ROMAI J., v.14, no.2(2018), 129–137

MULTI-INDEX TRANSPORT PROBLEM WITH NON-LINEAR COST FUNCTIONS

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Abstract In this paper we formulate the 4 index transport problem with the cost function described by a piecewise concave function. We describe the method to balance the problem and explain the economic significance of the manipulations. We describe a method of linearization of the non-linear objective function and bring a practical example to describe the implementation of the algorithm.

Keywords: non-linear programming, transport problem, index.

2010 MSC: 68W25, 90B06, 90C05, 90C26, 90C30, 90C35, 90C59.

1. INTRODUCTION

Supply management is becoming increasingly important in economy for the efficient operation of companies. To incur minimal costs, the implementation of an economical and fast transportation system is necessary. Transport models play an important role in supply logistics to reduce costs and improve services.

The problems involving the transportation of goods vary depending on the type of goods (construction materials, foodstuffs, petroleum products) transported, the number and type of transport [2] (train, ship, truck, airplane), number of sources and destinations, but also the route of transportation (air, water, roads, pipes, cables). There are 4 main groups of transport problems:

1. *Two index transport problem (2ITP):* The classical transport problem was formalized by the French mathematician Gaspard Monge in 1781 [11], studied by L. V. Kantorovich [8] and F. L. Hithcock [6] who used the properties of the transport problem only to determine the initial solution, but later, T. C. Koopmans [10] improves the solving method. Dj. Datzing [1] creates the simplex method which, although exponential, proved to be a very effective one. 2ITP is solved in most cases using the potential method. The potential method is the first precise method of solving the transport problem; it was proposed in 1949 by Kantorovici L. V. and Gavurin M. C. [9]. It is a modification of the simplex method that takes into consideration the specificity of the transport problem, thus this method differs from the simplex method only by the control step of the non-bounding function on the set of solutions. This problem involves knowing the sources and destinations of transportation of products, which are widely studied. The research directions are determining a good initial solution, solving the problem with constraints on the capacity of the transport route, with unitary transport costs, where the cost of transportation of a unit of product, the supply and demand are fuzzy or the multi-objective problem.

2. *Three index transport problem (3ITP):* assumes that the sources, destinations, the types of products or types of transportation used are known. It can be solved by applying an extended method of potential. Among the research directions, we can highlight the fuzzy and multi-objective problem.

3. *Four index transport problem (4ITP)* is a case that better describes the activity of an enterprise, which attracts the research of scientists. This problem assumes that the sources, the destinations, the set of products to be delivered and the types of transportation used are known. Throughout the rest of the paper we will study in detail this type of problem.

4. *Multi-index transport problem (nITP)*, the general case of nITP was formulated by Ph.-X. Ninh [12] who also proposed an exact method of solution, an extension of the potential method by coordinating the solution of the primary and dual problem. An important result was the formulation and proof of the theorem of the necessary and sufficient condition for a solution to exist. Among the authors who studied the multi-index transport problem are K. B. Hayley [4], who presented a solving algorithm, and W. Junginer [7], who conducted a study on the characteristics of the multi-index problem and proposed a number of logical problems to solve the multi-index transport problem.

The situation when a group of companies holds the raw materials that are necessary for another group to produce products needed in everyday life is well-known in economy. In a transport problem, the cost of transport is directly proportional to the number of units transported, which is not always true in the real world. The cost may be the cost of renting a vehicle, the landing charge at the airport, the parking taxes, fuel expenses, driver pay, etc. This type of 4ITP was studied by R. Zitouni. He proposes a solution algorithm [16] and compares it with other methods [15]. A modification of the classic algorithm has been described by A. Djamel et. al. [3]. T-H. Pham and Ph. Dott proposes an exact method to solve the problem [13], [14].

2. PROBLEM FORMULATION. BASIC CONCEPTS

In the following, we assume that we are given:
- m sources A_1, A_2, ... ,A_m which possess a quantity of the respective supply α_1, α_2, ...α_n;

- n destinations B_1, B_2, ... ,B_n with the respective demand β_1, β_2, ... , β_n;
- p different types of products P_1, P_2, ... ,P_p of respective quantities γ_1, γ_2, ..., γ_p;
- q different types of transport T_1, T_2, ... ,T_p with the respective transportation capacities of δ_1, δ_2, ... , δ_q;

The transportation cost is given by the nonlinear function $\varphi_{ijkl}(x_{ijkl})$, which means it will depend on the size of the flow of the transported product. The flow x_{ijkl} describes the quantity of product P_k transported from source A_i to destination B_j using the type of transport T_l. We aim to minimize the total transportation cost of all products available in sources to supply destinations. This problem is part of the 4ITP problem group where we have as the 4 indexes: the sources, the destinations, the types of transport and products.

The non-linear 4 index transport problem (N4ITP) consists in determining a flow x^* that minimizes the function:

$$F(x) = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q}\varphi_{ijkl}(x_{ijkl})$$

where $\varphi_{ijkl}(x_{ijkl})$ are piecewise non-decreasing concave cost functions.

We have to solve the non-linear problem:

$$F(x^*) = \min_{x\in X} F(x) \tag{1}$$

$$\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ijkl} = \alpha_i \qquad \forall i = 1,\ldots,m \tag{2}$$

$$\sum_{i=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ijkl} = \beta_j \qquad \forall j = 1,\ldots,n \tag{3}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{q} x_{ijkl} = \gamma_k \qquad \forall k = 1,\ldots,p \tag{4}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} x_{ijkl} = \delta_l \qquad \forall l = 1,\ldots,q \tag{5}$$

$$x_{ijkl} \geq 0 \qquad \forall (i,j,k,l) \tag{6}$$

To meet the conditions of positivity, the following variables α_1,$\alpha_2,\ldots$,α_m, β_1,$\beta_2,\ldots$,β_n, γ_1,$\gamma_2,\ldots$,γ_p and δ_1,$\delta_2,\ldots$,δ_q must be positive and non-zero. We will have no restrictions on the capacity of transport routes from source i to j destination.

Definition 2.1. *A function f with m variables is called separable if it can be written as a sum of m functions each by one variable $f(x) = \sum_{i=1}^{m} f_i(x_i)$.*

Definition 2.2. *A non-linear programming problem is called separable if the objective function and all restrictions are separable functions.*

Therefore we can say that problem (1)-(6) is a non-linear separable problem.

Any convex function can be approximated with some degree of accuracy to a piecewise linear function which implies the approximation of a nonlinear programming problem with some accuracy to a linear programming problem. Separable programming can also be applied to non-convex functions, but it should be noted that in this case we are not guaranteed the global minimum.

Definition 2.3. *We will consider N4ITP balanced and that it accepts solutions if it meets the conditions:*

$$\sum_{i=1}^{m} \alpha_i = \sum_{j=1}^{n} \beta_j = \sum_{k=1}^{p} \gamma_k = \sum_{l=1}^{q} \delta_l = H \tag{7}$$

with $\alpha_i > 0, \beta_j > 0, \gamma_k > 0, \delta_l > 0$.

If one of the equalities isnt satisfied, a fictive point will be added to offset the equality restriction and the transportation cost will be considered zero. We will determine the value of $H = \max\{\sum_{i=1}^{m} \alpha_i, \sum_{j=1}^{n} \beta_j, \sum_{k=1}^{p} \gamma_k, \sum_{l=1}^{q} \delta_l\}$ and depending on the situation we can have:

1. When $\sum_{i=1}^{m} \alpha_i < H$, we will add a node $m+1$ with $\alpha_{i+1} = H - \sum_{i=1}^{m} \alpha_i$. Because this node $m+1$ missing, products will not be transported from this source and the associated transport costs will be described by a function $\varphi(x) = 0$;

2. When $\sum_{i=1}^{n} \beta_j < H$, we will add a node $n+1$ with $\beta_{j+1} = H - \sum_{j=1}^{n} \beta_j$. Because this node $n+1$ is missing, products will not be transported to this destination and the associated transport costs will be described by a function $\varphi(x) = 0$;

3. When $\sum_{i=1}^{p} \gamma_k < H$, we will add a product $p+1$ with $\gamma_{p+1} = H - \sum_{k=1}^{p} \gamma_k$. Because this type of transport is missing, it cant be used to transport products and the associated transport costs will be described by a function $\varphi(x) = 0$;

4. When $\sum_{l=1}^{q} \delta_l < H$, we will add a product $q+1$ with $\delta_{p+1} = H - \sum_{l=1}^{q} \delta_l$. Because this type of transport is missing, it cant be used to transport products and the associated transport costs will be described by a function $\varphi(x) = 0$.

After the modifications we will solve the problem with the fictive node, product or type of transport and the system of admissible solutions (2)-(6) will also contain the new element.

If a non-zero value associated with the $m+1$ source, $n+1$ destination, $p+1$ product or $q+1$ type of transport occurs in the optimal solution, we will replace that value with zero when describing the optimal solution for the initial problem. For example, if there are $x_{12(p+1)4} = 5$ units of product for the $p+1$ product, we will replace them with $x_{12(p+1)4} = 0$. This is due to the fact that 5 units of product cant be transported from the source 2 to the destination 3 with the type of transport 4, because this $(p+1)$ product is missing.

3. NON-LINEARITY OF THE OBJECTIVE FUNCTION. DESCRIPTION OF THE LINEARIZATION ALGORITHM

Since we are interested in the case when the transportation cost of a product P_k, $k = 1,\ldots,p$ using type of transport T_l, $l = 1,\ldots,q$ from source A_i, $i = 1,\ldots,m$ to destination B_j, $j = 1,\ldots,n$ depend on the product and type of transport used, the non-linear function is given by the formula:

$$\varphi_{ijkl}(x_{ijkl}) = \begin{cases} \Gamma_k * x_{ijkl}, & 0 \le x_{ijkl} \le \Delta_l \\ \Gamma_k * \Delta_l, & x_{ijkl} > \Delta_l \end{cases} \tag{8}$$

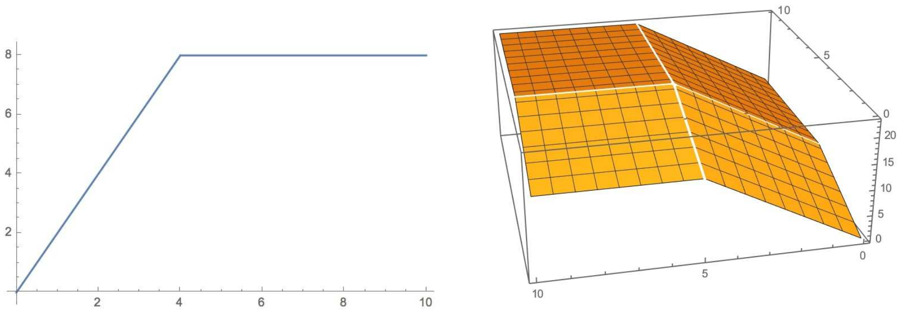
with $i = 1,\ldots,m$, $j = 1,\ldots,n$, $k = 1,\ldots,p$ and $l = 1,\ldots,q$, where Δ_l is the coefficient that describes the dependence between the transportation cost and the type of transport and Γ_k describes the dependence between the transportation cost and the type of product. If the transportation cost depends only on the type of transport, the cost function will be described as follows:

$$\varphi_{ijkl}(x_{ijkl}) = \begin{cases} x_{ijkl}, & 0 \le x_{ijkl} \le \Delta_l \\ \Delta_l, & x_{ijkl} > \Delta_l \end{cases} \tag{9}$$

with $i = 1,\ldots,m$, $j = 1,\ldots,n$, $k = 1,\ldots,p$, $l = 1,\ldots,q$.

Thus, in both of the studied cases, the cost functions will be piecewise non-decreasing concave functions of the form given in Figure 1.a.

For the sum of two functions of this type we obtain the non-linear function as in Figure 1.b. In the general case, when the objective function is given as the sum of several piecewise non-decreasing concave functions we have a non-linear programming problem with the admissible solutions described by the system (2)-(6), which describes a convex bounded set and the objective



a. Piecewise function.

b. Sum of two piecewise functions.

Fig. 1. Function graph

function is concave. An optimal solution will be one of the extremes of the set of restrictions [5]. If the problem is not degenerate and with all the cost functions positive, the extremes of the set are local minimums.

In the following, we will aim to linearize the objective function, which means that we will reduce the non-linear programming problem to a linear programming problem, the solution of which can be found using the simplex method adapted to the 4 index problem, such as the ones proposed by Djamel [3] or Zitouni [16].

Description of the algorithm:

Step 1. Determine an admissible solution of the non-linear problem that satisfies the following system of equations:

$$\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ijkl} = \alpha_i \qquad \forall i = 1,\ldots,m$$

$$\sum_{i=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ijkl} = \beta_j \qquad \forall j = 1,\ldots,n$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{q} x_{ijkl} = \gamma_k \qquad \forall k = 1,\ldots,p$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} x_{ijkl} = \delta_l \qquad \forall l = 1,\ldots,q$$

$$x_{ijkl} \geq 0 \qquad \forall (i,j,k,l)$$

Step 2. Calculate the coefficients C_{ijkl} as follows:

$$C_{ijkl}) = \begin{cases} \varphi_{ijkl}(x^0_{ijkl})/x^0_{ijkl}, & x^0_{ijkl} > 0 \\ \varphi'{}_{ijkl}(0), & x^0_{ijkl} = 0 \end{cases}$$

Step 3. Using the coefficients obtained in step 2, the linear function is constructed:

$$Z(x_{ijkl}) = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q} C_{ijkl}x_{ijkl} \tag{10}$$

We will solve the 4 index linear transport problem (10) in condition (2) - (5), which is a linear programming problem. In the case of linearization of the objective function, the optimal solution obtained will depend much on the initial admissible solution used for approximation. For this reason, the initial solution that will be at the base of linearization must be chosen with great care. It is very hard to do this for problems with big dimensions, as such, a random initial solution is chosen, which wont guarantee an optimal solution.

4. PRACTICAL RESULTS

We describe a transport problem with two sources A_1 and A_2 that contain respectively $\alpha_1 = 15$ and $\alpha_2 = 7$ units of products of type P_1 and P_2 of respectively $\gamma_1 = 6$ and $\gamma_2 = 16$ units each. The products will be transported using the types of transport T_1 and T_2 of the respective capacity $\delta_1 = 8$ and $\delta_2 = 14$ to the destinations B_1 and B_2 with the respective demand $\beta_1 = 12$ and $\beta_2 = 10$.

The transportation cost is described by the function:

$$\varphi_1(x) = \begin{cases} 2x & 0 \leq x \leq 4 \\ 8 & x > 4 \end{cases}$$

for the set of tuples $T_p(ijkl) = \{(1111), (1122), (1211), (1212), (1222), (2121), (2211), (2212), (2222)\}$ and by the function:

$$\varphi_2(x) = \begin{cases} 3x & 0 \leq x \leq 5 \\ 15 & x > 5 \end{cases}$$

for the set of tuples $T_p(ijkl) = \{(1112), (1121), (1221), (2111), (2112), (2122), (2221)\}$.

We have to determine the flow that minimizes the folowing non-linear cost function:

$$F(x) = \varphi_1(x_{1111})x_{1111} + \varphi_1(x_{1122})x_{1122} + \varphi_1(x_{1211})x_{1211} + \varphi_1(x_{1212})x_{1212} + \\ \varphi_1(x_{1222})x_{1222} + \varphi_1(x_{2121})x_{2121} + \varphi_1(x_{2211})x_{2211} + \varphi_1(x_{2212})x_{2212} +$$

$$\varphi_1(x_{2222})x_{2222} + \varphi_2(x_{1112})x_{1112} + \varphi_2(x_{1121})x_{1121} + \varphi_2(x_{1221})x_{1221} + \varphi_2(x_{2111})x_{2111} + \varphi_2(x_{2112})x_{2112} + \varphi_2(x_{2122})x_{2122} + \varphi_2(x_{2221})x_{2221}$$

By applying the procedure described above we will linearize the cost function.

The solving procedure:

An admissible solution of the non-linear problem must be determined, thus it is necessary to find a solution to the following system of equations:

$$\begin{cases} x_{1111} + x_{1112} + x_{1121} + x_{1122} + x_{1211} + x_{1212} + x_{1221} + x_{1222} = 14 \\ x_{2111} + x_{2112} + x_{2121} + x_{2122} + x_{2211} + x_{2212} + x_{2221} + x_{2222} = 8 \\ x_{1111} + x_{1112} + x_{1121} + x_{1122} + x_{2111} + x_{2112} + x_{2121} + x_{2122} = 16 \\ x_{1211} + x_{1212} + x_{1221} + x_{1222} + x_{2211} + x_{2212} + x_{2221} + x_{2222} = 6 \\ x_{1111} + x_{1112} + x_{1121} + x_{1212} + x_{2111} + x_{2112} + x_{2211} + x_{2212} = 12 \\ x_{1121} + x_{1122} + x_{1221} + x_{1222} + x_{1221} + x_{2122} + x_{2221} + x_{2222} = 10 \\ x_{1111} + x_{1121} + x_{1211} + x_{1221} + x_{2111} + x_{2121} + x_{2211} + x_{2221} = 18 \\ x_{1112} + x_{1122} + x_{1212} + x_{1222} + x_{2112} + x_{2122} + x_{2212} + x_{2222} = 4 \end{cases}$$

We obtain $x^0_{ijkl} = (3,0,2,7,3,0,0,0,0,0,0,0,0,0,0,7)$ and the coefficients $C_{ijkl} = (2,3,3,15/7,2,2,3,2,3,3,2,3,2,2,3,8/7)$ are calculated.

Using the coefficients obtained in step 2, the linear function is constructed: $Z(x_{ijkl}) = 2x_{1111}+3x_{1112}+\frac{15}{7}x_{1122}+2x_{1211}+2x_{1212}+3x_{1221}+2x_{1222}+3x_{2111}+3x_{2112}+2x_{2121}+3x_{2122}+2x_{2211}+2x_{2212}+3x_{2221}+\frac{8}{7}x_{2222}$.

5. CONCLUSION

The transport problem is a complex one that requires in-depth study, especially for non-linear cost functions. The optimum solution of the linear problem obtained from the algorithm described above depends much on the admissible solution used for approximation. Because the initial solution is taken at random, it is not guaranteed to obtain an optimal solution, often resulting in a local minimum.

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