ROMAI J., v.15, no.1(2019), 25–35

NEW IDENTITIES AND LOWER BOUNDS FOR RANDOM VARIABLES: APPLICATIONS FOR CUD AND BETA DISTRIBUTIONS

Zoubir Dahmani

Lab. Pure and Appl. Maths., Faculty of Exact Sciences and Informatics, UMAB,

University of Mostaganem, Algeria

zoubir.dahmani@univ-mosta.dz

Abstract Using Riemann-Liouville integration, some concepts on weighted continuous random variables are introduced and some applications on CUD and beta distributions are discussed. Some fractional bounds estimating the expectations and variances are established too. Also, some classical identities for covariances are fractionally generalised for any $\alpha \geq 1, \beta \geq 1$.

Keywords: integral inequalities, Riemann-Liouville integral, random variable, fractional $\omega-$weighted expectation, fractional $\omega-$weighted covariance.

2010 MSC: 26D15, 26A33, 60E15.

1. INTRODUCTION

The integral inequalities are very important in the theory of differential equations, probability theory and in applied sciences. For more details, we refer the reader to [2, 3, 17, 18, 20, 23] and the references therein. The study of the integral inequalities using fractional integration theory are also of great importance, we refer to [1, 8, 9, 10, 12, 21, 22] for some applications.

In this sense, in a recent work [8], by introducing new concepts on probability theory using fractional integration in the RL-sense, the author extended to $\alpha \geq 0$ some classical results of the papers [3, 17].

Then, based on [8], the authors of [13] introduced other classes of weighted concepts and generalized some classical results of [3, 18].

Very recently, in [11], the author presented some fractional applications for continuous random variables having probability density functions (p.d.f.) defined on some bonded real lines.

Motivated by the papers in [5, 8, 24, 19], in this work, we focus our attention on the applications of the RL-fractional integration on random variables. We begin by recalling some $\omega-$concepts on continuous random variables that can be found in [13]; these notions generalise the case where ω is equal to 1 over $[a, b]$, see [8]. Then, we introduce the notion of $\omega-$covariance. We present, for the first time, some applications on continuous uniform distribution (CUD)

as well as on beta random variable. To estimate the $\omega-$expectations and the variances, we present new lower bounds. Finally, some classical covariance identities, that correspond to $\alpha = 1$; are generalised for any $\alpha \geq 1; \beta \geq 1$.
To explain much clearly which definitions are new, we invite the reader to see the Introduction Section as well as Section 2 and Section 3. For more information on the proposed generalizations, the interested reader can see all the theorems of Sections 4,5 and 6 with their remarks.

2. RL-INTEGRATION AND WEIGHTED RANDOM VARIABLES

In this section, we recall some preliminaries that will be used in this work. We begin by the following definition.

Definition 2.1. [15] *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a continuous function f on $[a;b]$ is defined as*

$$J_a^\alpha[f(t)] = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \ \alpha > 0\ ,\ a < t \leq b, \tag{1}$$
$$J_a^0[f(t)] = f(t),$$

where $\Gamma(\alpha) := \int_0^\infty e^{-u} u^{\alpha-1} du$
For $\alpha > 0, \beta > 0$, we have:

$$J_a^\alpha J_a^\beta [f(t)] = J^\alpha + \beta_a [f(t)] \tag{2}$$

and

$$J_a^\alpha J_a^\beta [f(t)] = J_a^\beta J_a^\alpha [f(t)]. \tag{3}$$

Let us now consider a positive continuous function ω defined on $[a,b]$. We recall the $\omega - concepts$ [13] :

Definition 2.2. *The fractional $\omega - weighted$ expectation of order $\alpha > 0$, for a random variable X with a positive p.d.f. f defined on $[a,b]$ is defined as*

$$E_{\alpha,\omega}(X) := J_a^\alpha[t\omega f](b) = \frac{1}{\Gamma(\alpha)} \int_a^b (b-\tau)^{\alpha-1} \tau\omega(\tau) f(\tau) d\tau, \alpha > 0\ ,\ a < t \leq b, \tag{4}$$

Definition 2.3. *The fractional $\omega - weighted$ variance of order $\alpha > 0$ for a random variable X having a positive p.d.f. f on [a,b] is given by*

$$\sigma^2_{\alpha,\omega}(X) = V_{\alpha,\omega}(X) := \frac{1}{\Gamma(\alpha)} \int_a^b (b-\tau)^{\alpha-1} (\tau - E(X))^2 \omega(\tau) f(\tau) d\tau, \alpha > 0\ . \tag{5}$$

Definition 2.4. *The fractional* $\omega-$ *weighted moment of orders* $r>0,\alpha>0$ *for a continuous random variable* X *having a p.d.f.* f *defined on [a,b] is defined by*

$$E_{\alpha,\omega}(X^r):=\frac{1}{\Gamma(\alpha)}\int_a^b(b-\tau)^{\alpha-1}\tau^r\omega(\tau)f(\tau)d\tau,\alpha>0\ . \tag{6}$$

We introduce also:

Definition 2.5. *Let* f_1,f_2 *be two continuous on [a, b]. We define the fractional* $\omega-$ *weighted covariance of order* $\alpha>0$ *for* $(f_1(X),f_2(X))$ *by:*

$$Cov_{\alpha,\omega}(f_1(X),f_2(X)):=\frac{1}{\Gamma(\alpha)}\int_a^b(b-\tau)^{\alpha-1}(f_1(\tau)-f_1(\mu))(f_2(\tau)-f_2(\mu))\omega(\tau)f(\tau)d\tau, \tag{7}$$

$\alpha>0$, *where* μ *is the classical expectation of* X.

Based on the above definitions, we give the following properties:

3. WEIGHTED PROPERTIES

(1*) If we take $\alpha=1,\omega(t)=1,t\in[a,b]$ in Definition 2.2, we obtain the classical expectation: $E_{1,1}(X)=E(X)$.

(2*) If we take $\alpha=1,\omega(t)=1,t\in[a,b]$ in Definition 2.3, we obtain the classical variance: $\sigma^2_{1,1}(X)=\sigma^2(X)=\int_a^b(\tau-E(X))^2f(\tau)d\tau$.

(3*) If we take $\alpha=1,\omega(t)=1,t\in[a,b]$ in Definition 2.4, we obtain the classical moment of order $r>0, M_r:=\int_a^b\tau^rf(\tau)d\tau$.
We have also :

(4*) $E_{\alpha,\omega}(AX+B)=AE_{\alpha,\omega}(X)+BJ_a^\alpha[\omega f(b)]$.

(5*) $\sigma^2_{\alpha,\omega}(X)=E_{\alpha,\omega}(X^2)-2E(X)E_{\alpha,\omega}(X)+E^2(X)J_a^\alpha[\omega f(b)]$.

(6*) It is to note that in the case where $\omega(t)=1,t\in[a,b]$, we obtain the definitions proposed in the paper [8].

4. APPLICATIONS ON CUD AND BETA DISTRIBUTIONS

We begin by stating the following generalized property of the p.d.f. of X :

Proposition 4.1. *Let* X *be a continuous random variable having a p.d.f.* $f:[a,b]\to\mathbb{R}^+$. *Then we have:*

$$J^\alpha[f(b)]=\frac{1}{\Gamma(\alpha)}E((b-X)^{\alpha-1}),J^\alpha[\omega f(b)]=\frac{1}{\Gamma(\alpha)}E((b-X)^{\alpha-1}\omega(X)). \tag{8}$$

The proof is evident and hence, it is omitted.

Remark 4.1. *In the above proposition, if we take $\alpha = 1, \omega(x) = 1, x \in [a, b]$, we obtain the well known property $\int_a^b f(u)du = 1$.*

In what follows, we present some fractional applications for the uniform random variable as well as for the beta distribution.

4.1. NEW RESULTS ON CONTINUOUS UNIFORM DISTRIBUTION

Let us consider the continuous uniform distribution (CUD) whose $p.d.f.$ is defined for any $x \in [a, b]$ by $f(x) = (b-a)^{-1}$.

1: $\alpha-$Fractional Expectation for CUD:
We have:

$$E_\alpha(X) = (b-a)^{-1}[\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{a(b-a)^\alpha}{\Gamma(\alpha+1)}]; \alpha \geq 1. \tag{9}$$

Remark that if we take $\alpha = 1$ in the above formula, then we obtain the well known expectation of X:

$$E_1(X) = \frac{b+a}{2} = E(X).$$

2: $(2, \alpha)-$Fractional Moment for CUD:
We have, for $\alpha \geq 1$,

$$E_\alpha(X^2) = \frac{2(b-a)^{\alpha+1}}{\Gamma(\alpha+3)} + 2a(\frac{(b-a)^\alpha}{\Gamma(\alpha+2)} + \frac{a(b-a)^{\alpha-1}}{\Gamma(\alpha+1)}) - \frac{a^2(b-a)^{\alpha-1}}{\Gamma(\alpha+1)}]. \tag{10}$$

Taking $\alpha = 1$ in the above formula, we obtain:

$$E_1(X^2) = \frac{a^2 + b^2 + 2ab}{3} = E(X^2). \tag{11}$$

3: $\alpha-$Fractional Variance for CUD:
In this case, the quantity $J^\alpha f(b)$ of Theorem 3.1 of [11] is given by:

$$J^\alpha f(b) = \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha+1)}, \alpha \geq 1. \tag{12}$$

Then, we get, for $\alpha \geq 1$,

$$\sigma_\alpha^2(X) = \frac{2(b-a)^{\alpha+1}}{\Gamma(\alpha+3)} + 2a(\frac{(b-a)^\alpha}{\Gamma(\alpha+2)} + \frac{a(b-a)^{\alpha-1}}{\Gamma(\alpha+1)}) - \frac{a^2(b-a)^{\alpha-1}}{\Gamma(\alpha+1)}]. \tag{13}$$

If $\alpha = 1$, we obtain $\sigma_1^2(X) = \sigma^2(X) = \frac{(b-a)^2}{12}$.

4: $(r, \alpha)-$**Fractional Moment for CUD:**
In the particular case where the $p.d.f.$ of the uniform random X is defined on some positive real interval of type $[0, b]$, the fractional moment of X is given by:

$$E_\alpha(X^r) = \frac{\Gamma(r+1)}{\Gamma(\alpha+r+1)} b^{r+\alpha-1}. \tag{14}$$

Note that if $\alpha = 1$, we obtain the classical moment of order r for the uniform distribution of X:

$$E_1(X^r) = \frac{\Gamma(r+1)}{\Gamma(r+2)} b^r = E(X^r).$$

4.2. NEW RESULTS ON BETA DISTRIBUTION

Now, we consider the beta distribution which is defined, for any $x \in [0, 1]$, by$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}$.
Using the above fractional notions, we obtain:

1: $(r; \alpha)-$**Fractional Moment for Beta Distribution:**

$$E_\alpha(X^r) = \frac{B(\alpha+b-1, a+r)}{\Gamma(\alpha)B(a, b)}. \tag{15}$$

2: $\alpha-$**Fractional Expectation for Beta Distribution:**
Taking $r = 1$, in the above fractional moment formula, we get

$$E_{X,\alpha} = \frac{a\Gamma(a+b)\Gamma(\alpha+b-1)}{\Gamma(\alpha)\Gamma(b)\Gamma(\alpha+b+a)}, \tag{16}$$

which generalises the classical expectation$E(X) = \frac{a}{a+b}$ corresponding to $\alpha = 1$.

3: $(\alpha)-$**Fractional Variance for Beta Distribution:**
Taking into account that $J^\alpha f(b) = \frac{B(\alpha+b-1,a)}{\Gamma(\alpha)B(a,b)}$, we obtain

$$Var_\alpha(X) = E_\alpha(X^2) - 2\mu E_\alpha(X) + \mu^2 \frac{B(\alpha+b-1, a)}{\Gamma(\alpha)B(a, b)}. \tag{17}$$

We remark also that in this case, if we take $\alpha = 1$, we obtain

$$Var_1(X) = Var(X).$$

5. NEW BOUNDS OF FRACTIONAL MOMENTS OF BETA DISTRIBUTION

In this section, we present some fractional results on the beta distribution. We prove

Theorem 5.1. *Let m, n, p, q and α be positive real numbers, such that $(p-m)(q-n) \leq 0$. Then,*

$$B(p, q+\alpha-1)B(m, n+\alpha-1) \geq B(p, n+\alpha-1)B(m, q+\alpha-1), \alpha \geq 1.$$

The proof of this result is based on Definition 2.1 as well as on the beta distribution p.d.f. And, we remark that for $\alpha = 1$, we obtain Lemma 2.1 of [24].

Now, we shall prove another result on the fractional moments for beta distribution. We have:

Theorem 5.2. *Let X, Y, U and V be four random variables, such that$X \sim B(p,q), Y \sim B(m,n), U \sim B(p,n)$ and $V \sim B(m,q)$. If $(p-m)(q-n) \leq 0$, then,*

$$\frac{E_\alpha(X^r)E_\alpha(Y^r)}{E_\alpha(U^r)E_\alpha(V^r)} \geq \frac{B(p,n)B(m,q)}{B(p,q)B(m,n)}, \alpha \geq 1.$$

Proof. In the following weighted version of Chebyshev fractional inequality, (see [1] for no weighted version)

$$J^\alpha p(1)J^\alpha pfg(1) - J^\alpha p(1)J^\alpha pg(1) \geq 0, \tag{18}$$

we take$f(x) = x^{p-m}, g(x) = (1-x)^{q-n}, p(x) = x^{r+m-1}(1-x)^{n-1}; x \in [0,1]$. Then, it yields that

$$B(p,q)B(m,n)E_\alpha(X^r)E_\alpha(Y^r) - B(p,n)B(m,q)E_\alpha(U^r)E_\alpha(V^r). \tag{19}$$

∎

Remark 5.1. *The above theorem generalises Theorem 3.1 of [24].*

We propose also the following $(\alpha, \beta)-$version:

Theorem 5.3. *Let X, Y, U and V be four random variables, such that$X \sim B(p,q), Y \sim B(m,n), U \sim B(p,n)$ and $V \sim B(m,q)$. If $(p-m)(q-n) \leq 0$, then,*

$$\frac{E_\alpha(X^r)E_\beta(Y^r) + E_\beta(X^r)E_\alpha(Y^r)}{E_\alpha(U^r)E_\beta(V^r) + E_\beta(U^r)E_\alpha(V^r)} \geq \frac{B(p,n)B(m,q)}{B(p,q)B(m,n)}, \alpha, \beta \geq 1.$$

Proof. In the following fractional inequality, (see [12]):

$$J^\alpha p(1)J^\beta pfg(1) + J^\beta p(1)J^\alpha pfg(1) - J^\alpha p(1)J^\beta pg(1) + J^\beta p(1)J^\alpha pg(1) \geq 0, \tag{20}$$

we take$f(x) = x^{p-m}, g(x) = (1-x)^{q-n}, p(x) = x^{r+m-1}(1-x)^{n-1}; x \in [0,1]$. Then, we directly deduce the desired formula. ∎

Remark 5.2. *If $\alpha = \beta = 1$, then the above theorem reduces to Theorem 3.1 of [24].*

6. NEW IDENTITIES AND LOWER BOUNDS

We begin by proving a result that generalises a covariance identity in [5]. In our result, the fractional covariance of X and $g(X)$ can be expressed with the derivative of $g(X)$. Then, we use the established fractional identity to prove new lower bounds for the fractional variance of $g(X)$:

Theorem 6.1. *Let x be a random variable having ap.d.f defined on $[a,b]$; $\mu = E(X)$. Then, we have*

$$Cov_\alpha(X, g(X)) = \frac{1}{\Gamma(\alpha)} \int_a^b g'(x)dx \int_a^x (b-t)^{\alpha-1}(\mu - t)f(t)dt, \alpha \geq 1. \tag{21}$$

Proof. By *Definition 5*, and taking $\omega(x) = 1$, we write

$$Cov_\alpha(X, g(X)) = \frac{1}{\Gamma(\alpha)} \int_a^b (b-x)^{\alpha-1}(x-\mu)(g(x) - g(\mu))f(x)dx. \tag{22}$$

Then, we have

$$Cov_\alpha(X, g(X)) = \frac{1}{\Gamma(\alpha)} \int_a^b (b-x)^{\alpha-1}(x-\mu)f(x)dx \int_\mu^x g'(t)dt. \tag{23}$$

Therefore,

$$\begin{aligned} Cov_\alpha(X, g(X)) = &\frac{1}{\Gamma(\alpha)} \int_\mu^b g'(t)dt \int_t^b (b-x)^{\alpha-1}(\mu - x)f(x)dx \\ &- \frac{1}{\Gamma(\alpha)} \int_a^\mu g'(t)dt \int_a^t (b-x)^{\alpha-1}(\mu - x)f(x)dx. \end{aligned} \tag{24}$$

Consequently,

$$Cov_\alpha(X, g(X)) = \frac{1}{\Gamma(\alpha)} \int_a^b g'(t)dt \int_a^t (b-x)^{\alpha-1}(\mu - x)f(x)dx. \tag{25}$$

Theorem 6.1 is thus proved. ∎

The following theorem establishes a lower bound for$Ver_\alpha(g(X))$ of any $C^1([a,b])-$functions. We have:

Theorem 6.2. *Let X be a random variable having a p.d.f defined on $[a,b]$;$\mu = E(X)$. Then, we have*

$$Ver_\alpha(g(X)) \geq \frac{1}{Ver_{X,\alpha}}(\frac{1}{\Gamma(\alpha)}\int_a^b g'(x)dx\int_a^x (b-t)^{\alpha-1}(\mu-t)f(t)dt)^2. \quad (26)$$

For any $g \in C^1([a,b])$.

Proof. We use fractional Cauchy Schwarz inequality on Theorem 6.1. (See [7]). ∎

Let us consider $\omega \in C([a,b])$, that satisfies

$$\int_a^x (b-t)^{\alpha-1}(\mu-t)f(t)dt = (b-x)^{\alpha-1}\sigma^2\omega(x)f(x).$$

Then, we present the following result.

Theorem 6.3. *Let X be a random variable having a p.d.f defined on $[a,b]$ such that $\mu = E(X), \sigma^2 = Var(X)$ and $\omega \in C([a,b])$, such that*

$$\int_a^x (b-t)^{\alpha-1}(\mu-t)f(t)dt = (b-x)^{\alpha-1}\sigma^2\omega(x)f(x).$$

Then, we have

$$Ver_\alpha(g(X)) \geq \frac{\sigma^4(X)}{Ver_\alpha(X)}E_\alpha^2(g'(X)\omega(X)). \quad (27)$$

Proof. It is clear that

$$Cov_\alpha^2(X,g(X)) = [\frac{1}{\Gamma(\alpha)}\int_a^b g'(x)dx(b-x)^{\alpha-1}\sigma^2\omega(x)f(x)dx]^2. \quad (28)$$

On the other hand, we observe that

$$[\frac{1}{\Gamma(\alpha)}\int_a^b g'(x)dx(b-x)^{\alpha-1}\sigma^2\omega(x)f(x)dx]^2) = \sigma^4 E_\alpha^2(g'(X)\omega(X)) \quad (29)$$

Thanks to the fractional version of Cauchy Schwarz inequality [7], and since

$$Cov_\alpha^2(X,g(X)) \leq Ver_\alpha(X)Ver_\alpha(g(X)), \quad (30)$$

then, we obtain

$$\sigma^4 E_\alpha^2(g'(X)\omega(X)) \leq Ver_\alpha(X)Ver_\alpha(g(X)). \quad (31)$$

The proof is complete. ∎

Remark 6.1. *By the formulas (28) and (29), we obtain the following fractional covariance identity*

$$\sigma^2 E_\alpha(g'(X)\omega(X)) = Cov_\alpha(X, g(X)).$$

It generalises the good standard identity obtained in [5] that corresponds to $\alpha = 1$ *and given by*

$$\sigma^2 E(g'(X)\omega(X)) = Cov(X, g(X)).$$

Another generalisation for weighted distributions, by replacing X by $h(X)$, is given by:

Theorem 6.4. *Let* X *be a continuous random variable with support an interval* $[a,b]$*, mean* μ *and density function* f*. Then, for any* $\alpha \geq 1$*, the following general covariance identity holds*

$$Cov_\alpha(h(X), g(X)) = E_\alpha(g'(X)Z(X)), \tag{32}$$

where $g \in C^1([a,b])$*, with* $E|Z(X)g'(X)| < \infty, h(x)$ *is a given function and* $Z(x)f(x)\frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)} = \int_a^x (E(h(X)) - h(t))\frac{(b-t)^{\alpha-1}}{\Gamma(\alpha)} f(t)dt.$

Proof. By definition, we have

$$Cov_\alpha(h(X), g(X)) = \frac{1}{\Gamma(\alpha)} \int_a^b (b-x)^{\alpha-1}(h(x) - h(\mu))(g(x) - g(\mu))f(x)dx \tag{33}$$

and

$$E_\alpha(g'(X)Z(X)) = \frac{1}{\Gamma(\alpha)} \int_a^b (b-x)^{\alpha-1} g'(x)Z(x)f(x)dx. \tag{34}$$

Thanks to the definition of $Z(X)$, we can write

$$\begin{aligned} E_\alpha(g'(X)Z(X)) &= \frac{1}{\Gamma(\alpha)} \int_a^\mu g'(x)dx \int_a^x (b-t)^{\alpha-1}(h(\mu) - h(t))f(t)dt \\ &+ \frac{1}{\Gamma(\alpha)} \int_\mu^b g'(x)dx \int_x^b (b-t)^{\alpha-1}(h(t) - h(\mu))f(t)dt. \end{aligned} \tag{35}$$

This it to say that

$$\begin{aligned} E_\alpha(g'(X)Z(X)) &= \frac{1}{\Gamma(\alpha)} \int_a^\mu (g(\mu) - g(t))(b-t)^{\alpha-1}(h(\mu) - h(t))f(t)dt \\ &+ \frac{1}{\Gamma(\alpha)} \int_\mu^b (g(t) - g(\mu))(b-t)^{\alpha-1}(h(t) - h(\mu))f(t)dt. \end{aligned} \tag{36}$$

Hence, it yields that

$$E_\alpha(Z(X)g'(X)) = Cov_\alpha(g(X), h(X)). \tag{37}$$

∎

Remark 6.2. *Taking $\alpha = 1$, in the above theorem, we obtain* Theorem 2.2 *of [19].*

References

[1] S. Belarbi, Z. Dahmani: *On some new fractional integral inequalities.* J. Inequal. Pure Appl. Math, 10(3), (2009), 1-12.

[2] N.S. Barnett, P. Cerone, S.S. Dragomir, J. Roumeliotis: *Some inequalities for the expectation and variance of a random variable whose PDF is n-time diferentiable.* J. Inequal. Pure Appl. Math., 1(21), (2000), 1-29.

[3] N.S. Barnett, P. Cerone, S.S. Dragomir and J. Roumeliotis: *Some inequalities for the dispersion of a random variable whose PDF is defined on a finite interval.* J. Inequal. Pure Appl. Math., 2(1), (2001), 1-18.

[4] T. Caccoulos, V. Papathanasiou: *On upper bounds for the variance of functions of random variables*, Stat. Proba. Lett., 3, (1985), 175-184.

[5] T. Caccoulos, V. Papathanasiou: *Caracterizations of distribution by variance bounds*, Stat. Proba. Lett., 7, (1989), 351-356.

[6] T. Caccoulos, V. Papathanasiou: *Characterizations of distributions by generalizations of variance bounds and simple proofs of the CLT*, Journal of Statistical Planningand and inference, 63 (1997), 157-171

[7] Z. Dahmani: *About some integral inequalities using Riemann-Liouville integrals.* General Mathematics Vol. 20, No. 4 (2012), 63-69.

[8] Z. Dahmani: *Fractional integral inequalities for continuous random variables.* Malaya J. Mat., 2(2), (2014), 172-179.

[9] Z. Dahmani: *New inequalities in fractional integrals.* International Jourrnal of Nonlinear Sciences, 9(4), (2010), 493-497.

[10] Z. Dahmani: *On Minkowski and Hermite-Hadamad integral inequalities via fractional integration.* Ann. Funct. Anal., Vol. 1 Iss. 1 (2010), 51-58.

[11] Z. Dahmani: *New applications of fractional calculus on probabilistic random variables.* Acta Math. Univ. Comenianae Vol. LXXXVI, 2 (2017), pp. 299307.

[12] Z. Dahmani, L. Tabharit: *On weighted Gruss type inequalities via fractional integrals.* JARPM, Journal of Advanced Research in Pure Mathematics, 2(4), (2010), 31-38.

[13] Z. Dahmani, A.E. Bouziane, M. Houas, M.Z. Sarikaya: *New w-weighted concepts for continuous random variables with applications.* Note di Matematica, Note Mat. 37 (2017) no. 1, 2340.

[14] F. Goodarzi, M. Amini, G.R. Mohtashami Borzadaran: *On Upper Bounds for the Variance of Functions of Random Variables with Weighted Distributions*, Lobachevskii Journal of Mathematics, 37(4), (2016), 422-435.

[15] R. Gorenflo, F. Mainardi: Fractional calculus: *integral and diferential equations of fractional order.* Springer Verlag, Wien, (1997), 223-276.

[16] P. Kumar, S.P. Singh, S.S. Dragomir: *Some inequalities involving beta and gamma functions*, Nonlinear Analysis Forum, 6(1), (2001), 143150.

[17] P. Kumar: *Moment inequalities of a random variable deFIned over a finite interval.* J. Inequal. Pure Appl. Math., Vol. 3 Iss. 3, Art. 41 (2002), 1-24.

[18] P. Kumar: *Inequalities involving moments of a continuous random variable defined over a finite interval.* Computers and Mathematics with Applications, 48 (2004), 257-273.

[19] G.R. Mohtashami Borzadaran, D.N. Shanbhag: *Further results based on Chernoff-type inequalities*, Statistics and Probability Letters, 39 (1998), 109-117.

[20] M. Niezgoda: *New bounds for moments of continuous random varialbes*, Comput. Math. Appl., 60(12), (2010), 3130-3138.

[21] M.Z. Sarikaya, H. Yaldiz: *New generalization fractional inequalities of Ostrowski-Gruss type.* Lobachevskii Journal of Mathematics, 34(4), (2013), 326-331.

[22] M.Z. Sarikaya, H. Yaldiz and N. Basak: *New fractional inequalities of Ostrowski-Gruss type.* Le Matematiche, 2014, 227-235.

[23] R. Sharma, S. Devi, G. Kapoor, S. Ram, N.S. Barnett: *A brief note on some bounds connecting lower order moments for random variables defined on a finite interval.* Int. J. Theo. Appl. Sci., 1(2), (2009), 83-85.