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CERTAIN SUBCLASSES OF MULTIVALENTLY NON-BAZILEVIC FUNCTIONS INVOLVING A GENERALIZED MITTAG-LEFFLER FUNCTION

Mohamed Kamal Aouf[1], Tamer M. Seoudy[2]

[1] *Department of Mathematics, Faculty of Science, Mansoura University, Egypt*

[2] *Department of Mathematics, Faculty of Science, Fayoum University, Egypt*

Jamoum University College, Umm Al-Qura University, Makkah, Saudi Arabia

mkaouf127@yahoo.com, tms00@fayoum.edu.eg

Abstract In this paper we introduce a certain subclass of multivalently non-Bazilevic analytic functions defined by a generalized Mittag-Leffler function. Such results as subordination and superordination properties, inclusion theorem, distortion theorems and coefficient estimate for this subclass are proved.

Keywords: Analytic function, subordination, superordination, sandwich-type result, admissible class, integral operator.

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1. INTRODUCTION

Let $\mathcal{H}(\mathbb{U})$ be the class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}[a, m]$ be subclass of $\mathcal{H}(\mathbb{U})$ consisting of functions of the form:

$$f(z) = a + a_m z^m + a_{m+1} z^{m+1} + \dots \ (z \in \mathbb{U}).$$

Also, let $\mathcal{A}(p, m)$ denote the subclass of $\mathcal{H}(\mathbb{U})$ consisting of functions of the form:

$$f(z) = z^p + \sum_{n=p+m}^{\infty} a_n z^n \quad (p, m \in \mathbb{N} = \{1, 2, 3, \dots\}). \tag{1}$$

We write $\mathcal{A}(p, 1) = \mathcal{A}(p)$ and $\mathcal{A}(1, 1) = \mathcal{A}$. The Mittag-Leffler function $E_\alpha(z)\,(z \in \mathbb{C})$ is defined by (see [8] and [9]):

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \, (\alpha \in \mathbb{C}, Re\{\alpha\} > 0). \tag{2}$$

Srivastava and Tomovski [10] introduced the generalized Mittag-Leffler function $E_{\alpha,\beta}^{\gamma,k}(z)\,(z\in\mathbb{C})$ in the form:

$$E_{\alpha,\beta}^{\gamma,k}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{nk}\ z^n}{\Gamma(\alpha n+\beta)\ n!} \tag{3}$$

$$(\alpha,\beta,\gamma\in\mathbb{C};Re\{\alpha\}>\max\{0;Re\{k\}-1\};Re\{k\}>0),$$

where $(\gamma)_m$ denotes the Pochhammer symbol (or the shifted factorial). Srivastava and Tomovski [10] proved that the function $E_{\alpha,\beta}^{\gamma,k}(z)$ defined by (12) is an entire function in the complex $z-$plane .

We now define the function $Q_{p,\alpha,\beta}^{\gamma,k}(z)$ by

$$Q_{p,\alpha,\beta}^{\gamma,k}(z)=\frac{\Gamma(\alpha+\beta)}{(\gamma)_k}z^{p-1}\left(E_{\alpha,\beta}^{\gamma,k}(z)-\frac{1}{\Gamma(\beta)}\right) \tag{4}$$

$$(z\in\mathbb{U};\alpha,\beta,\gamma\in\mathbb{C};Re\{\alpha\}>\max\{0;Re\{k\}-1\};Re\{k\}>0).$$

Corresponding to the function $Q_{p,\alpha,\beta}^{\gamma,k}(z)$ defined by (13), we introduce the following linear operator $\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z):\mathcal{A}(p,m)\to\mathcal{A}(p,m)$ by

$$\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z)=Q_{p,\alpha,\beta}^{\gamma,k}(z)*f(z)$$

$$=z^p+\sum_{n=p+m}^{\infty}\frac{\Gamma(\gamma+(n-p+1)k)\ \Gamma(\alpha+\beta)}{\Gamma(\gamma+k)\Gamma(\beta+\alpha(n-p+1))\ (n-p+1)!}a_nz^n \tag{5}$$

$$(\alpha,\beta,\gamma\in\mathbb{C};Re\{\alpha\}>\max\{0;Re\{k\}-1\};Re\{k\}>0;z\in\mathbb{U}),$$

where the symbol $(*)$ denotes the Hadamard product (or convolution). We note that the operator $\mathcal{H}_{1,\alpha,\beta}^{\gamma,k}f(z)=\mathcal{H}_{\alpha,\beta}^{\gamma,k}$ was introduced by Attiya [1].

Also, it is easily verified from (14) that

$$k\ z\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z)\right)'=(\gamma+k)\ \mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k}f(z)-(\gamma+k-p)\ \mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z) \tag{6}$$

and

$$\alpha z\left(\mathcal{H}_{p,\alpha,\beta+1}^{\gamma,k}f(z)\right)'=(\alpha+\beta)\ \mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z)-(\alpha+\beta-p)\ \mathcal{H}_{p.\alpha,\beta+1}^{\gamma,k}f(z). \tag{7}$$

If $f\ (z)$ and $g(z)$ are analytic in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$, or $g(z)$ is superordinate to $f(z)$, written symbolically, $f\prec g$ in U or $f(z)\prec g(z)\,(z\in\mathbb{U})$, if there exists a Schwarz function $\omega(z)$, which (by definition) is analytic in $\mathbb{U}$ with $\omega(0)=0$ and $|\omega(z)|<1$ $(z\in\mathbb{U})$ such that

$f(z) = g(\omega(z))$ $(z \in \mathbb{U})$. Further more, if the function $g(z)$ is univalent in $\mathbb{U}$, then we have the following equivalence (see [14]):

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\phi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ and $h(z)$ be univalent in $\mathbb{U}$. If $p(z)$ is analytic in $\mathbb{U}$ and satisfies the first order differential subordination:

$$\phi\left(p(z), zp'(z); z\right) \prec h(z), \tag{8}$$

then $p(z)$ is a solution of the differential subordination (8). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (8) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (8). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants of (8) is called the best dominant. If $p(z)$ and $\phi\left(p(z), zp'(z); z\right)$ are univalent in $\mathbb{U}$ and if $p(z)$ satisfies first order differential superordination:

$$h(z) \prec \phi\left(p(z), zp'(z); z\right), \tag{9}$$

then $p(z)$ is a solution of the differential superordination (9). An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination (9) if $q(z) \prec p(z)$ for all $p(z)$ satisfying (9). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants of (9) is called the best subordinant.

Definition 1. *A function $f \in \mathcal{A}(p,m)$ is said to be the class $\mathcal{N}_{p,\alpha,\beta}^{\gamma,k}(\lambda,\mu;A,B)$ if it satisfies the following subordination condition:*

$$(1+\lambda)\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} - \lambda \frac{\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z)}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} \prec \frac{1+Az}{1+Bz}, \tag{10}$$

where $Re\{\alpha\} > \max\{0; Re\{k\}-1\}$, $Re\{k\} > 0$, $0 < \mu < 1$, $\lambda \in \mathbb{C}$, $-1 \leq B \leq 1$, $A \neq B$ and all the powers are principal values.

Furthermore, the function $f \in \mathcal{N}_{p,\alpha,\beta}^{\gamma,k}(\lambda,\mu;\delta)$ if and only if $f \in \mathcal{A}(p,m)$ and

$$Re\left\{(1+\lambda)\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} - \lambda \frac{\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z)}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu}\right\} > \delta$$

$$(0 \leq \delta < 1; z \in \mathbb{U}),$$

we write $\mathcal{N}_{p,\alpha,\beta}^{\gamma,k}(0,\mu;\delta) = \mathcal{N}_{p,\alpha,\beta}^{\gamma,k}(\mu;\delta)$.

In order to establish our main results, we need the following definition and lemmas.

Definition 2. *[9] Denote by Ω the set of all functions f that are analytic and injective on $\overline{\mathbb{U}}\backslash E(f)$, where*

$$E(f) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty \right\},$$

and such that $f^{'}(\zeta) \neq 0$ for $\zeta \in \overline{\mathbb{U}}\backslash E(f)$.

Lemma 1. *[14] Let the function h be analytic and convex (univalent) in $\mathbb{U}$ with $h(0) = 1$. Suppose also that the function $p(z)$ given by*

$$p(z) = 1 + c_m z^m + c_{m+1} z^{m+1} + ... \tag{11}$$

is analytic in $\mathbb{U}$. If

$$p(z) + \frac{zp^{'}(z)}{\eta} \prec h(z) \qquad (\mathbf{Re}(\eta) \geq \mathbf{0}; \eta \neq 0;\ z \in \mathbb{U}), \tag{12}$$

then

$$p(z) \prec q(z) = \frac{\eta}{m} z^{-\frac{\eta}{m}} \int_0^z h(t)\, t^{\frac{\eta}{m}-1}\, dt \prec h(z),$$

and $q(z)$ is the best dominant.

Lemma 2. *[17] Let q be a convex univalent function in $\mathbb{U}$ and let $\sigma \in \mathbb{C}$, $\eta \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ with*

$$Re\left(1 + \frac{zq^{''}(z)}{q^{'}(z)}\right) > \max\left\{0, -Re\left(\frac{\sigma}{\eta}\right)\right\}.$$

If the function p is analytic in $\mathbb{U}$ and

$$\sigma p(z) + \eta z p^{'}(z) \prec \sigma q(z) + \eta z q^{'}(z),$$

then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 3. *[9] Let q be convex univalent in $\mathbb{U}$ and $\kappa \in \mathbb{C}$. Further assume that $Re(\bar{\kappa}) > 0$. If $p(z) \in \mathcal{H}[q(0), 1] \cap \Omega$, and $p(z) + \kappa z p^{'}(z)$ is univalent in $\mathbb{U}$, then*

$$q(z) + \kappa z q^{'}(z) \prec p(z) + \kappa z p^{'}(z),$$

implies $q(z) \prec p(z)$ and q is the best subordinant.

Lemma 4. *[12] Let $\mathcal{F}$ be analytic and convex in $\mathbb{U}$. If $f, g \in \mathcal{A}$ and $f, g \prec \mathcal{F}$ then*

$$\lambda f + (1 - \lambda) g \prec \mathcal{F} \qquad (0 \leq \lambda \leq 1).$$

Lemma 5. *[16] Let $f(z)=\sum_{n=1}^{\infty}a_nz^n$ be analytic in $\mathbb{U}$ and $g(z)=\sum_{n=1}^{\infty}b_nz^n$ be analytic and convex in $\mathbb{U}$. If $f\prec g$, then*

$$|a_n|<|b_1| \quad (n\in\mathbb{N}).$$

In the present paper, we obtain some subordination and superordination properties, distortion theorems and inequality properties of the class $\mathcal{N}_{p,\alpha,\beta}^{\gamma,k}(\lambda,\mu;A,B)$. Several other new results are also obtained.

2. MAIN RESULTS

Unless otherwise mentioned, we assume throughout this paper that $Re\{\alpha\}>\max\{0;Re\{k\}-1\}$, $Re\{k\}>0$, $0<\mu<1$, $\lambda\in\mathbb{C}$, $-1\leq B<A\leq 1$ and $p\in\mathbb{N}$.

We begin by presenting our first subordination property given by

Theorem 2.1. *Let $f\in\mathcal{N}_{p,\alpha,\beta}^{\gamma,k}(\lambda,\mu;A,B)$ with $Re\left(\frac{\gamma+k}{\lambda k}\right)>0$. Then*

$$\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z)}\right)^{\mu}\prec q(z)=\frac{\mu(\gamma+k)}{m\lambda k}\int_0^1\frac{1+Azu}{1+Bzu}u^{\frac{\mu(\gamma+k)}{m\lambda k}-1}du\prec\frac{1+Az}{1+Bz}\quad(z\in\mathbb{U}), \tag{13}$$

and $q(z)$ is the best dominant.

Proof. Define the function $g(z)$ by

$$g(z)=\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z)}\right)^{\mu}\quad(z\in\mathbb{U}). \tag{14}$$

Then the function $g(z)$ is of the form (3) and analytic in $\mathbb{U}$. Taking the derivatives in the both sides of (14), we get

$$(1+\lambda)\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z)}\right)^{\mu}-\lambda\frac{\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k}f(z)}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z)}\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z)}\right)^{\mu}=g(z)+\frac{\lambda k}{\mu(\gamma+k)}zg'(z). \tag{15}$$

Since $f\in\mathcal{N}_{p,\alpha,\beta}^{\gamma,k}(\lambda,\mu;A,B)$, we have

$$g(z)+\frac{\lambda k}{\mu(\gamma+k)}zg'(z)\prec\frac{1+Az}{1+Bz}.$$

Applying Lemma 1 to (15) with $\eta=\frac{\mu(\gamma+k)}{\lambda k}$, we get

$$\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z)}\right)^{\mu}\prec q(z)=\frac{\mu(\gamma+k)}{m\lambda k}\,z^{-\frac{\mu(\gamma+k)}{m\lambda k}}\int_0^z\frac{1+At}{1+Bt}\,t^{\frac{\mu(\gamma+k)}{m\lambda k}-1}dt$$

$$= \frac{\mu(\gamma+k)}{m\lambda k}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{\mu(\gamma+k)}{m\lambda k}-1}du \prec \frac{1+Az}{1+Bz} \tag{16}$$

and $q(z)$ is the best dominant. The proof of Theorem 2.1 is thus completed. ∎

Theorem 2.2. *Let $q(z)$ be univalent in $\mathbb{U}$, $\lambda \in \mathbb{C}^*$. Suppose also that $q(z)$ satisfies the following inequality:*

$$Re\left(1+\frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -Re\left(\frac{\mu(\gamma+k)}{\lambda k}\right)\right\}. \tag{17}$$

If $f \in \mathcal{A}(p,m)$ satisfies the following subordination condition:

$$(1+\lambda)\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z)}\right)^{\mu} - \lambda\frac{\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k}f(z)}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z)}\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z)}\right)^{\mu} \prec q(z) + \frac{\lambda k}{\mu(\gamma+k)}zq'(z), \tag{18}$$

then

$$\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z)}\right)^{\mu} \prec q(z),$$

and $q(z)$ is the best dominant.

Proof. Let the function $g(z)$ be defined by (14). We know that (15) holds true. Combining (15) and (18), we find that

$$g(z) + \frac{\lambda k}{\mu(\gamma+k)}zg'(z) \prec q(z) + \frac{\lambda k}{\mu(\gamma+k)}zq'(z). \tag{19}$$

By using Lemma 2 and (19), we easily get the assertion of Theorem 2.2. ∎

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ in Theorem 2.2, we get the following result.

Corollary 6. *Let $\lambda \in \mathbb{C}^*$ and $-1 \le B < A \le 1$. Suppose also that*

$$Re\left(\frac{1-Bz}{1+Bz}\right) > \max\left\{0, -Re\left(\frac{\mu(\gamma+k)}{\lambda k}\right)\right\}.$$

If $f \in \mathcal{A}(p,m)$ satisfies th following subordination condition:

$$(1+\lambda)\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z)}\right)^{\mu} - \lambda\frac{\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k}f(z)}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z)}\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z)}\right)^{\mu} \prec \frac{1+Az}{1+Bz} + \frac{\lambda k}{\mu(\gamma+k)}\frac{(A-B)z}{(1+Bz)^2},$$

then

$$\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k}f(z)}\right)^{\mu} \prec \frac{1+Az}{1+Bz},$$

and the function $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Theorem 2.3. *Let* q *be convex univalent in* $\mathbb{U}$, $\lambda \in \mathbb{C}$ *with* $Re(\lambda) > 0$. *Also let* $\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}$ *and* $(1+\lambda)\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} - \lambda \frac{\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z)}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu}$ *be univalent in* $\mathbb{U}$. *If* $f \in \mathcal{A}(p)$ *satisfies the following superordination condition:*

$$q(z) + \frac{\lambda k}{\mu(\gamma+k)} z q'(z) \prec (1+\lambda)\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} - \lambda \frac{\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z)}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu},$$

then

$$q(z) \prec \left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu},$$

and the function $q(z)$ *is the best subordinant.*

Proof. Let the function $g(z)$ be defined by (14). Then

$$\begin{aligned} q(z) + \frac{\lambda k}{\mu(\gamma+k)} z q'(z) &\prec (1+\lambda)\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} - \lambda \frac{\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z)}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} \\ &= g(z) + \frac{\lambda k}{\mu(\gamma+k)} z g'(z). \end{aligned}$$

An application of Lemma 3 yields the assertion of Theorem 2.3. ∎

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 2.3, we get the following corollary.

Corollary 7. *Let* $-1 \leq B < A \leq 1$, $\lambda \in \mathbb{C}$ *with* $Re(\lambda) > 0$. *Also let* $\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} \in \mathcal{H}[1, 1] \cap \mathcal{Q}$, *and* $(1+\lambda)\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} - \lambda \frac{\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z)}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu}$ *be univalent in* $\mathbb{U}$. *If* $f \in \mathcal{A}(p)$ *satisfies the following superordination condition:*

$$\frac{1+Az}{1+Bz} + \frac{\lambda k}{\mu(\gamma+k)} \frac{(A-B)z}{(1+Bz)^2} \prec (1+\lambda)\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} - \lambda \frac{\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z)}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu},$$

then

$$\frac{1+Az}{1+Bz} \prec \left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu},$$

and the function $\frac{1+Az}{1+Bz}$ *is the best subordinant.*

Combining the above results of subordination and superordination. we easily get the following "Sandwich-type result".

Theorem 2.4. *Let q_1 be convex univalent and let q_2 be univalent in $\mathbb{U}$, $\lambda \in \mathbb{C}$ with $Re(\lambda) > 0$. Let q_2 satisfies (17). If $\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} \in \mathcal{H}[q_1(0), 1] \cap \mathcal{Q}$, and $(1+\lambda)\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} - \lambda \frac{\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z)}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu}$ be univalent in $\mathbb{U}$, also*

$$\begin{aligned} q_1(z) + \frac{\lambda k}{\mu(\gamma+k)} z q_1'(z) &\prec (1+\lambda)\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} - \lambda \frac{\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z)}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} \\ &\prec q_2(z) + \frac{\lambda k}{\mu(\gamma+k)} z q_2'(z), \end{aligned}$$

then

$$q_1(z) \prec \left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} \prec q_2(z),$$

and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and the best dominant.

Theorem 2.5. *If $\lambda > 0$ and $f \in \mathcal{N}_{p,\alpha,\beta}^{\gamma,k}(\mu;\delta)$ $(0 \le \beta < 1)$. Then $f \in \mathcal{N}_{p,\alpha,\beta}^{\gamma,k}(\lambda,\mu;\delta)$ for $|z| < R$, where*

$$R = \left(\sqrt{\left(\frac{m\lambda k}{\mu(\gamma+k)}\right)^2 + 1} - \frac{m\lambda k}{\mu(\gamma+k)}\right)^{\frac{1}{n}}. \tag{20}$$

The bound R is the best possible.

Proof. We begin by writing

$$\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} = \theta + (1-\theta) g(z) \quad (z \in \mathbb{U}). \tag{21}$$

Then clearly, the function $g(z)$ is of the form (3), is analytic and has a positive real part in $\mathbb{U}$. By taking the derivatives in the both sides in equality (21), we get

$$\frac{1}{1-\theta}\left\{(1+\lambda)\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} - \lambda \frac{\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z)}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} - \theta\right\}.$$

$$= g(z) + \frac{\lambda k}{\mu(\gamma+k)} z g'(z). \tag{22}$$

By making use of the following well-known estimate (see [11, Theorem 1]):

$$\frac{\left|zg'(z)\right|}{Re\{g(z)\}} \leq \frac{2mr^m}{1-r^{2m}} \qquad (|z| = r < 1)$$

in (22), we obtain that

$$Re\left(\frac{1}{1-\theta}\left\{(1+\lambda)\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} - \lambda \frac{\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z)}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} - \theta\right\}\right)$$

$$\geq Re\{g(z)\}\left(1 - \frac{2\lambda k m r^m}{\mu(\gamma+k)(1-r^{2m})}\right). \tag{23}$$

It is seen that the right-hand side of (23) is positive, provided that $r < R$, where R is given by (20).

In order to show that the bound R is the best possible, we consider the function $f \in \mathcal{A}(p,m)$ defined by

$$\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} = \beta + (1-\beta)\frac{1+z^m}{1-z^m} \qquad (z \in \mathbb{U}).$$

By noting that

$$\frac{1}{1-\beta}\left\{(1+\lambda)\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} - \lambda \frac{\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z)}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} - \beta\right\}$$

$$= \frac{1+z^m}{1-z^m} + \frac{2\lambda k m z^m}{\mu(\gamma+k)(1-z^m)^2} = 0, \tag{24}$$

for $z = R\ \exp\left(\frac{\pi i}{m}\right)$, we conclude that the bound is the best possible. Theorem 2.5 is thus proved. ∎

Theorem 2.6. *Let $\lambda_2 \geq \lambda_1 \geq 0$ and $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$. Then*

$$\mathcal{N}_{p,\alpha,\beta}^{\gamma,k}(\lambda_2, \mu; A_2, B_2) \subset \mathcal{N}_{p,\alpha,\beta}^{\gamma,k}(\lambda_1, \mu; A_1, B_1). \tag{25}$$

Proof. Suppose that $f \in \mathcal{N}_{p,\alpha,\beta}^{\gamma,k}(\lambda_2, \mu; A_2, B_2)$. We know that

$$(1+\lambda_2)\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} - \lambda_2 \frac{\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z)}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} \prec \frac{1+A_2 z}{1+B_2 z}.$$

Since $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$, we easily find that

$$(1+\lambda_2)\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} - \lambda_2 \frac{\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z)}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} \prec \frac{1+A_2 z}{1+B_2 z} \prec \frac{1+A_1 z}{1+B_1 z}, \tag{26}$$

that is $f \in \mathcal{N}_{p,\alpha,\beta}^{\gamma,k}(\lambda_2, \mu; A_1, B_1)$. Thus the assertion of Theorem 2.6 holds for $\lambda_2 = \lambda_1 \geq 0$. If $\lambda_2 > \lambda_1 \geq 0$, by Theorem 2.1 and (26), we know that $f \in \mathcal{N}_{p,\alpha,\beta}^{\gamma,k}(0, \mu; A_1, B_1)$, that is,

$$\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} \prec \frac{1+A_1 z}{1+B_1 z}, \tag{27}$$

At the same time, we have

$$(1+\lambda_1)\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} - \lambda_1 \frac{\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z)}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} = \left(1-\frac{\lambda_1}{\lambda_2}\right)\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu}$$

$$+\frac{\lambda_1}{\lambda_2}\left[(1+\lambda_2)\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} - \lambda_2 \frac{\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z)}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu}\right]. \tag{28}$$

Moreover, since $0 \leq \frac{\lambda_1}{\lambda_2} < 1$,and the function $\frac{1+A_1 z}{1+B_1 z}$ $(-1 \leq B_1 < A_1 \leq 1; z \in \mathbb{U})$ is analytic and convex in $\mathbb{U}$. Combining (26)-(28) and Lemma 4, we find that

$$(1+\lambda_1)\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} - \lambda_1 \frac{\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z)}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} \prec \frac{1+A_1 z}{1+B_1 z},$$

that is $f \in \mathcal{N}_{p,\alpha,\beta}^{\gamma,k}(\lambda_1, \mu; A_1, B_1)$, which implies that the assertion (25) of Theorem 2.6 holds. ∎

Theorem 2.7. *Let $f \in \mathcal{N}_{p,\alpha,\beta}^{\gamma,k}(\lambda, \mu; A, B)$ with $\frac{\gamma+k}{\lambda k} > 0$ and $-1 \leq B < A \leq 1$. Then*

$$\frac{\mu(\gamma+k)}{m\lambda k}\int_0^1 \frac{1-Au}{1-Bu}\, u^{\frac{\mu(\gamma+k)}{m\lambda k}-1} du < Re\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu}$$

$$< \frac{\mu(\gamma+k)}{m\lambda k}\int_0^1 \frac{1+Au}{1+Bu}\, u^{\frac{\mu(\gamma+k)}{m\lambda k}-1} du. \tag{29}$$

Proof. Let $f \in \mathcal{N}_{p,\alpha,\beta}^{\gamma,k}(\lambda,\mu;A,B)$ with $\frac{\gamma+k}{\lambda k} > 0$. From Theorem 2.1, we know that (13) holds, which implies that

$$\begin{aligned} Re\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} &< \sup_{z\in\mathbb{U}} Re\left\{\frac{\mu(\gamma+k)}{m\lambda k}\int_0^1 \frac{1+Azu}{1+Bzu}\ u^{\frac{\mu(\gamma+k)}{m\lambda k}-1}du\right\} \\ &\leq \frac{\mu(\gamma+k)}{m\lambda k}\int_0^1 \sup_{z\in\mathbb{U}} Re\left(\frac{1+Azu}{1+Bzu}\right) u^{\frac{\mu(\gamma+k)}{m\lambda k}-1}du \\ &< \frac{\mu(\gamma+k)}{m\lambda k}\int_0^1 \frac{1+Au}{1+Bu}\ u^{\frac{\mu(\gamma+k)}{m\lambda k}-1}du, \end{aligned} \tag{30}$$

and

$$\begin{aligned} Re\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} &> \inf_{z\in\mathbb{U}} Re\left\{\frac{\mu(\gamma+k)}{m\lambda k}\int_0^1 \frac{1+Azu}{1+Bzu}\ u^{\frac{\mu(\gamma+k)}{m\lambda k}-1}du\right\} \\ &\geq \frac{\mu(\gamma+k)}{m\lambda k}\int_0^1 \inf_{z\in\mathbb{U}} Re\left(\frac{1+Azu}{1+Bzu}\right) u^{\frac{\mu(\gamma+k)}{m\lambda k}-1}du \\ &> \frac{\mu(\gamma+k)}{m\lambda k}\int_0^1 \frac{1-Au}{1-Bu}\ u^{\frac{\mu(\gamma+k)}{m\lambda k}-1}du. \end{aligned} \tag{31}$$

Combining (30) and (31), we get (29). The proof of Theorem 2.7 is evidently completed. ∎

Theorem 2.8. *Let*

$$f(z) = z^p + \sum_{n=p+m}^{\infty} a_n z^n \in \mathcal{N}_{p,\alpha,\beta}^{\gamma,k}(\lambda,\mu;A,B). \tag{32}$$

Then

$$|a_{p+m}| \leq \left|\frac{(\gamma+k)\,\Gamma(\gamma+k)\,\Gamma(\beta+\alpha(m+1))\ (m+1)!\,(A-B)}{[\mu(\gamma+k)+m\lambda k]\ \Gamma(\gamma+(m+1)k)\ \Gamma(\alpha+\beta)}\right|. \tag{33}$$

Proof. Combining (2) and (32), we obtain

$$(1+\lambda)\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu} - \lambda\frac{\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,k} f(z)}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\left(\frac{z^p}{\mathcal{H}_{p,\alpha,\beta}^{\gamma,k} f(z)}\right)^{\mu}$$

$$= 1+\left(-\mu-\frac{m\lambda k}{\gamma+k}\right)\frac{\Gamma(\gamma+(m+1)k)\ \Gamma(\alpha+\beta)}{\Gamma(\gamma+k)\,\Gamma(\beta+\alpha(m+1))\ (m+1)!}a_{p+m}z^n+\ldots \prec \frac{1+Az}{1+Bz}. \tag{34}$$

An application of Lemma 5 to (34) yields

$$\left|\left(-\mu-\frac{m\lambda k}{\gamma+k}\right)\frac{\Gamma\left(\gamma+(m+1)k\right)\ \Gamma\left(\alpha+\beta\right)}{\Gamma\left(\gamma+k\right)\Gamma\left(\beta+\alpha\left(m+1\right)\right)\ (m+1)!}a_{p+m}\right|<\left|A-B\right|. \quad (35)$$

Thus, from (35), we easily arrive at (33) asserted by Theorem 2.8. ∎

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