forall t\in\Gamma, \tag{44}$$

where $\tilde{f}(t) = f(\omega(t))$, *then the operator*

$$M = \frac{\tilde{\beta}}{\Delta_3}P + \frac{\tilde{\alpha}}{\tilde{\Delta}_3}Q - (\frac{\gamma}{\Delta_3}P + \frac{\delta}{\tilde{\Delta}_3}Q)V$$

is a regularization for the operator A *in the space* $L_p(\Gamma)$.

The proof of Theorems 4.3 and 4.4 is similar to that of Theorem 4.2 and will be not given here. The remark to Theorem 4.2 can be related also to Theorems 4.3 and 4.4.

Naturally the problem arises: are the conditions (33), (43) and (44) necessary for the operator A_0 to admit the regularization in the space $L_p(\Gamma)$? We shall prove that the answer to this problem is positive. We shall consider in more details the case, when the operator V is an operator of complex conjugation. The case when V is an operator with shift $(V\varphi)(t) = \varphi(\omega(t))$ is investigated similarly. In this connection we use the properties of involution of the operator V and the properties $VPV = P + T$ or $VQV = Q + T$.

Let $(V\varphi)(t) = \overline{\varphi(t)}$ and consider the operator

$$A_0 = \alpha P + \beta Q + (\gamma P + \delta Q)V. \tag{45}$$

Theorem 4.5. *Let* $\alpha,\ \beta,\ \gamma,\ \delta \in C(\Gamma)$. *Then the operator* A_0 *is Fredholm in the space* $L_p(\Gamma)$ *if and only if*

$$\Delta(t) = \alpha(t)\overline{\beta(t)} - \gamma(t)\overline{\delta(t)} \neq 0, \ \ \forall t \in \Gamma. \tag{46}$$

If the condition (46) *is satisfied, then*

$$IndA_0 = -ind\Delta(t). \tag{47}$$

Proof. According to works [2], [5], the operator A_0 is Fredholm in $L_p(\Gamma)$ if and only if the same property has the operator

$$\tilde{A}_0 = \left\| \begin{array}{cc} \alpha P + \beta Q & V(\alpha P + \beta Q)V \\ \gamma P + \delta Q & V(\gamma P + \delta Q)V \end{array} \right\| = \left\| \begin{array}{cc} \alpha & \overline{\delta} \\ \gamma & \overline{\beta} \end{array} \right\| P + \left\| \begin{array}{cc} \beta & \overline{\gamma} \\ \delta & \overline{\alpha} \end{array} \right\| Q + T,$$

in the space $L_p^2(\Gamma)$, where T is a compact operator in $L_p^2(\Gamma)$. Moreover, $IndA_0 = \frac{1}{2}Ind\tilde{A}_0$. The operator $\tilde{A}_0$ is a singular (without conjugation) operator with matrix coefficients. The operator $\tilde{A}_0$ is Fredholm in $L_p^2(\Gamma)$ if and only if $\Delta(t) = \alpha(t)\overline{\beta(t)} - \gamma(t)\overline{\delta(t)} \neq 0$ and $\overline{\Delta}(t) = \overline{\alpha(t)}\beta(t) - \overline{\gamma(t)}\delta(t) \neq 0, \ \ \forall t \in \Gamma$, and

$$Ind\tilde{A}_0 = -ind\frac{\Delta(t)}{\overline{\Delta}(t)}.$$

Hence the operator A_0 is Fredholm in the space $L_p(\Gamma)$ if and only if $\Delta(t) \neq 0$, moreover

$$IndA_0 = \frac{1}{2} Ind\tilde{A}_0 = -\frac{1}{2}\frac{\Delta(t)}{\overline{\Delta}(t)} = -ind\Delta(t)$$

and Theorem 4.5 is proved. ∎

Remark 4.2. *From Theorems 4.2 and 4.5 it follows that the operator* $A_0 = \alpha P + \beta Q + (\gamma P + \delta Q)V$ $((V\varphi)(t) = \overline{\varphi}(t))$ *admits the regularization in the space* $L_p(\Gamma)$ *if and only if* $\Delta(t) = \alpha(t)\overline{\beta(t)} - \gamma(t)\overline{\delta(t)} \neq 0,\ \forall t \in \Gamma$. *In this case the operator*

$$M = \frac{\bar{\beta}}{\Delta}P + \frac{\bar{\alpha}}{\overline{\Delta}}Q - (\frac{\gamma}{\Delta}P + \frac{\delta}{\overline{\Delta}}Q)V$$

is a regularization for A_0.

Remark 4.3. *The operator* $A_0 = \alpha P + \beta Q + (\gamma P + \delta Q)V$ *admits an equivalent regularization if and only if* $ind\,\Delta(t) \leq 0$.

The similar assertions are true for the operator A_0 with shift $((V\varphi)(t) = \varphi(\omega(t)))$. Let us state these assertions.

Theorem 4.6. *Let* ω *keep the orientation on* Γ *and* $(V\varphi)(t) = \varphi(\omega(t))$. *The operator* $A_0 = \alpha P + \beta Q + (\gamma P + \delta Q)V$ *admits the regularization in* $L_p(\Gamma)$ *if and only if*

$$\Delta_1(t) = \alpha(t)\tilde{\alpha}(t) - \gamma(t)\tilde{\gamma}(t) \neq 0,\ \ \Delta_2(t) = \beta(t)\tilde{\beta}(t) - \delta(t)\tilde{\delta}(t) \neq 0$$

on Γ. *If these conditions are satisfied, then the index of the operator* A_0 *is calculated by formula*

$$IndA_0 = -\frac{1}{2} ind\frac{\Delta_1(t)}{\Delta_2(t)}$$

Remark 4.4. *The operator* A_0 *admits an equivalent regularization if and only if*

$$ind\frac{\Delta_1(t)}{\Delta_2(t)} \leq 0.$$

Theorem 4.7. *Let* ω *change the orientation on* Γ *and* $(V\varphi)(t) = \varphi(\omega(t))$. *The operator* $A_0 = \alpha P + \beta Q + (\gamma P + \delta Q)V$ *admits the regularization in* $L_p(\Gamma)$ *if and only if*

$$\Delta_3(t) = \alpha(t)\tilde{\beta}(t) - \gamma(t)\tilde{\delta}(t) \neq 0,\ \forall t \in \Gamma.$$

If this condition is realized, then $IndA_0 = -ind\Delta_3(t)$ *and the operator*

$$M = \frac{\tilde{\beta}}{\Delta_3}P + \frac{\tilde{\alpha}}{\tilde{\Delta}_3}Q - (\frac{\gamma}{\Delta_3}P + \frac{\delta}{\tilde{\Delta}_3}Q)V$$

is the regularization for A_0.

Remark 4.5. *If we assume that* $\gamma(t) = \delta(t) \equiv 0$, *then we get the known results for singular operator* $\alpha P + \beta Q$.

Consider a particular case, i.e. the operator A_0 is of the form $A_0 = \alpha P + \beta Q$.

Theorem 4.8. *Let* α, $\beta \in C(\bar{\Gamma})$. *The operator* A_0 *is Fredholm in the space* $L_p(\Gamma, \rho)$ *if and only if the following relation* $\gamma(t) \neq 0, \beta(t) \neq 0$ $(t \in \bar{\Gamma})$ *is satisfied. If this condition is realized, then* $IndA_0 = -ind\alpha(t)\beta^{-1}(t)$ *and the operator*

$$M = \frac{1}{\alpha}P + \frac{1}{\beta}Q$$

is regularizing for A_0.

By condition $IndA_0 \geq 0$, the equations $A_0\varphi = f$ and $MA_0\varphi = Mf$ are equivalent for any right hand sides $f \in L_p(\Gamma, \rho)$. If $IndA_0 < 0$, then all the solutions of the equation $A_0\varphi = f$ are obtained by formula $\varphi = Mh$, where h passes over all the solutions of the equation $A_0Mh = f$.

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