ROMAI J., v.15, no.1(2019), 37–44

ON THE DOUBLE PELL SEQUENCES

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Abstract In this paper, we define the double Pell sequence and extend it to groups and rings. Then we examine the behaviour of the periods of the double Pell sequences in finite 2-generator groups and rings. Also, we give the lenghts of the periods of the double Pell sequences in the generalized quaternion group Q_{2^n} and the ring F for the generating pair (a, b) as applications of the results obtained.

Keywords: the double Pell sequence, group, ring, length.

2010 MSC: 11B39, 16P10, 20D60.

1. INTRODUCTION

For $n \geq 2$, the well-known Pell sequence $\{P_n\}$ is defined as

$$P_n = 2P_{n-1} + P_{n-2},$$

where $P_0 = 0$ and $P_1 = 1$.

In [11], Kilic and Tasci defined the k sequences of the generalized order-k Pell numbers as follows:
for $n > 0$ and $1 \leq \mathrm{i} \leq k$

$$P_n^i = 2P_{n-1}^i + P_{n-2}^i + \cdots + P_{n-k}^i,$$

with initial conditions

$$P_n^i = \begin{cases} 1 & \text{if } n = 1 - i, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1 - k \leq n \leq 0,$$

where P_n^i is the nth term of the ith sequence. If $k = 2$, the generalized order-k Pell sequence, $\left\{P_n^k\right\}$, is reduced to the usual Pell sequence, $\{P_n\}$.

Definition 3. *(Gülec and Taskara [8]) For any real numbers s, t and $n \geq 2$, let $s^2 + t > 0$, $s > 0$ and $t \neq 0$. Then the (s,t)-Pell sequence $\{p_n(s,t)\}_{n \in N}$ is defined by*

$$p_n(s,t) = 2sp_{n-1}(s,t) + tp_{n-2}(s,t)$$

with initial conditions $p_0(s,t) = 0$, $p_1(s,t) = 1$. *If* $s = t = 1$, *the classic Pell sequence is obtained.*

Let G be a finite n-generator group and suppose that $X = \{(x_1, x_2, \ldots, x_n) \in \underbrace{G \times G \times \cdots \times G}_{n} \mid \; < \{x_1, x_2, \ldots, x_n\} >= G\}$. We call $(x_1, x_2, \ldots, x_n)$ a generating n-tuple for G.

Definition 4. *(Deveci and Karaduman [5])A generalized order-k Pell sequence in a finite group is a sequence of group elements* $x_0,\ x_1, \ldots, x_n, \ldots$ *for which, given an initial (seed) set* $x_0, \ldots, x_{j-1}$, *each element is defined by*

$$x_n = \begin{cases} x_0 x_1 \cdots (x_{n-1})^2 & \text{for } j \leq n < k, \\ x_{n-k} x_{n-k+1} \cdots (x_{n-1})^2 & \text{for } n \geq k. \end{cases}$$

It is require that the initial elements of the sequence, $x_0, \ldots, x_{j-1}$, generate the group, thus, forcing the generalized order-k Pell sequence to reflect the structure of the group. We denote the generalized order-k Pell sequence of a group G generated by $x_0, \cdots \ldots,\ x_{j-1}$ by $Q_k(G; x_0, \ldots, x_{j-1})$. If $k = 2$, then a generalized order-2 Pell sequence of group elements is said to be a Pell sequence or Pell orbit of a finite group [5].

Recenlty, the Pell sequences in special groups have been investigated by some authors; see for example, [4, 6, 9].

Definition 5. *Let R be a ring with identity I, the sequence* $\{M_n\}$ *of elements of R is defined by*

$$M_{n+2} = A_1 M_{n+1} + A_0 M_n,\ n \geq 0 \tag{1}$$

where M_0, M_1, A_0 *and* A_1 *are arbitrary elements of R.*

Special cases of (1) were studied in [1, 2, 10, 9].

Definition 6. *(Tasyurdu, et al. [12]).We define the Pell polynomail-type orbit* $P^R_{(\alpha,\beta)}(x) = x_i$ *of* (α, β) *by*

$$x_0 = \alpha,\ x_1 = \beta \text{ and } x_{n+1} = 2\beta x_n + x_{n-1} \text{ for } n \geq 1$$

where R is a 2-generator ring and (α, β) *is a generating pair of the ring R.*

It is well-known that a sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the shortest repeating subsequence is called the period of the sequence. For example, the sequence $a, b, c, d, c, d, c, d, \ldots$ is periodic after the initial element a and has period 2. In particular, if the first n elements in the sequence form a repeating subsequence, then this sequence is simply periodic and its period is n. For example, the sequence $a, b, c, a, b, c, a, b, c, ...$ is simply periodic with period 3.

For given an integer matrix $M = [m_{ij}]$, $M \,(\mathrm{mod}\, \alpha)$ means that all entries of M are modulo α, that is, $M \,(\mathrm{mod}\, \alpha) = (m_{ij} \,(\mathrm{mod}\, \alpha))$. Let us consider the set $\langle M \rangle_\alpha = \{M^n \,(\mathrm{mod}\, \alpha) \mid n \geq 0\}$. If $\gcd(|\det M|, \alpha) = 1$, then $\langle M \rangle_\alpha$ is a cyclic group; if $\gcd(|\det M|, \alpha) \neq 1$, then $\langle M \rangle_\alpha$ is a semigroup. Let the notation $|\langle M \rangle_\alpha|$ denote the cardinality of the set $\langle M \rangle_\alpha$.

The generalized quaternion group $Q_{2^n}, (n \geq 3)$ is defined by presentation

$$Q_{2^n} = \left\langle a, b \mid a^{2^{n-1}} = e,\ b^2 = a^{2^{n-2}},\ y^{-1}xy = x^{-1} \right\rangle.$$

Note that the order of the generalized quaternion group Q_{2^n} is 2^n.

For any prime p, up to isomorphism, the 2-generator ring F of order p^2, which is not field is given by the following presentation

$$F = \langle a, b \mid pa = pb = 0,\ a^2 = a,\ b^2 = b,\ ab = b,\ ba = a \rangle.$$

For more information on the ring F, see [7].

The study of the double-type sequences in groups and rings began with the work of Deveci [3] where the double Fibonacci sequence and the basic double Fibonacci in groups and rings were investigated. Now we extend the concept to the double Pell sequences.

In this paper, we define the double Pell orbits and the basic double Pell orbits of groups and rings and then we give the lengths of the periods of the double Pell orbits and the basic double Pell orbits of the generalized quaternion group Q_{2^n} and the ring F for the generating pair (a, b).

2. THE MAIN RESULTS

Now we define the double Pell sequence $\{P(n,k)\}$ as shown:

$$P(n,k) = P_{n+2}P_{k+1} + P_{n+1}P_k \text{ for } n, k \geq 0. \tag{2}$$

It is important to note that $P(n,k) = P_{n+k+2}$.

Definition 7. *Let G be a 2-generator group and suppose that (x, y) is a generating pair of G. For generating pair (x, y), we define the kth double Pell orbit $P^k_{(x,y)}(G) = \{a^k_n\}$ by*

$$a^k_{n+2} = \left(a^k_{n+1}\right)^{P_k} \left(a^k_n \left(a^k_{n+1}\right)^2\right)^{P_{k+1}},\ n \geq 1$$

in which $a^k_1 = x$, $a^k_2 = y$ and $k \geq 1$.

Theorem 2.1. *A kth double Pell orbit of a finite 2-generator group is periodic for every integer $k \geq 1$.*

Proof. Let $|G| = m$, then it is easy to see that there are m^2 distinct 2-tuples of elements of G. Thus it is verified that at least one of the 2-tuples appears twice in a kth double Pell orbit. Because of the repeating, the kth double Pell orbit is periodic. ∎

Let $LP^k_{(x,y)}(G)$ be the notation used to denote the length of the period of the kth double Pell orbit $P^k_{(x,y)}(G)$.

Definition 8. *Let $k \geq 1$ be an integer and let u be smallest positive integer such that $a^k_u = a^k_{u+LP^k_{(x,y)}(G)}$ and $a^k_{u+1} = a^k_{u+LP^k_{(x,y)}(G)+1}$. For a generating pair (x, y), the kth basic double Pell orbit $\overline{P^k_{\left(b^k_1, b^k_2\right)}}(G)$ of the basic period m is a sequence of group elements $b^k_1, b^k_2, \ldots, b^k_n, \ldots$ for which, given the initial (seed) set $b^k_1 = a^k_u$, $b^k_2 = a^k_{u+1}$, each element is defined by*

$$b^k_{n+2} = \left(b^k_{n+1}\right)^{P_k}\left(b^k_n\left(b^k_{n+1}\right)^2\right)^{P_{k+1}}, \; n \geq 1$$

in which $m \geq 2$ is the smallest integer with $b^k_1 = b^k_{m+1}\theta$ and $b^k_2 = b^k_{m+2}\theta$ for some $\theta \in AutG$ (where $AutG$ means that the authomorphism group of G).

We denote the lenght of the period of the kth basic double Pell orbit $\overline{P^k_{\left(b^k_1, b^k_2\right)}}(G)$ by $\overline{BLP^k_{\left(b^k_1, b^k_2\right)}}(G)$ and call the basic length of the kth basic double Pell orbit.

Theorem 2.2. *Let G be a finite 2-generator group and suppose that (x, y) is a generating pair of G. If $LP^k_{(x,y)}(G) = \lambda$ and $\overline{BLP^k_{\left(a^k_u, a^k_{u+1}\right)}}(G) = m$, then m divides λ.*

Proof. Let u be smallest positive integer such that $a^k_u = a^k_{u+LP^k_{(x,y)}(G)}$ and $a^k_{u+1} = a^k_{u+LP^k_{(x,y)}(G)+1}$. Then we have $\lambda = \eta \cdot m$ where η is order of automorphism $\theta \in AutG$ since

$$P^k_{\left(a^k_u, a^k_{u+1}\right)}(G) = \overline{P^k_{\left(a^k_u, a^k_{u+1}\right)}}(G) \cup \overline{P^k_{\left(a^k_u\theta, a^k_{u+1}\theta\right)}}(G) \cup \overline{P^k_{\left(a^k_u\theta^2, a^k_{u+1}\theta^2\right)}}(G) \cup \cdots$$

and

$$\overline{BLP^k_{\left(a^k_u, a^k_{u+1}\right)}}(G) = \overline{BLP^k_{\left(a^k_u\theta, a^k_{u+1}\theta\right)}}(G).$$

∎

Reducing the double Pell sequence $\{P(n,k)\}$ by a modulus α, then we get the repeating sequence, denoted by

$$\{P^{\alpha}(n,k)\} = \{P^{\alpha}(1,k), P^{\alpha}(2,k), \ldots, P^{\alpha}(i,k), \ldots\}$$

where $P^{\alpha}(i,k) = P(i,k) \,(\mathrm{mod}\, \alpha)$. It has the same recurrence relation as in (2).

Theorem 2.3. *The sequence $\{P^{\alpha}(n,k)\}$ is periodic for every positive integers k and $\alpha \geq 2$.*

Proof. Let $Q = \{(q_1, q_2) \mid q_i\text{'s are integers such that } 0 \leq q_i \leq \alpha - 1\}$. Since there are α^2 distinct 2-tuples of elements of Z_{α}, at least one of the 2-tuples appears twice in the sequence $\{P^{\alpha}(n,k)\}$. Thus, the subsequence following this 2-tuple repeats; hence, the sequence is periodic. ∎

We next denote the period of the sequence $\{P^{\alpha}(n,k)\}$ by $PerP^{\alpha}(n,k)$. Now we consider the lengths of the periods of the kth double Pell orbits and the kth basic double Pell orbits of the generalized quaternion group $Q_{2^n}, (n \geq 3)$ for $k \geq 1$.

Theorem 2.4. *For $k \geq 1$, $n \geq 3$ and generating pair (a,b), the lengths of the periods of the kth double Pell orbits and the kth basic double Pell orbits of the generalized quaternion group Q_{2^n} are as follows:*

(i). If k is odd, then $LP^{k}_{(a,b)}(Q_{2^n}) = PerP^{\left|ba^{P_{k+1}}\right|}$ (where the order of the element $ba^{P_{k+1}}$ is denoted by $\left|ba^{P_{k+1}}\right|$).

(ii). If k is even, then $\overline{BLP^{k}_{(a,b)}}(Q_{2^n}) = 2$ and $LP^{k}_{(a,b)}(Q_{2^n}) = 2\lambda$ where λ is the least integer with $(P_{k+1})^{\lambda} \equiv 1 \bmod 2^{n-1}$.

Proof. (i). If k is odd, then P_k is odd and P_{k+1} is even. So, we have the sequence

$$a^k_1 = a,\ a^k_2 = b,\ a^k_3 = ba^{P_{k+1}},\ a^k_4 = ba^{P_{k+1}}, \ldots.$$

Then, it is clear that the length of the period of the sequence is $PerP^{\left|ba^{P_{k+1}}\right|}$.
(ii). If k is even, then P_k is even and P_{k+1} is odd. So, we have the sequence

$$\begin{aligned}
a^k_1 &= a,\ a^k_2 = b,\ a^k_3 = a^{P_{k+1}},\ a^k_4 = b, \\
a^k_5 &= a^{(P_{k+1})^2},\ a^k_4 = b, \ldots \\
a^k_i &= a^{(P_{k+1})^{\frac{i-1}{2}}},\ a^k_{i+1} = b, \ldots.
\end{aligned}$$

Since $|a| = 2^{n-1}$, it is easy to see that the length of the period of the sequence is 2λ where λ is the least integer such that $(P_{k+1})^{\lambda} \equiv 1 \bmod 2^{n-1}$. Also, we obtain $\overline{BLP^{k}_{(a,b)}}(Q_{2^n}) = 2$ since $a\theta = a^{\beta}$ and $b\theta = b$ where β is the least integer with $(P_{k+1})\,\beta \equiv 1 \bmod 2^{n-1}$ and θ is a automorphism of order λ. ∎

Definition 9. *Let R be a 2-generator ring and suppose that (x, y) be a generating pair of R. For generating pair (x, y), we define the double Pell orbit $P^{r}_{(x,y)}(R) = \{x_n\}$ by*

$$x_1 = x, x_2 = y, x_{n+2} = xx_n + (x + 2y)\,x_{n+1}, n \geq 1.$$

Theorem 2.5. *If R is a finite 2-generator ring and (x, y) is a generating pair of R, then the double Pell orbit $P^{r}_{(x,y)}(R)$ is periodic.*

Proof. The proof is similar the proof of Theorem 2.1 and is omitted. ∎

Let $LP^{r}_{(x,y)}(R)$ be the notation used to denote the length of the period of the sequence $P^{r}_{(x,y)}(R)$.

Definition 10. *Let u be smallest positive integer such that $x_u = x_{u+LP^{r}_{(x,y)}(R)}$ and $x_{u+1} = x_{u+LP^{r}_{(x,y)}(R)+1}$. For a generating pair (x, y), the basic double Pell orbit $\overline{P^{r}_{(c_1,c_2)}}(R)$ of the basic period m is a sequence of ring elements $c_1, c_2, \ldots, c_n, \ldots$ for which, given the initial (seed) set $c_1 = x_u$, $c_2 = x_{u+1}$, each element is defined by*

$$c_{n+2} = xc_n + (x + 2y)\,c_{n+1},\ n \geq 1$$

where $m \geq 2$ is the smallest integer with $c_1 = c_{m+1}\theta$ and $c_2 = c_{m+2}\theta$ for some $\theta \in AutR$ (where $AutR$ means that the set of all authomorphisms of the ring R).

We denote the length of the period of the sequence $\overline{P^{r}_{(c_1,c_2)}}(R)$ by $\overline{BLP^{r}_{(c_1,c_2)}}(R)$. The period $\overline{BLP^{r}_{(c_1,c_2)}}(R)$ is called the basic double length of the ring R with respect to the initial (seed) set c_1, c_2.

Theorem 2.6. *Let R be a finite 2-generator ring and let (x, y) be a generating pair of R. If $LP^{r}_{(x,y)}(R) = \beta$ and $\overline{BLP^{r}_{(x_u,x_{u+1})}}(R) = m$, then m divides β.*

Proof. The proof is similar the proof of Theorem 2.2 and is omitted. ∎

Let M be a 2×2 companion matrix as follows:

$$M = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}.$$

Since $\det M = -1$, it is clear that $\langle M \rangle_p$ is a cyclic group for every prime p. Now we give lengths of the periods of the double Pell orbits and the basic double Pell orbits of the ring F by the aid of the order of the cyclic group $\langle M\rangle_p$.

Theorem 2.7. *The lengths of the double Pell periods and the basic double Pell periods of the ring F for the generating pairs* (a,b) *are as follows:*

$$LP^r_{(a,b)}(F) = \left|\langle M\rangle_p\right|$$

and for $p \neq 3$,

$$\overline{BLP^r_{(a,b)}}(F) = \frac{\left|\langle M\rangle_p\right|}{2}.$$

Proof. It is important to note that $pa = pb = 0$, $a^2 = a$, $b^2 = b$, $ab = b$ and $ba = a$. We have the sequence

$$x_1 = a,\ x_2 = b,\ x_3 = 3b + a,\ x_4 = 10b + 3a, \ldots$$

$$x_{\frac{\left|\langle M\rangle_p\right|}{2}+1} = (p-1)\,a,\ x_{\frac{\left|\langle M\rangle_p\right|}{2}+2} = (p-1)\,b, \ldots$$

$$x_{\left|\langle M\rangle_p\right|+1} = (p+1)\,a,\ x_{\left|\langle M\rangle_p\right|+2} = (p+1)\,b, \ldots.$$

Using the above we obtain

$$x_{\left|\langle M\rangle_p\right|+1} = a,\ x_{\left|\langle M\rangle_p\right|+2} = b.$$

Then, it is readily seen that the length of period of the sequence $\{x_n\}$ is $\left|\langle M\rangle_p\right|$. Also, from the sequence $\{x_n\}$, we obtain $\overline{BLP^r_{(a,b)}}(F) = \frac{\left|\langle M\rangle_p\right|}{2}$ for $p \neq 3$ since $a\theta = (p-1)\,a$ and $b\theta = (p-1)\,b$ where θ is an automorphism of order 2. ∎

3. CONCLUSION

In [3, 4, 5, 6, 9, 12], the authors studied some linear recurrence sequences in groups and rings. In this paper, we expanded the theory the double Pell sequence which is related to Pell sequence. We defined the double Pell orbits and the basic double Pell orbits of 2-generator groups and 2-generator rings and then we discussed these sequences in finite groups and finite rings. Furthermore, we gave the lengths of the periods of the double Pell sequences and the double basic Pell sequences in the generalized quaternion group Q_{2^n} and the ring F.

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