ROMAI J., v.15, no.1(2019), 73–82

THE NUMERICAL SOLUTION OF THIRD-ORDER NON-LOCAL BOUNDARY VALUE PROBLEMS IN ODES BY THE FINITE DIFFERENCE METHOD

Pramod Kumar Pandey

Department of Mathematics, Dyal Singh College, University of Delhi, India

pramod_10p@hotmail.com

Abstract In this article we have proposed a finite difference method for the numerical solution of third order nonlocal boundary value problem. We have established the order of convergence of the proposed method theoretically which is quadratic. The computational results in numerical experiment on linear and nonlinear model test problems, verify the efficiency and theoretically established order of the convergence of the proposed method.

Keywords: boundary value problem, finite difference method, explicit inverse, integral boundary condition, second order convergence, third order two point BVP.
2010 MSC: 65L10, 65L12.

1. INTRODUCTION

A third order differential equation and corresponding boundary value problems arise in applied mathematics and physics. Understanding and explaining the concept in fluid dynamics, theory of boundary layer and theory of beam are some specific areas of study yields third order differential equations [1].

In this article we consider following third order boundary value problem and a boundary conditions,

$$u'''(x) = f(x, u), \quad a < x < b, \tag{1}$$

subject to the boundary conditions

$$u(a) = \alpha, \quad u''(a) = \beta \quad and \qquad u(b) = \int_a^b g(x)u(x)dx$$

where α, β and $u(b)$ are real constant and $f(x, u)$ is continuous.

There are several research works in the literature on the existence and uniqueness / multiplicity of solution of third-order differential equations and corresponding boundary value problems with two point boundary conditions especially in of [2, 3, 4] and references therein using different methods. Thus we have preassumed the existence and uniqueness of the solution of the problem (1).

Recently, numerical solution of third order multi point boundary value problems gained some importance and works on numerical solution were reported using, reproducing kerenel method [5], spline approximation method [6, 7, 8], superposition and chasing method [9] and finite difference method [9, 10, 11] and references therein. But multi point boundary value problem is a particular case of boundary value problem with integral boundary condition/s.

To the best of our knowledge a little work reported in the literature on numerical solution on third order nonlocal/ integral boundary condition/s. Hence the purpose and emphasis in this article will be on to develop and discuss a finite difference method for the numerical solution of the third order nonlocal boundary value problems.

We have presented our work in this article as follows. In the next section we proposed a finite difference method. We have discussed derivation and convergence of the proposed method under appropriate condition in Section 3 and Section 4 respectively. The application of the proposed method on the model problems and numerical results so produced to show the efficiency in Section 5. Discussion and conclusion on the performance of the proposed method are presented in Section 6.

2. THE DIFFERENCE METHOD

We substitute domain $[a, b]$ by a discrete set of points and call these points as nodal points. We wish to determine the numerical solution of the problem at these nodal points. Thus we define nodal points $a \le x_0 < x_1 < \cdots < x_N \le b$ in $[a, b]$ the domain of the considered problem using a uniform step length h such that $x_i = a + ih$, $i = 0, 1, \cdots, N+1$. We denote the numerical approximation of $u(x)$ at node $x = x_i$ as u_i, $i = 1, 2, \cdots, N$. The approximate value of the theoretical value of the forcing function $f(x, u(x))$ at node $x = x_i$, $i = 0, 1, 2, \cdots, N$ denoted by f_i. Thus, the continuous problem (1) transformed into a discrete problem by the application of the finite difference. Thus, at node $x = x_i$ we obtained following discrete problem,

$$u_i''' = f_i, \quad a < x_i < b, \tag{2}$$

subject to the boundary conditions

$$u_0 = \alpha, \quad u_0'' = \beta \quad \text{and} \qquad u_{N+1} = \int_a^b g(x)y(x)dx$$

The numerical solution of the problem (1) subject to local boundary conditions by the method of spline reported in [8]. Though the idea of discretization in [8] uses boundary conditions in a natural way. Following the idea of discretization of the problem (1) in [8], we discretize the problem (2) in (a, b) at nodes x_i,

$$
\begin{aligned}
-u_{i-1} + 2u_i - u_{i+1} &= h^2 u''_{i-1} + \frac{h^3}{12}(f_{i-1} - f_i), \quad i = 1 \qquad (2)\\
u_{i-2} - 3u_{i-1} + 3u_i - u_{i+1} &= \frac{h^3}{2}(f_{i+1} - 3f_i), \quad 2 \le i \le N.
\end{aligned}
$$

But in considered problem boundary condition is non-local, so we need to discretize boundary condition. Thus, we approximate the non local boundary condition, i.e. the integral that appeared at the end point x_{N+1} by the composite / repeated trapezoidal quadrature method [14]. Let us define nodal points $t_0, t_1, ..., t_{N+1}$ in $[a, b]$ using uniform step length h such that $t_j = a + jh$ and quadrature nodes $\lambda_j, \quad j = 0, 1, 2, ..., N + 1$. Thus,

$$
u_{N+1} = \int_a^b g(t)u(t)dt = \sum_{j=0}^{N}[g_j\lambda_j u_j + E_{tj}] + g_{N+1}\lambda_{N+1}u_{N+1} \qquad (4)
$$

where E_{tj} is the truncation error in j^{th} interval. The numerical coefficients in (4) i.e. λ_j are defined as,

$$
\lambda_j = \begin{cases} \frac{1}{2}h & \text{if } j = 0, N+1 \\ h & \text{otherwise } j = 1, 2, ..., N \ . \end{cases}
$$

The quadrature nodes $\lambda_i = h, i = 1, 2, ..., N$, do not depend on the functions $g(t), u(t)$. The term E_{tj} in (4) depend on h and small value of h reduces E_{tj} considerably.

Thus we obtained a system of linear equations or nonlinear equations in u_i, $i = 1, 2, \cdots, N$ if the forcing function $f(x, u)$ is linear or nonlinear respectively. Thus the solution of a system of equations is the solution of the considered problem at discrete nodes x_i in the considered region.

3. DEVELOPMENT OF THE FINITE DIFFERENCE METHOD

In this section we outline the development of the proposed finite difference method (3). Consider the following linear combination of solution, the second derivative of the solution of the problem (1) and forcing function $f(x, u)$ at the discrete points of $[a, b]$

$$
a_0 u_{i-1} + a_1 u_i + a_2 u_{i+1} + h^2 b_0 u''_{i-1} + h^3(c_0 f_{i-1} + c_1 f_i) + T_i = 0, \qquad i = 1. \quad (5)
$$

where $a_0, a_1, a_2, b_0, c_0, c_1$ are constant to be determined under appropriate conditions and T_i is the truncation error. Let us assume $u^{(4)}(x)$ is continuous and differentiable in [a,b] and $f(x,u)$ is differentiable in $(a,b)\times R$. Let us write each term of (5) in a Taylor series about point x_i. Compare the coefficients $h^p, p = 0,1,..,4$ in the so obtained series and using (2). Thus we find following system of equations,

$$\begin{aligned} a_0 + a_1 + a_2 &= 0 \\ a_0 - a_2 &= 0 \\ a_0 + a_2 + 2b_0 &= 0 \\ -a_0 + a_2 &= 6c_0 + 6c_1 \\ a_0 + a_2 &= -24c_0 \end{aligned} \tag{5}$$

Solving the system of equations (6), we have

$$(a_0, a_1, a_2, b_0, c_0, c_1) = (1, -2, 1, -1, \frac{-1}{12}, \frac{1}{12}) \tag{7}$$

Substituting these constants $(1, -2, 1, -1, \frac{-1}{12}, \frac{1}{12})$ in (5), we have obtained

$$u_{i-1} - 2u_i + u_{i+1} - h^2 u''_{i-1} - \frac{h^3}{12}(f_i - f_{i-1}) + T_i = 0, \qquad i = 1. \tag{8}$$

So,

$$\frac{h^5}{24} u_i^{(5)} + T_i = 0, \quad i = 1 \tag{9}$$

Neglecting the term T_i in (8), we obtained

$$u_{i-1} - 2u_i + u_{i+1} = h^2 u''_{i-1} + \frac{h^3}{12}(f_i - f_{i-1}), \qquad i = 1. \tag{10}$$

Following, the above method of Taylor series expansion and undetermined coefficients, we have following equations,

$$u_{i-2} - 3u_{i-1} + 3u_i - u_{i+1} = \frac{h^3}{2}(f_{i+1} - 3f_i), \tag{11}$$

and

$$\frac{h^5}{2} u_i^{(5)} + T_i = 0, \quad 2 \le i \le N. \tag{12}$$

Thus, in the discretization of the problem (1) at discrete nodal points of $[a,b]$ we conclude from (9) and (12) that the order of local truncation error at nodes $x_i, \quad i = 1, 2, ..., N$ is of $O(h^5)$.

4. CONVERGENCE ANALYSIS

In this section we discuss convergence of the proposed finite difference method (3) for the following linear test problem,

$$u'''(x) = f(x), \qquad a < x < b. \tag{13}$$

subject to the boundary conditions $u_0 = \alpha$, $u_0'' = \beta$ and $u_{N+1} = \int_a^b g(t)u(t)dt$.

For the exact solution $\mathbf{U}$ of the considered problem, we write the proposed finite method (3) in matrix form for the solution of the considered problem in the following form,

$$\mathbf{JU} = \mathbf{F} + \mathbf{T} \tag{14}$$

where the matrices in (14) are defined as,
$\mathbf{J} = \mathbf{D} + \mathbf{D}_1$ and

$$\mathbf{D} = \begin{pmatrix} 2 & -1 & & & & 0 \\ -3 & 3 & -1 & & & \\ 1 & -3 & 3 & -3 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & -3 & 3 & -1 \\ 0 & & & 1 & -3 & 3 \end{pmatrix}_{N\times N},$$

$$\mathbf{D}_1 = -\frac{2h}{2 - hg_{N+1}} \begin{pmatrix} & & 0 & \\ & & \vdots & \\ & & 0 & \\ g_1 & g_2 & \cdots & g_N \end{pmatrix}_{N\times N},$$

$\mathbf{U} = (U_i)_{N\times 1}$, $\mathbf{F} = (F_i)_{N\times 1}$ where

$$F_i = \begin{cases} \alpha + h^2\beta + \frac{h^3}{12}(f_{i-1} - f_i), & i = 1 \\ \frac{h^3}{2}(f_{i+1} - 3f_i), & 1 < i < N-1 \\ u_{N+1} + \frac{h^3}{2}(f_{i+1} - 3f_i) & i = N \end{cases}$$

and $\mathbf{T} = (T_i)_{N\times 1}$ where

$$T_i = -\frac{h^5}{24} \begin{cases} u_i^{(5)}, & i = 1 \\ 12u_i^{(5)}, & 2 \le i \le N \end{cases} \tag{15}$$

Let matrix $\mathbf{u} = (u_i)_{N\times 1}$ be the approximate solution of the considered problem. Thus we write the proposed finite method (3) in the matrix form,

$$\mathbf{Ju} = \mathbf{F} \tag{16}$$

Let us define an error ϵ_i at each node $x_i, i = 1, 2, .., N$, such that

$$\epsilon_i = U_i - u_i \tag{17}$$

and matrix $\boldsymbol{\epsilon} = (\epsilon_i)_{N\times1}$.
Subtracting (16) from (14) and using (17), we obtained following error equation in matrix form,

$$\mathbf{J}\boldsymbol{\epsilon} = \mathbf{T} \tag{18}$$

The matrix $\mathbf{D}$ is invertible [12, 13]. Let matrix $\mathbf{B} = (b_{l,m})_{N\times N}$ be the explicit inverse of matrix $\mathbf{D}$ where

$$b_{l,m} = \frac{1}{2(N+1)}\begin{cases}(N(N+1) - (m-1)(2N-m+2))l, & l \le m\\ (N-l+1)(l(N+1) - m(m-1)), & l > m\end{cases}$$

Let matrix $\mathbf{R} = (R_l)_{N\times1}$, denotes the matrix of the row sum of the matrix $\mathbf{B} = (b_{l,m})_{N\times N}$ where,

$$R_l = \sum_{m=1}^{N} b_{l,m}, \quad l = 1, 2, .., N. \tag{19}$$

Thus,

$$R_l = \frac{l}{6}((N+1)^2 - l^2), \quad l = 1, 2, .., N.$$

It is clear that $\mathbf{R}_l$ is maximum at $l = \frac{(N+1)\sqrt{3}}{3}$. Hence we have obtained,

$$\max_{1\le l\le N}(R_l) = \frac{1}{9\sqrt{3}}(N+1)^3 \tag{20}$$

Thus, we have for large N,

$$\|\mathbf{B}\| = \frac{1}{9\sqrt{3}}\frac{(b-a)^3}{h^3} \tag{21}$$

For same order any square matrix $\mathbf{S}$ and identity matrix $\mathbf{I}$ such that $||\mathbf{S}|| < 1$ then matrix $(\mathbf{I} + \mathbf{S})$ is invertible [14, 15], and

$$\|(\mathbf{I} + \mathbf{S})^{-1}\| < \frac{1}{1 - \|\mathbf{S}\|} \quad .$$

Let us assume,

$$\frac{2N^2hM^*}{(N+1)(2-hM^*)} < 1, \quad M^* = \max_{x\in[a,b]} g(x) \quad \text{and} \quad M = \max_{x\in[a,b]}\left|u^{(5)}(x)\right|.$$

From (18) and (21) we have,

$$\|\boldsymbol{\epsilon}\| \leq \frac{(1+b-a)(b-a)^3}{9\sqrt{3}h^3}\|\mathbf{T}\| \tag{22}$$

From (15) and (22) we have,

$$\|\boldsymbol{\epsilon}\| \leq \frac{(1+b-a)(b-a)^3}{18\sqrt{3}}h^2 M \tag{23}$$

It follows from (23) that $\|\boldsymbol{\epsilon}\| \to 0$ as $h \to 0$. Thus we have established the convergence of the proposed method (3) and the order of convergence of the proposed method (3) is at least $O(h^2)$.

5. NUMERICAL RESULTS

To verify the theoretical development and computational efficiency of the proposed method, we have considered three linear and nonlinear model problems. We have presented numerical results in Tables. In each tabulated numerical results, we have shown MAE the maximum absolute error in the approximate solution $u(x)$ of the problems (1) for different values of N and uniform step size h. We have used the following formula in computation of MAE,

$$MAE = \max_{1\leq i\leq N} |U(x_i) - u_i|.$$

For the solution of system of equations (3), we have used Gauss Seidel and Newton-Raphson method respectively for linear and nonlinear system of equations. All computations were performed on a Windows 2007 Home Basic operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Core i3-2330M, 2.20 Ghz PC. The solutions are computed on N nodes and iteration is continued until either the maximum difference between two successive iterates is less than 10^{-10} or the number of iterations reached 10^4.

Problem 1. The linear model problem in [11] with different boundary conditions is given by

$$u'''(x) = xu(x) + f(x), \quad 0 < x < 1$$

subject to the boundary conditions

$$u(0) = 0 \quad , \quad u''(0) = 0 \qquad \text{and} \quad u(1) = \int_0^1 g(x)u(x)dx$$

Table 1 Maximum absolute error (Problem 1).

	N+1				
	128	256	512	1024	2048
MAE	.28495770(-3)	.69653892(-4)	.17408880(-4)	.43131377(-5)	.10286080(-5)
Order	-	2.03247	2.00038	2.01301	2.06804

where $f(x)$ is calculated so that the analytical solution of the problem is $u(x) = x(1-x)\exp(x)$ and $g(x) = \sin(1.0 - 2x - x^2) - 1.0$. The MAE computed by method (3) for different values of N are presented in Table 1.

Problem 2. The linear model problem in [16] with different boundary conditions is given by,

$$u'''(x) = u(x) + f(x), \quad 0 < x < 1$$

subject to the boundary conditions

$$u(0) = 1 \quad , \quad u''(0) = -1 \qquad \text{and} \quad u(1) = \int_0^1 g(x)u(x)dx$$

where $f(x)$ is calculated so that the analytical solution of the problem is $u(x) = (1-x)\exp(x)$ and $g(x) = \frac{\exp(x^2)-1.0}{2.0}$. The MAE computed by method (3) for different values of N are presented in Table 2.

Problem 3. The nonlinear model problem in [16] with different boundary conditions is given by,

$$u'''(x) = e^{-x}u^2(x) + f(x), \quad 0 < x < 1$$

subject to the boundary conditions

$$u(0) = 0 \quad , \quad u''(0) = -2 \qquad \text{and} \quad u(1) = \int_0^1 g(x)u(x)dx$$

where $f(x)$ is calculated so that the analytical solution of the problem is $u(x) = (1-x)\sin(x)$ and $g(x) = \frac{\exp(x)-1.0}{\cos(x-2)}$. The MAE computed by method (3) for different values of N are presented in Table 3.

Table 2 Maximum absolute error (Problem 2).

	N+1				
	128	256	512	1024	2048
MAE	.12575090(-3)	.32846816(-4)	.76917931(-5)	.18957071(-5)	.67334622(-6)
Order	-	1.93674	2.09436	2.02058	1.49332

Table 3 Maximum absolute error (Problem 3).

	N+1				
	128	256	512	1024	2048
MAE	.15833229(-3)	.38277125(-4)	.93068229(-5)	.23112516(-5)	.55402052(-6)
Order	-	2.04840	2.04012	2.00961	2.06066

We have considered linear and nonlinear nonlocal boundary value model problems corresponding to third order differential equations in ODEs to test the computational efficiency of the proposed method. We observed in numerical results obtained for problems considered in numerical experiment for different values of N presented in tables, maximum absolute error in solution decreases with decrease in h. The numerical results also approve the theoretically established order of accuracy of the proposed method. Thus, it is evident from the tabulated results that method (3) is accurate, efficient and convergent.

6. CONCLUSION

A nonlocal boundary value problem corresponding to third order differential equations in ODEs considered for the numerical solution in this article. A continuous problem transformed into the discrete problem. For the solution of discrete problem, a finite difference method has been developed and discussed. Thus, we have obtained a system of algebraic equations (3). The solution of the system of equations (3) is the discrete solution of the problem (1). The numerical results those we obtained by the application of the proposed method (3) are in good agreement to the theoretically developed order and accuracy of the method. Improvement in the order of accuracy or idea in proposed finite difference method is possible. Works in these directions are in progress.

References

[1] Gregus M., *Third Order Linear Differential Equations*, Mathematics and its Applications (East European Series), D. Reidel Publishing Company, Dordrecht / Boston / Lancaster / Tokyo 1987.

[2] Agarwal, R. P., *On boundary value problems for* $y''' = f(x, y, y', y'')$. Bull. Inst. Math. Acad. Sinica 12 (2), 153-157 (1984).

[3] A. Boucherif, A., Al- Maliki, N., *Nonlinear three-point third order boundary value problems*, Appl. Math. Comput. 190, 1168-1177 (2007).

[4] Jackson, *Existence and Uniqueness of Solutions of Boundary Value Problems for Third Order Differential Equations*, J. Diff. Eqs., 13, 432-437 (1973).

[5] Akaram, G., Tehseen, M., Siddiqi, S. S., Rehman, U. H., *Solution of a Linear Third order Multi-Point Boundary Value Problem using RKM*, British Journal of Mathematics & Computer Science 3(2), 180-194, (2013).

[6] Khan, A., Aziz, T., *The Numerical Solution of Third- Order Boundary-Value Problems Using Quintic Splines*, Applied Mathematics and Computation, Vol. 137, No. 2-3, 253-260 (2003).

[7] Islam, S., Khan, M. A., Tirmizi, I. A., Twizell, E. H., *Non-polynomial splines approach to the solution of a system of third-order boundary-value problems*, Applied Mathematics and Computation, vol. 168, no. 1, 152-163 (2005).

[8] Pandey, P. K., *Solving third-order boundary value problems with quartic splines*, Springerplus 5(1), 1-10 (2016).

[9] Na, T. Y., *Computational Methods in Engineering Boundary Value Problems*, Academic Press, New York 1979.

[10] Pandey, P. K., *A finite difference method for the numerical solution of general third order boundary value problem with an internal boundary condition*, Russian Mathematics 61(12),29-38 (2017).

[11] Salama, A. A., Mansour, A. A., *Fourth-order finite-difference method for third-order boundary-value problems*, Numerical Heat Transfer, Part B, 47, 383-401 (2005).

[12] Varga, R. S., *Matrix Iterative Analysis*, Second Revised and Expanded Edition. Springer-Verlag, Heidelberg, 2000.

[13] Horn, R. A. and Johnson C. R., Matrix Analysis. Cambridge University Press, New York, NY 10011, USA, 1990.

[14] Jain, M.K., Iyenger, S. R. K., Jain, R. K., *Numerical Methods for Scientific and Engineering Computation*, (2/e), Willey Eastern Limited, New Delhi, 1987.

[15] Froberg, C. E., *Introduction to Numerical Analysis*, 2nd ed.. Addison-Wesley, New York, 1969.

[16] Hasan, Y. Q., *The numerical solution of third-order boundary value problems by the modified decomposition method*, Advances in Intelligent Transportation Systems (AITS),Vol.1, No.3,71-74 (2012).