CERTAIN SUBCLASSES OF MULTIVALENTLY
NON-BAZILEVIC FUNCTIONS INVOLVING A
GENERALIZED MITTAG-LEFFLER FUNCTION
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Abstract
In this paper we introduce a certain subclass of multivalently non-Bazilevic
analytic functions defined by a generalized Mittag-Leffler function. Such results
as subordination and superordination properties, inclusion theorem, distortion
theorems and coefficient estimate for this subclass are proved.

Keywords: Analytic function, subordination, superordination, sandwich-type result, ad-
missible class, integral operator.

2010 MSC: Primary 30C45; Secondary 30D30, 33D20.

1. INTRODUCTION
Let \( \mathcal{H}(\mathbb{U}) \) be the class of analytic functions in the open unit disk
\( \mathbb{U} = \{z \in \mathbb{C} : |z| < 1\} \) and \( \mathcal{H}[a, m] \) be subclass of \( \mathcal{H}(\mathbb{U}) \) consisting of functions of the form:

\[
f(z) = a + a_m z^m + a_{m+1} z^{m+1} + \ldots \quad (z \in \mathbb{U}).
\]

Also, let \( \mathcal{A}(p, m) \) denote the subclass of \( \mathcal{H}(\mathbb{U}) \) consisting of functions of the form:

\[
f(z) = z^p + \sum_{n=p+m}^{\infty} a_n z^n \quad (p, m \in \mathbb{N} = \{1, 2, 3, \ldots\}). \quad (1)
\]

We write \( \mathcal{A}(p, 1) = \mathcal{A}(p) \) and \( \mathcal{A}(1, 1) = \mathcal{A} \). The Mittag-Leffler function
\( E_\alpha(z) (z \in \mathbb{C}) \) is defined by (see [8] and [9]):

\[
E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \quad (\alpha \in \mathbb{C}, Re \{\alpha\} > 0) .
\]
Srivastava and Tomovski [10] introduced the generalized Mittag-Leffler function $E_{\alpha,\beta}^{\gamma,k}(z)$ ($z \in \mathbb{C}$) in the form:

$$E_{\alpha,\beta}^{\gamma,k}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}$$

(3)

$(\alpha, \beta, \gamma \in \mathbb{C}; \Re \{\alpha\} > \max \{0; \Re \{\beta\} - 1\}; \Re \{\beta\} > 0),$ where $(\gamma)_n$ denotes the Pochhammer symbol (or the shifted factorial).

Srivastava and Tomovski [10] proved that the function $E_{\alpha,\beta}^{\gamma,k}(z)$ defined by (12) is an entire function in the complex $z-$plane.

We now define the function $Q_{\gamma,k}^{p,\alpha,\beta}(z)$ by

$$Q_{\gamma,k}^{p,\alpha,\beta}(z) = \frac{\Gamma(\alpha + \beta)}{(\gamma)_k} z^{p-1} \left( E_{\alpha,\beta}^{\gamma,k}(z) - \frac{1}{\Gamma(\beta)} \right)$$

(4)

$$(z \in U; \alpha, \beta, \gamma \in \mathbb{C}; \Re \{\alpha\} > \max \{0; \Re \{\beta\} - 1\}; \Re \{\beta\} > 0).$$

Corresponding to the function $Q_{\gamma,k}^{p,\alpha,\beta}(z)$ defined by (13), we introduce the following linear operator $H_{\gamma,k}^{p,\alpha,\beta}f(z) : A(p,m) \rightarrow A(p,m)$ by

$$H_{\gamma,k}^{p,\alpha,\beta}f(z) = Q_{\gamma,k}^{p,\alpha,\beta}(z) \ast f(z)$$

$$= z^p + \sum_{n=p+m}^{\infty} \frac{\Gamma(\alpha + (n - p + 1)k)}{\Gamma(\alpha + (n - p + 1)k)} \frac{\Gamma(\alpha)}{\Gamma(\alpha + (n - p + 1)k)} a_n z^n$$

(5)

$(\alpha, \beta, \gamma \in \mathbb{C}; \Re \{\alpha\} > \max \{0; \Re \{\beta\} - 1\}; \Re \{\beta\} > 0; z \in U),$ where the symbol $(\ast)$ denotes the Hadamard product (or convolution). We note that the operator $H_{1,\alpha,\beta}^{\gamma,k}f(z) = H_{\alpha,\beta}^{\gamma,k}$ was introduced by Attiya [1].

Also, it is easily verified from (14) that

$$k z \left( H_{\alpha,\beta}^{\gamma,k}f(z) \right) = (\gamma + k) H_{\alpha,\beta}^{\gamma+1,k}f(z) - (\gamma + k - p) H_{\alpha,\beta}^{\gamma,k}f(z)$$

(6)

and

$$\alpha z \left( H_{\alpha,\beta+1}^{\gamma,k}f(z) \right) = (\alpha + \beta) H_{\alpha,\beta}^{\gamma,k}f(z) - (\alpha + \beta - p) H_{\alpha,\beta}^{\gamma,k}f(z).$$

(7)

If $f(z)$ and $g(z)$ are analytic in $U,$ we say that $f(z)$ is subordinate to $g(z),$ or $g(z)$ is superordinate to $f(z)$, written symbolically, $f < g$ in $U$ or $f(z) \prec g(z)$ ($z \in U$), if there exists a Schwarz function $\omega(z),$ which (by definition) is analytic in $U$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ ($z \in U$) such that
$f(z) = g(ω(z))$ ($z ∈ U$). Further more, if the function $g(z)$ is univalent in $U$, then we have the following equivalence (see [14]):

$$f(z) \prec g(z) \quad (z ∈ U) ⇐⇒ f(0) = g(0) \quad \text{and} \quad f(U) ⊂ g(U).$$

Let $ϕ : \mathbb{C}^2 × U → \mathbb{C}$ and $h(z)$ be univalent in $U$. If $p(z)$ is analytic in $U$ and satisfies the first order differential subordination:

$$ϕ \left( p(z), zp'(z) ; z \right) \prec h(z), \quad (8)$$

then $p(z)$ is a solution of the differential subordination (8). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (8) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (8). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants of (8) is called the best dominant. If $p(z)$ and $ϕ \left( p(z), zp'(z) ; z \right)$ are univalent in $U$ and if $p(z)$ satisfies first order differential superordination:

$$h(z) \prec ϕ \left( p(z), zp'(z) ; z \right), \quad (9)$$

then $p(z)$ is a solution of the differential superordination (9). An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination (9) if $q(z) \prec p(z)$ for all $p(z)$ satisfying (9). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants of (9) is called the best subordinant.

**Definition 1.** A function $f ∈ A(p, m)$ is said to be the class $\mathcal{N}^{γ, k}_{p, α, β}(λ, μ; A, B)$ if it satisfies the following subordination condition:

$$(1 + λ) \left( \frac{z^p}{Z^{γ, k}_{p, α, β} f(z)} \right)^μ - λ \left( \frac{Z^{γ+1, k}_{p, α, β} f(z)}{Z^{γ, k}_{p, α, β} f(z)} \right)^μ \prec 1 + Az + Bz, \quad (10)$$

where $\text{Re} \{α\} > \max \{0; \text{Re} \{k\} - 1\}$, $\text{Re} \{k\} > 0$, $0 < μ < 1$, $λ ∈ \mathbb{C}$, $-1 ≤ B ≤ 1$, $A ≠ B$ and all the powers are principal values.

Furthermore, the function $f ∈ \mathcal{N}^{γ, k}_{p, α, β}(λ, μ; δ)$ if and only if $f ∈ A(p, m)$ and

$$\text{Re} \left\{ (1 + λ) \left( \frac{z^p}{Z^{γ, k}_{p, α, β} f(z)} \right)^μ - λ \left( \frac{Z^{γ+1, k}_{p, α, β} f(z)}{Z^{γ, k}_{p, α, β} f(z)} \right)^μ \right\} > δ$$

$$(0 ≤ δ < 1; z ∈ U),$$

we write $\mathcal{N}^{γ, k}_{p, α, β}(0, μ; δ) = \mathcal{N}^{γ, k}_{p, α, β}(μ; δ).$
In order to establish our main results, we need the following definition and lemmas.

**Definition 2.** [9] Denote by $\mathcal{Q}$ the set of all functions $f$ that are analytic and injective on $\mathbb{U}\setminus E(f)$, where

$$E(f) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty \right\},$$

and such that $f'(\zeta) \neq 0$ for $\zeta \in \mathbb{U}\setminus E(f)$.

**Lemma 1.** [14] Let the function $h$ be analytic and convex (univalent) in $\mathbb{U}$ with $h(0) = 1$. Suppose also that the function $p(z)$ given by

$$p(z) = 1 + c_mz^m + c_{m+1}z^{m+1} + \ldots$$

is analytic in $\mathbb{U}$. If

$$p(z) + \frac{zp'(z)}{\eta} \prec h(z) \quad (\text{Re}(\eta) \geq 0; \eta \neq 0; \ z \in \mathbb{U}),$$

then

$$p(z) < q(z) = \frac{\eta}{m}z^\frac{m}{m} \int_0^z h(t) t^{\frac{n}{m} - 1} dt < h(z),$$

and $q(z)$ is the best dominant.

**Lemma 2.** [17] Let $q$ be a convex univalent function in $\mathbb{U}$ and let $\sigma \in \mathbb{C}$, $\eta \in \mathbb{C}^* = \mathbb{C}\setminus\{0\}$ with

$$\text{Re} \left( 1 + \frac{zq''(z)}{q'(z)} \right) > \max \left\{ 0, -\text{Re} \left( \frac{\sigma}{\eta} \right) \right\}.$$

If the function $p$ is analytic in $\mathbb{U}$ and

$$\sigma p(z) + \eta zq'(z) < \sigma q(z) + \eta zq'(z),$$

then $p(z) \prec q(z)$ and $q$ is the best dominant.

**Lemma 3.** [9] Let $q$ be convex univalent in $\mathbb{U}$ and $\kappa \in \mathbb{C}$. Further assume that $\text{Re}(\kappa) > 0$. If $p(z) \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}$, and $p(z) + \kappa zq'(z)$ is univalent in $\mathbb{U}$, then

$$q(z) + \kappa zq'(z) < p(z) + \kappa zq'(z),$$

implies $q(z) \prec p(z)$ and $q$ is the best subordinant.

**Lemma 4.** [12] Let $\mathcal{F}$ be analytic and convex in $\mathbb{U}$. If $f, g \in \mathcal{A}$ and $f, g \prec \mathcal{F}$ then

$$\lambda f + (1 - \lambda) g \prec \mathcal{F} \quad (0 \leq \lambda \leq 1).$$
Lemma 5. [16] Let \( f(z) = \sum_{n=1}^{\infty} a_n z^n \) be analytic in \( U \) and \( g(z) = \sum_{n=1}^{\infty} b_n z^n \) be analytic and convex in \( U \). If \( f \prec g \), then
\[
|a_n| < |b_1| \quad (n \in \mathbb{N}).
\]

In the present paper, we obtain some subordination and superordination properties, distortion theorems and inequality properties of the class \( N_{p,\alpha,\beta}^{\gamma,k} (\lambda; \mu; A, B) \). Several other new results are also obtained.

2. MAIN RESULTS

Unless otherwise mentioned, we assume throughout this paper that \( \text{Re} \{\alpha\} > \max \{0; \text{Re} \{k\} - 1\}, \text{Re} \{k\} > 0, 0 < \mu < 1, \lambda \in \mathbb{C}, -1 \leq B < A \leq 1 \) and \( p \in \mathbb{N} \).

We begin by presenting our first subordination property given by

Theorem 2.1. Let \( f \in N_{p,\alpha,\beta}^{\gamma,k} (\lambda; \mu; A, B) \) with \( \text{Re} \left( \frac{\gamma + k}{m\lambda k} \right) > 0 \). Then
\[
\left( \frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)} \right)^{\mu} \prec q(z) = \frac{\mu(\gamma + k)}{m\lambda k} \int_0^1 \frac{1 + Az u^{\frac{\mu(\gamma + k)}{m\lambda k} - 1}}{1 + Bz u} du \prec \frac{1 + Az}{1 + Bz} (z \in U),
\]
and \( q(z) \) is the best dominant.

Proof. Define the function \( g(z) \) by
\[
g(z) = \left( \frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)} \right)^{\mu} (z \in U). \tag{14}
\]

Then the function \( g(z) \) is of the form (3) and analytic in \( U \). Taking the derivatives in both sides of (14), we get
\[
(1 + \lambda) \left( \frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)} \right)^{\mu} - \lambda \frac{\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z)}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)} \left( \frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)} \right)^{\mu} = \frac{\lambda k}{\mu(\gamma + k)} z g'(z). \tag{15}
\]

Since \( f \in N_{p,\alpha,\beta}^{\gamma,k} (\lambda; \mu; A, B) \), we have
\[
g(z) + \frac{\lambda k}{\mu(\gamma + k)} z g'(z) \prec \frac{1 + Az}{1 + Bz}.
\]

Applying Lemma 1 to (15) with \( \eta = \frac{\mu(\gamma + k)}{m\lambda k} \), we get
\[
\left( \frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)} \right)^{\mu} \prec q(z) = \frac{\mu(\gamma + k)}{m\lambda k} z - \frac{\mu(\gamma + k)}{m\lambda k} \int_0^z \frac{1 + At}{1 + Bt} t^{-\frac{\mu(\gamma + k)}{m\lambda k} - 1} dt.
\]
and \(q(z)\) is the best dominant. The proof of Theorem 2.1 is thus completed. □

**Theorem 2.2.** Let \(q(z)\) be univalent in \(U\), \(\lambda \in \mathbb{C}^*\). Suppose also that \(q(z)\) satisfies the following inequality:

\[
\text{Re} \left( 1 + \frac{zq''(z)}{q'(z)} \right) > \max \left\{ 0, -\frac{\mu(\gamma + k)}{\lambda k} \right\}. \tag{17}
\]

If \(f \in A(p,m)\) satisfies the following subordination condition:

\[
(1 + \lambda) \left( \frac{z^p}{\mathcal{H}_{p.a.\beta}^\gamma f(z)} \right)^\mu - \frac{\lambda k}{\mu(\gamma + k)} \left( \frac{z^p}{\mathcal{H}_{p.a.\beta}^\gamma f(z)} \right)^\mu < q(z) + \frac{\lambda k}{\mu(\gamma + k)} zq'(z), \tag{18}
\]

then

\[
\left( \frac{z^p}{\mathcal{H}_{p.a.\beta}^\gamma f(z)} \right)^\mu < q(z),
\]

and \(q(z)\) is the best dominant.

**Proof.** Let the function \(g(z)\) be defined by (14). We know that (15) holds true. Combining (15) and (18), we find that

\[
g(z) + \frac{\lambda k}{\mu(\gamma + k)} zg'(z) < q(z) + \frac{\lambda k}{\mu(\gamma + k)} zq'(z). \tag{19}
\]

By using Lemma 2 and (19), we easily get the assertion of Theorem 2.2. □

Taking \(q(z) = \frac{1 + A_2}{1 + B_2} (-1 \leq B < A \leq 1)\) in Theorem 2.2, we get the following result.

**Corollary 6.** Let \(\lambda \in \mathbb{C}^*\) and \(-1 \leq B < A \leq 1\). Suppose also that

\[
\text{Re} \left( \frac{1 - Bz}{1 + Bz} \right) > \max \left\{ 0, -\frac{\mu(\gamma + k)}{\lambda k} \right\}. \tag{17}
\]

If \(f \in A(p,m)\) satisfies the following subordination condition:

\[
(1 + \lambda) \left( \frac{z^p}{\mathcal{H}_{p.a.\beta}^\gamma f(z)} \right)^\mu - \frac{\lambda k}{\mu(\gamma + k)} \left( \frac{z^p}{\mathcal{H}_{p.a.\beta}^\gamma f(z)} \right)^\mu < \frac{1 + A_2}{1 + B_2} + \frac{\lambda k}{\mu(\gamma + k)} (A - B) \frac{z}{(1 + B)^2},
\]

then

\[
\left( \frac{z^p}{\mathcal{H}_{p.a.\beta}^\gamma f(z)} \right)^\mu < \frac{1 + A_2}{1 + B_2}.
\]
and the function \( \frac{1+Az}{1+Bz} \) is the best dominant.

**Theorem 2.3.** Let \( q \) be convex univalent in \( U \), \( \lambda \in \mathbb{C} \) with \( Re(\lambda) > 0 \). Also let
\[
\left( \frac{z^p}{H_{p,\alpha,\beta}^{\gamma,k}(z)} \right)^\mu \in \mathcal{H}[q(0),1] \cap \mathcal{Q} \text{ and } (1 + \lambda) \left( \frac{z^p}{\lambda H_{p,\alpha,\beta}^{\gamma,k}(z)} \right)^\mu - \lambda \frac{g_{p,\alpha,\beta}^{\gamma+1,k}(z)}{H_{p,\alpha,\beta}^{\gamma,k}(z)} \left( \frac{z^p}{H_{p,\alpha,\beta}^{\gamma,k}(z)} \right)^\mu \]
be univalent in \( U \). If \( f \in \mathcal{A}(p) \) satisfies the following superordination condition:
\[
q(z) + \frac{\lambda k}{\mu(\gamma + k)} zq'(z) < (1 + \lambda) \left( \frac{z^p}{H_{p,\alpha,\beta}^{\gamma,k}(z)} \right)^\mu - \lambda \frac{g_{p,\alpha,\beta}^{\gamma+1,k}(z)}{H_{p,\alpha,\beta}^{\gamma,k}(z)} \left( \frac{z^p}{H_{p,\alpha,\beta}^{\gamma,k}(z)} \right)^\mu,
\]
then
\[
q(z) \prec \left( \frac{z^p}{H_{p,\alpha,\beta}^{\gamma,k}(z)} \right)^\mu,
\]
and the function \( q(z) \) is the best subordinator.

**Proof.** Let the function \( g(z) \) be defined by (14). Then
\[
q(z) + \frac{\lambda k}{\mu(\gamma + k)} zq'(z) < (1 + \lambda) \left( \frac{z^p}{H_{p,\alpha,\beta}^{\gamma,k}(z)} \right)^\mu - \lambda \frac{g_{p,\alpha,\beta}^{\gamma+1,k}(z)}{H_{p,\alpha,\beta}^{\gamma,k}(z)} \left( \frac{z^p}{H_{p,\alpha,\beta}^{\gamma,k}(z)} \right)^\mu
\]
\[
= g(z) + \frac{\lambda k}{\mu(\gamma + k)} zq'(z).
\]
An application of Lemma 3 yields the assertion of Theorem 2.3. \( \blacksquare \)

Taking \( q(z) = \frac{1+Az}{1+Bz} (-1 \leq B < A \leq 1) \) in Theorem 2.3, we get the following corollary.

**Corollary 7.** Let \(-1 \leq B < A \leq 1\), \( \lambda \in \mathbb{C} \) with \( Re(\lambda) > 0 \). Also let
\[
\left( \frac{z^p}{H_{p,\alpha,\beta}^{\gamma,k}(z)} \right)^\mu \in \mathcal{H}[1,1] \cap \mathcal{Q}, \text{ and } (1 + \lambda) \left( \frac{z^p}{\lambda H_{p,\alpha,\beta}^{\gamma,k}(z)} \right)^\mu - \lambda \frac{g_{p,\alpha,\beta}^{\gamma+1,k}(z)}{H_{p,\alpha,\beta}^{\gamma,k}(z)} \left( \frac{z^p}{H_{p,\alpha,\beta}^{\gamma,k}(z)} \right)^\mu \]
be univalent in \( U \). If \( f \in \mathcal{A}(p) \) satisfies the following superordination condition:
\[
\frac{1+Az}{1+Bz} + \frac{\lambda k}{\mu(\gamma + k)} \frac{(A-B)z}{(1+Bz)^2} < (1 + \lambda) \left( \frac{z^p}{H_{p,\alpha,\beta}^{\gamma,k}(z)} \right)^\mu - \lambda \frac{g_{p,\alpha,\beta}^{\gamma+1,k}(z)}{H_{p,\alpha,\beta}^{\gamma,k}(z)} \left( \frac{z^p}{H_{p,\alpha,\beta}^{\gamma,k}(z)} \right)^\mu,
\]
then
\[
\frac{1+Az}{1+Bz} \prec \left( \frac{z^p}{H_{p,\alpha,\beta}^{\gamma,k}(z)} \right)^\mu,
\]
and the function \( \frac{1+Az}{1+Bz} \) is the best subordinator.
Combining the above results of subordination and superordination, we easily get the following “Sandwich-type result”.

**Theorem 2.4.** Let $q_1$ be convex univalent and let $q_2$ be univalent in $U$, $\lambda \in \mathbb{C}$ with $\text{Re} (\lambda) > 0$. Let $q_2$ satisfies (17). If $\left( \frac{z^p}{\mathcal{I}_{p,\alpha,\beta}^{\gamma,k}(z)} \right)^{\mu} \in \mathcal{H} [q_1(0), 1] \cap \Omega$, and

\[
(1 + \lambda) \left( \frac{z^p}{\mathcal{I}_{p,\alpha,\beta}^{\gamma,k}(z)} \right)^{\mu} - \lambda \frac{\mathcal{C}^{\gamma+1,k}_{p,\alpha,\beta} f(z)}{\mathcal{I}_{p,\alpha,\beta}^{\gamma,k}(z)} \left( \frac{z^p}{\mathcal{I}_{p,\alpha,\beta}^{\gamma,k}(z)} \right)^{\mu} \text{ be univalent in } U,
\]

also

\[
q_1(z) + \frac{\lambda k}{\mu (\gamma + k)} z q_1'(z) \prec (1 + \lambda) \left( \frac{z^p}{\mathcal{I}_{p,\alpha,\beta}^{\gamma,k}(z)} \right)^{\mu} - \lambda \frac{\mathcal{C}^{\gamma+1,k}_{p,\alpha,\beta} f(z)}{\mathcal{I}_{p,\alpha,\beta}^{\gamma,k}(z)} \left( \frac{z^p}{\mathcal{I}_{p,\alpha,\beta}^{\gamma,k}(z)} \right)^{\mu} \prec q_2(z) + \frac{\lambda k}{\mu (\gamma + k)} z q_2'(z),
\]

then

\[
q_1(z) \prec \left( \frac{z^p}{\mathcal{I}_{p,\alpha,\beta}^{\gamma,k}(z)} \right)^{\mu} \prec q_2(z),
\]

and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and the best dominant.

**Theorem 2.5.** If $\lambda > 0$ and $f \in \mathcal{N}_{p,\alpha,\beta}^\gamma (\mu, \delta)$ ($0 \leq \beta < 1$). Then $f \in \mathcal{N}_{p,\alpha,\beta}^\gamma (\lambda, \mu; \delta)$ for $|z| < R$, where

\[
R = \left( \frac{1}{\sqrt{\frac{m\lambda k}{\mu (\gamma + k)} + 1 - \frac{m\lambda k}{\mu (\gamma + k)}}} \right)^\frac{1}{n}. \tag{20}
\]

The bound $R$ is the best possible.

**Proof.** We begin by writing

\[
\left( \frac{z^p}{\mathcal{I}_{p,\alpha,\beta}^{\gamma,k}(z)} \right)^{\mu} = \theta + (1 - \theta) g(z) \quad (z \in U). \tag{21}
\]

Then clearly, the function $g(z)$ is of the form (3), is analytic and has a positive real part in $U$. By taking the derivatives in the both sides in equality (21), we get

\[
\frac{1}{1 - \theta} \left\{ (1 + \lambda) \left( \frac{z^p}{\mathcal{I}_{p,\alpha,\beta}^{\gamma,k}(z)} \right)^{\mu} - \lambda \frac{\mathcal{C}^{\gamma+1,k}_{p,\alpha,\beta} f(z)}{\mathcal{I}_{p,\alpha,\beta}^{\gamma,k}(z)} \left( \frac{z^p}{\mathcal{I}_{p,\alpha,\beta}^{\gamma,k}(z)} \right)^{\mu} - \theta \right\}.\]
Certain subclasses of multivalently non-Bazilevic functions...

\[ g(z) + \frac{\lambda k}{\mu (\gamma + k)} z g'(z). \]  

(22)

By making use of the following well-known estimate (see [11, Theorem 1]):

\[ \left| \frac{z g'(z)}{Re \{g(z)\}} \right| \leq \frac{2mr^m}{1 - r^{2m}} \quad (|z| = r < 1) \]

in (22), we obtain that

\[ Re \left( \frac{1}{1 - \theta} \left\{ (1 + \lambda) \left( \frac{z^p}{\mathcal{H}_{p,\alpha,\beta} f(z)} \right)^\mu - \lambda \frac{\mathcal{H}_{\gamma+1,k} f(z)}{\mathcal{H}_{\gamma,k} f(z)} \left( \frac{z^p}{\mathcal{H}_{p,\alpha,\beta} f(z)} \right)^\mu - \theta \right\} \right) \]

\[ \geq Re \{g(z)\} \left( 1 - \frac{2\lambda kmr^m}{\mu (\gamma + k) (1 - r^{2m})} \right). \]  

(23)

It is seen that the right-hand side of (23) is positive, provided that \( r < R \), where \( R \) is given by (20).

In order to show that the bound \( R \) is the best possible, we consider the function \( f \in \mathcal{A}(p, m) \) defined by

\[ \left( \frac{z^p}{\mathcal{H}_{p,\alpha,\beta} f(z)} \right)^\mu = \beta + (1 - \beta) \frac{1 + z^m}{1 - z^m} \quad (z \in \mathbb{U}). \]

By noting that

\[ \frac{1}{1 - \beta} \left\{ (1 + \lambda) \left( \frac{z^p}{\mathcal{H}_{p,\alpha,\beta} f(z)} \right)^\mu - \lambda \frac{\mathcal{H}_{\gamma+1,k} f(z)}{\mathcal{H}_{\gamma,k} f(z)} \left( \frac{z^p}{\mathcal{H}_{p,\alpha,\beta} f(z)} \right)^\mu - \beta \right\} \]

\[ = \frac{1 + z^m}{1 - z^m} + \frac{2\lambda kmz^m}{\mu (\gamma + k) (1 - z^m)^2} = 0, \]  

(24)

for \( z = R \exp \left( \frac{\pi i}{m} \right) \), we conclude that the bound is the best possible. Theorem 2.5 is thus proved.

Theorem 2.6. Let \( \lambda_2 \geq \lambda_1 \geq 0 \) and \(-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1\). Then

\[ \mathcal{N}_{p,\alpha,\beta}^\gamma (\lambda_2; \mu; A_2, B_2) \subset \mathcal{N}_{p,\alpha,\beta}^\gamma (\lambda_1; \mu; A_1, B_1). \]  

(25)

Proof. Suppose that \( f \in \mathcal{N}_{p,\alpha,\beta}^\gamma (\lambda_2; \mu; A_2, B_2) \). We know that

\[ (1 + \lambda_2) \left( \frac{z^p}{\mathcal{H}_{p,\alpha,\beta} f(z)} \right)^\mu - \lambda_2 \frac{\mathcal{H}_{\gamma+1,k} f(z)}{\mathcal{H}_{\gamma,k} f(z)} \left( \frac{z^p}{\mathcal{H}_{p,\alpha,\beta} f(z)} \right)^\mu \leq \frac{1 + A_2 z}{1 + B_2 z}. \]
Since \(-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1\), we easily find that

\[
(1 + \lambda_2) \left( \frac{z^p}{\mathcal{G}^{\gamma,k}_{p,\alpha,\beta} f(z)} \right)^\mu - \lambda_2 \frac{\mathcal{G}^{\gamma+1,k}_{p,\alpha,\beta} f(z)}{\mathcal{G}^{\gamma,k}_{p,\alpha,\beta} f(z)} \left( \frac{z^p}{\mathcal{G}^{\gamma,k}_{p,\alpha,\beta} f(z)} \right)^\mu \leq \frac{1 + A_2 z}{1 + B_2 z} < \frac{1 + A_1 z}{1 + B_1 z},
\]

that is \(f \in \mathcal{N}^{\gamma,k}_{p,\alpha,\beta} (\lambda_2, \mu; A_1, B_1)\). Thus the assertion of Theorem 2.6 holds for \(\lambda_2 = \lambda_1 \geq 0\). If \(\lambda_2 > \lambda_1 \geq 0\), by Theorem 2.1 and (26), we know that \(f \in \mathcal{N}^{\gamma,k}_{p,\alpha,\beta} (0, \mu; A_1, B_1)\), that is,

\[
\left( \frac{z^p}{\mathcal{G}^{\gamma,k}_{p,\alpha,\beta} f(z)} \right)^\mu < \frac{1 + A_1 z}{1 + B_1 z},
\]

(27)

At the same time, we have

\[
(1 + \lambda_1) \left( \frac{z^p}{\mathcal{G}^{\gamma,k}_{p,\alpha,\beta} f(z)} \right)^\mu - \lambda_1 \frac{\mathcal{G}^{\gamma+1,k}_{p,\alpha,\beta} f(z)}{\mathcal{G}^{\gamma,k}_{p,\alpha,\beta} f(z)} \left( \frac{z^p}{\mathcal{G}^{\gamma,k}_{p,\alpha,\beta} f(z)} \right)^\mu = \left( 1 - \frac{\lambda_1}{\lambda_2} \right) \left( \frac{z^p}{\mathcal{G}^{\gamma,k}_{p,\alpha,\beta} f(z)} \right)^\mu
\]

\[
+ \frac{\lambda_1}{\lambda_2} \left( 1 + \lambda_2 \left( \frac{z^p}{\mathcal{G}^{\gamma,k}_{p,\alpha,\beta} f(z)} \right)^\mu - \lambda_2 \frac{\mathcal{G}^{\gamma+1,k}_{p,\alpha,\beta} f(z)}{\mathcal{G}^{\gamma,k}_{p,\alpha,\beta} f(z)} \left( \frac{z^p}{\mathcal{G}^{\gamma,k}_{p,\alpha,\beta} f(z)} \right)^\mu \right).
\]

(28)

Moreover, since \(0 \leq \frac{\lambda_1}{\lambda_2} < 1\) and the function \(\frac{1 + A_1 z}{1 + B_1 z} \quad (-1 \leq B_1 < A_1 \leq 1; \ z \in U)\) is analytic and convex in \(U\). Combining (26)-(28) and Lemma 4, we find that

\[
(1 + \lambda_1) \left( \frac{z^p}{\mathcal{G}^{\gamma,k}_{p,\alpha,\beta} f(z)} \right)^\mu - \lambda_1 \frac{\mathcal{G}^{\gamma+1,k}_{p,\alpha,\beta} f(z)}{\mathcal{G}^{\gamma,k}_{p,\alpha,\beta} f(z)} \left( \frac{z^p}{\mathcal{G}^{\gamma,k}_{p,\alpha,\beta} f(z)} \right)^\mu \leq \frac{1 + A_1 z}{1 + B_1 z},
\]

that is \(f \in \mathcal{N}^{\gamma,k}_{p,\alpha,\beta} (\lambda_1, \mu; A_1, B_1)\), which implies that the assertion (25) of Theorem 2.6 holds.

**Theorem 2.7.** Let \(f \in \mathcal{N}^{\gamma,k}_{p,\alpha,\beta} (\lambda, \mu; A, B)\) with \(\frac{\gamma+k}{\lambda k} > 0\) and \(-1 \leq B < A \leq 1\). Then

\[
\frac{\mu(\gamma+k)}{m\lambda k} \int_0^1 \frac{1 - Au}{1 - Bu} u^{\frac{\mu(\gamma+k)}{m\lambda k} - 1} du < Re \left( \frac{z^p}{\mathcal{G}^{\gamma,k}_{p,\alpha,\beta} f(z)} \right)^\mu \leq \frac{\mu(\gamma+k)}{m\lambda k} \int_0^1 \frac{1 + Au}{1 + Bu} u^{\frac{\mu(\gamma+k)}{m\lambda k} - 1} du.
\]

(29)
Proof. Let $f \in N_{p,\alpha,\beta}^{\gamma,k} (\lambda, \mu; A, B)$ with $\frac{\gamma+k}{\lambda k} > 0$. From Theorem 2.1, we know that (13) holds, which implies that

$$\text{Re} \left( \frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f (z)} \right)^\mu < \sup_{z \in \mathbb{U}} \text{Re} \left\{ \frac{\mu (\gamma+k)}{m \lambda k} \int_0^1 \frac{1 + A z u}{1 + B z u} u^{\frac{\mu (\gamma+k)}{m \lambda k} - 1} du \right\}$$

$$\leq \frac{\mu (\gamma+k)}{m \lambda k} \int_0^1 \sup_{z \in \mathbb{U}} \text{Re} \left( \frac{1 + A z u}{1 + B z u} \right) u^{\frac{\mu (\gamma+k)}{m \lambda k} - 1} du$$

$$< \frac{\mu (\gamma+k)}{m \lambda k} \int_0^1 1 + A u 1 + B u u^{\frac{\mu (\gamma+k)}{m \lambda k} - 1} du,$$

and

$$\text{Re} \left( \frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f (z)} \right)^\mu > \inf_{z \in \mathbb{U}} \text{Re} \left\{ \frac{\mu (\gamma+k)}{m \lambda k} \int_0^1 \frac{1 + A z u}{1 + B z u} u^{\frac{\mu (\gamma+k)}{m \lambda k} - 1} du \right\}$$

$$\geq \frac{\mu (\gamma+k)}{m \lambda k} \int_0^1 \inf_{z \in \mathbb{U}} \text{Re} \left( \frac{1 + A z u}{1 + B z u} \right) u^{\frac{\mu (\gamma+k)}{m \lambda k} - 1} du$$

$$> \frac{\mu (\gamma+k)}{m \lambda k} \int_0^1 1 - A u 1 - B u u^{\frac{\mu (\gamma+k)}{m \lambda k} - 1} du.$$  \hspace{1cm} (30)

Combining (30) and (31), we get (29). The proof of Theorem 2.7 is evidently completed.

Theorem 2.8. Let

$$f (z) = z^p + \sum_{n=p+m}^{\infty} a_n z^n \in N_{p,\alpha,\beta}^{\gamma,k} (\lambda, \mu; A, B).$$

Then

$$|a_{p+m}| \leq \frac{(\gamma+k) \Gamma (\gamma+k) \Gamma (\beta + \alpha (m+1)) \Gamma (m+1)! (A-B)}{[\mu (\gamma+k) + m \lambda k] \Gamma (\gamma + (m+1) k) \Gamma (\alpha + \beta)}.$$  \hspace{1cm} (32)

Proof. Combining (2) and (32), we obtain

$$1 + (1 + \lambda) \left( \frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f (z)} \right)^\mu - \lambda \frac{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k+1} f (z)}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f (z)} \left( \frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f (z)} \right)^\mu$$

$$= 1 + \left( -\mu - \frac{m \lambda k}{\gamma+k} \right) \frac{\Gamma (\gamma + (m+1) k) \Gamma (\alpha + \beta)}{\Gamma (\gamma+k) \Gamma (\beta + \alpha (m+1)) \Gamma (m+1)!} a_{p+m} z^n + \ldots < \frac{1 + A z}{1 + B z}.$$  \hspace{1cm} (34)
An application of Lemma 5 to (34) yields
\[
\left| \left( -\mu - \frac{m\lambda k}{\gamma + k} \right) \frac{\Gamma (\gamma + (m + 1)k) \Gamma (\alpha + \beta)}{\Gamma (\gamma + k) \Gamma (\beta + \alpha (m + 1))} a_{p+m} \right| < |A - B|. \tag{35}
\]
Thus, from (35), we easily arrive at (33) asserted by Theorem 2.8.

Acknowledgement. The authors are grateful to the referees for their valuable suggestions.

References