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ON SOLUTIONS OF SOME CLASSES OF FRACTIONAL INTEGRO-DIFFERENTIAL INCLUSIONS WITH INTEGRAL AND MULTI-POINT BOUNDARY CONDITIONS

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Abstract Existence of solutions for three classes of fractional integro-differential inclusions with integral and multi-point boundary conditions is investigated in the case when the values of the set-valued map are not convex.

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1. INTRODUCTION

In the last years one may see a strong development of the theory of differential equations and inclusions of fractional order ([2, 8, 10, 11, 12] etc.). The main reason is that fractional differential equations are very useful tools in order to model many physical phenomena.

In some recent papers [4, 13, 14] etc. the attention was focused on special classes of boundary value problems associated to fractional differential equations; namely, problems with both integral and multi-point boundary conditions. This is the explanation for the study in the present paper of some fractional integro-differential inclusions with integral and multi-point boundary conditions.

We consider first the problem

$$D^q x(t) \in F(t, x(t), V(x)(t)) \quad a.e. \ ([0,1]), \tag{1.1}$$

$$x^{(i)}(0) = 0, \ i = \overline{0, n-2}, \quad x(1) = \sum_{i=1}^{m-2} \beta_i \int_0^{\eta_i} x(s)ds + \sum_{i=1}^{m-2} \gamma_i x(\eta_i), \tag{1.2}$$

where D^q is s the standard Riemann-Liouville fractional derivative, $n-1 < q \leq n$, $n \geq 3$, $0 < \eta_1 < ... < \eta_{m-2} < 1$, $\beta_i, \gamma_i > 0$, $i = \overline{1, m-2}$, $m \geq 3$, $F : [0,1] \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map,
$V : C([0,1], \mathbf{R}) \to C([0,1], \mathbf{R})$ is a nonlinear Volterra operator
$V(x)(t) = \int_0^t k(t, s, x(s))ds$ with $k(.,.,.) : [0,1] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ a given function.

If F is single-valued and does not depend on the last variable, fractional differential inclusion (1.1) reduces to the fractional differential equation

$$D^q x(t) = f(t, x(t)), \tag{1.3}$$

where $f : [0,1] \times \mathbf{R} \to \mathbf{R}$.

Existence results for problem (1.3)-(1.2) are obtained in [14] and are based on some suitable theorems of fixed point theory.

Our goal is to extend the study in [14] to the more general problem (1.1)-(1.2) and to show that Filippov's ideas ([9]) can be suitably adapted in order to obtain the existence of solutions for this problem. Recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values, Filippov's theorem ([9]) consists in proving the existence of a solution starting from a given "quasi" solution. At the same time, the result provides an estimate between the "quasi" solution and the solution obtained.

Secondly, we obtain similar results for problem

$$D_c^q x(t) \in F(t, x(t), V(x)(t)) \quad a.e. \ ([0,1]) \tag{1.4}$$

$$\begin{array}{l} \int_0^1 x(s)ds = \sum_{j=1}^p \beta_j x(\sigma_j), \ x'(0) = x''(0) = ... = x^{(n-2)}(0) = 0 \\ a_1 x(1) + a_2 D_c^{q-1} x(1) = \sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^{\eta_i} x(s)ds, \end{array} \tag{1.5}$$

where D_c^q is the Caputo fractional derivative of order q, $n-1 < q \leq n$, $n \geq 2$, $0 < \sigma_1 < ... < \sigma_p < \xi_1 < \eta_1 < \xi_2 < \eta_2 < ... < \xi_{m-2} < \eta_{m-2} < 1$, $\alpha_i, \beta_j, a_1, a_2 \in \mathbf{R}$, $i = \overline{1, m-2}$, $j = \overline{1, p}$.

In the case when F does not depend on the last variable, fractional integro-differential inclusion (1.4) reduces to the fractional differential inclusion

$$D_c^q x(t) \in F(t, x(t)). \tag{1.6}$$

In [13] fixed point techniques are employed to obtain the existence of solutions for problem (1.6)-(1.5).

Finally, we establish corresponding results for problem

$$D^\alpha x(t) + \lambda D^{\alpha-1} x(t) \in F(t, x(t), V(x)(t)) \quad a.e. \ ([0,1]), \tag{1.7}$$

$$I^\beta x(t)|_{t=0} = 0, \quad x(1) = I^\gamma x(1), \tag{1.8}$$

where $\alpha \in (1,2]$, $\beta \in [0, 2-\alpha]$, $\lambda \geq 0$, $\gamma > 0$, F and V are as above, $I^\beta x(t)|_{t=0} := \lim_{t \to 0} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} u(s)ds$ and $I^\gamma x(1) = \int_0^1 \frac{(1-s)^{\gamma-1}}{\Gamma(\gamma)} u(s)ds$.

In the situation when $V(x)(t) \equiv D^{\alpha-1}x(t)$ existence results and topological properties of the solution set of problem (1.7)-(1.8) may be found in [4].

We note that existence results of the type provided in the present paper exists in the literature ([5, 6, 7] etc.), but their exposure in the framework of problems (1.1)-(1.2), (1.4)-(1.5) and (1.7)-(1.8) is new.

The paper is organized as follows: in Section 2 we recall some preliminary results that we need in the sequel and in Section 3 we prove our results.

2. PRELIMINARIES

Let (X, d) be a metric space. Recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\},\ d^*(A,B) = \sup\{d(a,B); a \in A\},$$

where $d(x, B) = \inf_{y\in B} d(x, y)$.

Let $I = [0, 1]$, we denote by $C(I, \mathbf{R})$ the Banach space of all continuous functions from I to $\mathbf{R}$ with the norm $||x(.)||_C = \sup_{t\in I} |x(t)|$ and $L^1(I, \mathbf{R})$ is the Banach space of integrable functions $u(.) : I \to \mathbf{R}$ endowed with the norm $||u(.)||_1 = \int_0^1 |u(t)|dt$.

The fractional integral of order $q > 0$ of a integrable function $f : (0, \infty) \to \mathbf{R}$ is defined by

$$I^q f(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds,$$

provided the right-hand side is pointwise defined on $(0, \infty)$ and Γ is the (Euler's) Gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$.

The Riemann-Liouville fractional derivative of order $q > 0$ of a integrable function $f : (0, \infty) \to \mathbf{R}$ is defined by

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t (t-s)^{-q+n-1} f(s)ds,$$

where $n = [\alpha] + 1$, provided the right-hand side is pointwise defined on $(0, \infty)$.

The Caputo fractional derivative of order $q > 0$ of a function $f : [0, \infty) \to \mathbf{R}$ is defined by

$$\mathrm{D}_c^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{-q+n-1} f^{(n)}(s)\mathrm{d}s,$$

where $n = [q] + 1$. It is assumed implicitly that f is n times differentiable whose n-th derivative is absolutely continuous.

The next technical result is proved in [14].

Lemma 2.1. *For a given* $f(.) \in L^1(I, \mathbf{R})$, $n - 1 < q \le n$, $n \ge 3$ *the unique solution* $x(.)$ *of problem* $D^q x(t) = f(t)$ *a.e.* $([0, 1])$ *with boundary conditions (1.2) is given by*

$$x(t) = \int_0^1 G(t,s)f(s)ds + \tfrac{t^{q-1}}{\xi} \sum_{i=1}^{m-2} \beta_i \int_0^1 H(\eta_i, s)f(s)ds + \tfrac{t^{q-1}}{\xi} \sum_{i=1}^{m-2} \gamma_i \int_0^1 G(\eta_i, s)f(s)ds \tag{2.1}$$

where $\xi = 1 - \frac{1}{\sigma} \sum_{i=1}^{m-2} \beta_i \eta_i^q - \sum_{i=1}^{m-2} \gamma_i \eta_i^{q-1} > 0$ *and*

$$G(t,s) := \frac{1}{\Gamma(q)} \begin{cases} [t(1-s)]^{q-1} - (t-s)^{q-1}, & if \quad 0 \le s < t \le 1, \\ [t(1-s)]^{q-1}, & if \quad 0 \le t < s \le 1, \end{cases}$$

$$H(t,s) := \frac{1}{\Gamma(q+1)} \begin{cases} t^q(1-s)^{q-1} - (t-s)^q, & \text{if} \quad 0 \le s < t \le 1, \\ t^q(1-s)^{q-1}, & \text{if} \quad 0 \le t < s \le 1. \end{cases}$$

If we denote $G_1(t,s) = G(t,s) + \frac{t^{q-1}}{\xi}\sum_{i=1}^{m-2}\beta_i H(\eta_i,s) + \frac{t^{q-1}}{\xi}\sum_{i=1}^{m-2}\gamma_i G(\eta_i,s)$, where $\chi_S(\cdot)$ denotes the characteristic function of the set S, then solution $x(\cdot)$ in Lemma 2.1 may be written as $x(t) = \int_0^1 G_1(t,s)f(s)ds$.

Taking into account that (e.g., Lemma 2.6 in [14]) $0 \le G(t,s) \le \frac{1}{\Gamma(q)}$, $0 \le H(t,s) \le \frac{1}{\Gamma(q+1)}$, we deduce that for any $t,s \in I$

$$|G_1(t,s)| \le \frac{1}{\Gamma(q)} + \frac{1}{\xi}\sum_{i=1}^{m-2}|\beta_i|\frac{1}{\Gamma(q+1)} + \frac{1}{\xi}\sum_{i=1}^{m-2}|\gamma_i|\frac{1}{\Gamma(q)} =: M_1.$$

Definition 2.1. *A function $x(.) \in C(I,\mathbf{R})$ with its Riemann-Liouville derivative of order q existing on $[0,1]$ is a solution of problem (1.1)-(1.2) if there exists a function $f(.) \in L^1(I,\mathbf{R})$ that satisfies $f(t) \in F(t,x(t),V(x)(t))$ a.e. (I) and $x(.)$ is given by (2.1).*

The proof of the following lemma may be found in [13].

Define $\lambda_1(t) = \frac{A_2 - A_1 t^{m-1}}{\Delta}$, $\lambda_2(t) = \frac{A_4 - A_3 t^{m-1}}{\Delta}$, $A_1 = 1 - \sum_{j=1}^p \beta_j$, $A_2 = \frac{1}{m} - \sum_{j=1}^p \beta_j \sigma_j^{m-1}$, $A_3 = \sum_{i=1}^{m-2}\alpha_i(\eta_i - \xi_i) - a_1$, $A_4 = \sum_{i=1}^{m-2}\alpha_i \frac{\eta_i^n - \xi_i^n}{n} - (a_1 + a_2\frac{(n-1)!}{\Gamma(n-q+1)})$, $\Delta = A_2A_3 - A_1A_4$.

Lemma 2.2. *Assume that $\Delta \neq 0$, $n-1 < q \le n$, $n \ge 2$,. For a given $f(.) \in L^1(I,\mathbf{R})$, the unique solution $x(.)$ of problem $D_c^q x(t) = f(t)$ a.e. $([0,1])$ with boundary conditions (1.5) is given by*

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s)ds - \lambda_1(t)[\sum_{i=1}^{m-2}\alpha_i \int_{\xi_i}^{\eta_i}(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)}f(u)du)ds - a_1\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}f(s)ds - a_2\int_0^1 f(s)ds] - \lambda_2(t)[\int_0^1(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)}f(u)du)ds - \sum_{j=1}^p \beta_j \int_0^{\sigma_j}\frac{(\sigma_j-u)^{q-1}}{\Gamma(q)}f(u)du]. \tag{2.2}$$

If we denote

$$G_2(t,s) = \frac{(t-s)^{q-1}}{\Gamma(q)}\chi_{[0,t]}(s) - \lambda_1(t)\sum_{i=1}^{m-2}\alpha_i \frac{\eta_i^q - \xi_i^q}{\Gamma(q+1)}\chi_{[0,\eta_i]}(s) + a_1\lambda_1(t)\frac{(1-s)^{q-1}}{\Gamma(q)} + (\lambda_1(t)a_2 - \frac{\lambda_2(t)}{\Gamma(q+1)}) + \sum_{j=1}^p \lambda_2(t)\beta_j \frac{(\sigma_j-s)^{q-1}}{\Gamma(q)}\chi_{[0,\sigma_j]}(s)$$

then solution $x(\cdot)$ in Lemma 2.2 may be written as $x(t) = \int_0^1 G_2(t,s)f(s)ds$.

Moreover, since $q > n-1 \ge 1$ for any $t,s \in I$ we have

$$|G_2(t,s)| \le \frac{1}{\Gamma(q)} + \overline{\lambda_1}\sum_{i=1}^{m-2}\frac{|\alpha_i(\eta_i^q - \xi_i^q)|}{\Gamma(q+1)} + \frac{\overline{\lambda_1}|a_1|}{\Gamma(q)} + (\overline{\lambda_1}|a_2| + \frac{\overline{\lambda_2}}{\Gamma(q+1)}) + \overline{\lambda_2}\sum_{j=1}^p \frac{|\beta_j\sigma_j^{q-1}|}{\Gamma(q)} =: M_2$$

where $\overline{\lambda_1} = \max_{t\in[0,1]} |\lambda_1(t)| = \frac{|A_1|+|A_2|}{|\Delta|}$, $\overline{\lambda_2} = \max_{t\in[0,1]} |\lambda_2(t)| = \frac{|A_3|+|A_4|}{|\Delta|}$.

Definition 2.2. *A function $x(.) \in C(I, \mathbf{R})$ with its Caputo derivative of order q existing on $[0,1]$ is a solution of problem (1.4)-(1.5) if there exists a function $f(.) \in L^1(I, \mathbf{R})$ that satisfies $f(t) \in F(t, x(t), V(x)(t))$ a.e. (I), and $x(.)$ is given by (2.2).*

Lemma 2.3. *For a given $f(.) \in L^1(I, \mathbf{R})$, the unique solution $x(.)$ of problem $D^\alpha x(t) + \lambda D^{\alpha-1}x(t) = f(t)$ a.e. $([0,1])$ with boundary conditions (1.8) is given by*

$$x(t) = \int_0^1 G_3(t,s)f(s)ds \quad t \in [0,1] \tag{2.3}$$

where

$$G_3(t,s) := \varphi(s)I^{\alpha-1}(e^{-\lambda t}) + \begin{cases} e^{\lambda s} I_s^{\alpha-1}(e^{-\lambda t}), & \text{if} \quad 0 \le s \le t \le 1, \\ 0, \quad \text{if} \quad 0 \le t \le s \le 1, \end{cases}$$

$$\varphi(s) = \frac{e^{\lambda s}}{\mu_0}[(I_s^{\alpha-1+\gamma}(e^{-\lambda t}))(1) - (I_s^{\alpha-1}(e^{-\lambda t}))(1)],$$

$$\mu_0 = (I^{\alpha-1}(e^{-\lambda t}))(1) - (I^{\alpha-1+\gamma}(e^{-\lambda t}))(1), \; I_s^q g(t) = \int_s^t \frac{(t-u)^{q-1}}{\Gamma(q)} g(u)du.$$

According to Lemma 2.3 in [4]

$$|G_3(t,s)| \le \frac{1}{\Gamma(\alpha)}\left(\frac{1+\Gamma(\gamma+1)}{|\mu_0|\Gamma(\alpha)\Gamma(\gamma+1)} + 1\right) =: M_3 \quad \forall t, s \in I.$$

Definition 2.3. *A function $x(.) \in C(I, \mathbf{R})$ with its Riemann-Liouville derivative of order $\alpha - 1$ existing on $[0,1]$ is a solution of problem (1.7)-(1.8) if there exists a function $f(.) \in L^1(I, \mathbf{R})$ that satisfies $f(t) \in F(t, x(t), V(x)(t))$ a.e. (I) and $x(.)$ is given by (2.3).*

Finally, we recall a selection result ([1]) which is a version of the celebrated Kuratowski and Ryll-Nardzewski selection theorem.

Lemma 2.4. *Consider X a separable Banach space, B is the closed unit ball in X, $G : I \to \mathcal{P}(X)$ is a set-valued map with nonempty closed values and $c : I \to X, r : I \to \mathbf{R}_+$ are measurable functions. If*

$$G(t) \cap (c(t) + r(t)B) \neq \emptyset \quad a.e.(I),$$

then the set-valued map $t \to G(t) \cap (c(t) + r(t)B)$ has a measurable selection.

3. THE MAIN RESULTS

In order to prove our results we need the following hypotheses.

Hypothesis H1. i) $F(.,.,.) : I \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ *has nonempty closed values and is* $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R} \times \mathbf{R})$ *measurable.*

ii) *There exists* $l(.) \in L^1(I, (0, \infty))$ *such that, for almost all* $t \in I$, $F(t,.,.)$ *is* $l(t)$*-Lipschitz in the sense that*

$$d_H(F(t,x_1,y_1), F(t,x_2,y_2)) \le l(t)(|x_1 - x_2| + |y_1 - y_2|) \ \forall\, x_1, x_2, y_1, y_2 \in \mathbf{R}.$$

We use next the following notations

$$L(t) := l(t)(1 + \int_0^t l(u)du), \quad t \in I, \tag{3.1}$$

$$L_0 = \int_0^1 L(t)dt. \tag{3.2}$$

Theorem 3.1. *Assume that Hypothesis H1 is satisfied and* $M_1 L_0 < 1$. *Consider* $y(.) \in C(I, \mathbf{R})$ *with its Riemann-Liouville derivative of order* q *existing on* $[0,1]$ *such that* $x^{(i)}(0) = 0$, $i = \overline{0, n-2}$, $x(1) = \sum_{i=1}^{m-2} \beta_i \int_0^{\eta_i} x(s)ds + \sum_{i=1}^{m-2} \gamma_i x(\eta_i)$ *and there exists* $p(.) \in L^1(I, \mathbf{R}_+)$ *verifying* $d(D^q y(t), F(t, y(t), V(y)(t))) \le p(t)$ *a.e.* (I).

Then there exists $x(.)$ *a solution of problem (1.1)-(1.2) satisfying for all* $t \in I$

$$|x(t) - y(t)| \le \frac{M_1}{1 - M_1 L_0} \int_0^1 p(t)dt. \tag{3.3}$$

Proof. The multifunction $t \to F(t, y(t), V(y)(t))$ has closed values, is measurable and from hypothesis of theorem one has

$$F(t, y(t), V(y)(t)) \cap \{D^q y(t) + p(t)[-1,1]\} \ne \emptyset \quad a.e.\ (I).$$

We apply Lemma 2.4 to find a measurable function $f_1(t) \in F(t, y(t), V(y)(t))$ *a.e.* (I) such that

$$|f_1(t) - D^q y(t)| \le p(t) \quad a.e.\ (I) \tag{3.4}$$

Define $x_1(t) = \int_0^1 G_1(t,s) f_1(s)ds$ and one has $|x_1(t) - y(t)| \le M_1 \int_0^1 p(t)dt$.

We point out that it is enough to construct the sequences $x_n(.) \in C(I, \mathbf{R})$, $f_n(.) \in L^1(I, \mathbf{R})$, $n \ge 1$ with the following properties

$$x_n(t) = \int_0^1 G_1(t,s) f_n(s)ds, \quad t \in I, \tag{3.5}$$

$$f_n(t) \in F(t, x_{n-1}(t), V(x_{n-1})(t)) \quad a.e.\,(I), \tag{3.6}$$

$$|f_{n+1}(t)-f_n(t)| \leq l(t)(|x_n(t)-x_{n-1}(t)|+\int_0^t l(s)|x_n(s)-x_{n-1}(s)|ds)\, a.e.\, (I) \tag{3.7}$$

If this construction is realized then from (3.4)-(3.7) we have for almost all $t \in I$

$$|x_{n+1}(t)-x_n(t)| \leq M_1(M_1L_0)^n \int_0^1 p(t)dt \quad \forall n \in \mathbf{N}.$$

Indeed, assume that the last inequality is true for $n-1$ and we prove it for n. One has

$$|x_{n+1}(t)-x_n(t)| \leq \int_0^1 |G_1(t,t_1)|.|f_{n+1}(t_1)-f_n(t_1)|dt_1 \leq$$

$$M_1 \int_0^1 l(t_1)[|x_n(t_1)-x_{n-1}(t_1)|+\int_0^{t_1} l(s)|x_n(s)-x_{n-1}(s)|ds]dt_1 \leq M_1$$

$$\int_0^1 l(t_1)(1+\int_0^{t_1} l(s)ds)dt_1.M_1^nL_0^{n-1}\int_0^1 p(t)dt = M_1(M_1L_0)^n\int_0^1 p(t)dt$$

Thus, $\{x_n(.)\}$ is Cauchy in the Banach space $C(I,\mathbf{R})$, therefore, converging uniformly to some $x(.) \in C(I,\mathbf{R})$. Hence, by (3.7), for almost all $t \in I$, the sequence $\{f_n(t)\}$ is Cauchy in $\mathbf{R}$. Denote $f(.)$ the pointwise limit of $f_n(.)$.

At the same time, one has

$$\begin{aligned} &|x_n(t)-y(t)| \leq |x_1(t)-y(t)|+\textstyle\sum_{i=1}^{n-1}|x_{i+1}(t)-x_i(t)| \leq \\ &M_1\textstyle\int_0^1 p(t)dt+\sum_{i=1}^{n-1}(M_1\int_0^1 p(t)dt)(M_1L_0)^i = \frac{M_1\int_0^1 p(t)dt}{1-M_1L_0}. \end{aligned} \tag{3.8}$$

Moreover, from (3.4), (3.7) and (3.8) we obtain for almost all $t \in I$

$$\begin{aligned} &|f_n(t)-D^q y(t)| \leq \\ &\textstyle\sum_{i=1}^{n-1}|f_{i+1}(t)-f_i(t)|++|f_1(t)-D^q y(t)| \leq L(t)\frac{M_1\int_0^1 p(t)dt}{1-M_1L_0}+p(t) \end{aligned}$$

In particular, the sequence $f_n(.)$ is integrably bounded and thus $f(.) \in L^1(I,\mathbf{R})$.

From Lebesgue's dominated convergence theorem and passing the limit in (3.5), (3.6) we obtain that $x(.)$ is a solution of (1.1). Finally, passing to the limit in (3.8) we obtained the desired estimate on $x(.)$.

In order to finish the proof it remains to realize the construction of the sequences $x_n(.), f_n(.)$ with the properties in (3.5)-(3.7). This will be done by induction.

Since the first step is already realized, assume that for some $N \geq 1$ we already constructed $x_n(.) \in C(I,\mathbf{R})$ and $f_n(.) \in L^1(I,\mathbf{R})$, $n=1,2,...N$ satisfying (3.5), (3.7) for $n=1,2,...N$ and (3.6) for $n=1,2,...N-1$. The set-valued map $t \to F(t, x_N(t), V(x_N)(t))$ is measurable; as well as the map $t \to l(t)(|x_N(t) - x_{N-1}(t)| + \int_0^t l(s)|x_N(s) - x_{N-1}(s)|ds)$ is measurable. By the lipschitzianity of $F(t, .)$ we have that for almost all $t \in I$

$$F(t, x_N(t)) \cap \{f_N(t) + l(t)(|x_N(t) - x_{N-1}(t)| + \int_0^t l(s)|x_N(s) - x_{N-1}(s)|ds)[-1, 1]\} \neq \emptyset.$$

Lemma 2.4 allows to find a measurable selection $f_{N+1}(.)$ of $F(., x_N(.), V(x_N)(.))$ such that for almost all $t \in I$

$$|f_{N+1}(t) - f_N(t)| \leq l(t)(|x_N(t) - x_{N-1}(t)| + \int_0^t l(s)|x_N(s) - x_{N-1}(s)|ds).$$

We define $x_{N+1}(.)$ as in (3.5) with $n = N + 1$. Thus $f_{N+1}(.)$ satisfies (3.6) and (3.7) and the proof is complete. ∎

The assumptions in Theorem 3.1 are satisfied, in particular, for $y(.) = 0$ and therefore with $p(.) = l(.)$. We obtain the following consequence of Theorem 3.1.

Corollary 3.1. *Assume that Hypothesis H1 is satisfied,* $d(0, F(t, 0, 0)) \leq l(t)$ *a.e.* (I), $M_1 L_0 < 1$. *Then there exists* $x(.)$ *a solution of problem (1.1)-(1.2) satisfying for all* $t \in I$

$$|x(t)| \leq \frac{M_1}{1 - M_1 L_0} \int_0^1 l(t)dt.$$

If F does not depend on the last variable, Hypothesis H1 becames

Hypothesis H2. i) $F(., .) : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ *has nonempty closed values and is* $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R})$ *measurable.*

ii) *There exists* $l(.) \in L^1(I, (0, \infty))$ *such that, for almost all* $t \in I$, $F(t, .)$ *is* $L(t)$*-Lipschitz in the sense that*

$$d_H(F(t, x_1), F(t, x_2)) \leq l(t)|x_1 - x_2| \quad \forall\, x_1, x_2 \in \mathbf{R}.$$

Denote $M_0 = \int_0^1 l(t)dt$ and consider the fractional differential inclusion

$$D^q x(t) \in F(t, x(t)) \quad a.e.\ ([0, 1]), \tag{3.9}$$

Corollary 3.2. *Assume that Hypothesis H2 is satisfied,* $d(0, F(t, 0)) \leq l(t)$ *a.e.* (I) *and* $M_1 M_0 < 1$. *Then there exists* $x(.)$ *a solution of problem (3.9)-(1.2) satisfying for all* $t \in I$

$$|x(t)| \leq \frac{M_1 M_0}{1 - M_1 M_0}.$$

Remark 3.1. *If in (3.9) F is single-valued, then a similar result to the one in Corollary 3.2 may be found in [14]; namely, Theorem 3.6.*

We are concern next with problem (1.4)-(1.5).

Theorem 3.2. *Assume that Hypothesis H1 is satisfied, $\Delta \neq 0$ and $M_2L_0 < 1$. Consider $y(.) \in C(I, \mathbf{R})$ with its Caputo derivative of order q existing on $[0, 1]$ such that $\int_0^1 y(s)ds = \sum_{j=1}^p \beta_j y(\sigma_j)$, $y'(0) = y''(0) = ... = y^{(n-2)}(0) = 0$, $a_1y(1) + a_2D_c^{q-1}y(1) = \sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^{\eta_i} y(s)ds$ and there exists $q(.) \in L^1(I, \mathbf{R}_+)$ verifying $d(D_c^q y(t), F(t, y(t), V(y)(t))) \leq q(t)$ a.e. (I).*

Then there exists $x(.)$ a solution of problem (1.4)-(1.5) satisfying for all $t \in I$

$$|x(t) - y(t)| \leq \frac{M_2}{1 - M_2L_0} \int_0^1 q(t)dt.$$

Proof. The proof is similar to the proof of Theorem 3.1. ∎

If in Theorem 3.2, $y(.) = 0$ and $q(.) = l(.)$ we get the following consequence of Theorem 3.2.

Corollary 3.3. *Assume that Hypothesis H1 is satisfied, $d(0, F(t, 0, 0)) \leq l(t)$ a.e. (I), $\Delta \neq 0$ and $M_2L_0 < 1$. Then there exists $x(.)$ a solution of problem (1.4)-(1.5) satisfying for all $t \in I$*

$$|x(t)| \leq \frac{M_2}{1 - M_2L_0} \int_0^1 l(t)dt.$$

Next F does not depend on the last variable. Set $K_0 = \int_0^1 l(t)dt$.

Corollary 3.4. *Assume that Hypothesis H2 is satisfied, $d(0, F(t, 0)) \leq (t)$ a.e. (I), $\Delta \neq 0$ and $M_2K_0 < 1$. Then there exists $x(.)$ a solution of problem (1.6)-(1.2) satisfying for all $t \in I$*

$$|x(t)| \leq \frac{M_2K_0}{1 - M_2K_0}.$$

Remark 3.2. *Problem (1.6)-(1.2) is studied in [13]. It is assumed that F is upper semicontinuous with compact convex values and the existence result; namely, Theorem 6.3, is obtained using Bohenenblust-Karlin fixed point theorem. As one may see our approach does not requires that the values of the set-valued map are convex and also provides a priori bounds for the solution.*

Finally, we deal with problem (1.7)-(1.8).

Theorem 3.3. *Assume that Hypothesis H1 is satisfied and $M_3L_0 < 1$. Consider $y(.) \in C(I, \mathbf{R})$ with its Riemann-Liouville derivative of order $\alpha - 1$*

existing on $[0,1]$ *such that* $I^{\beta}y(t)|_{t=0} = 0$, $y(1) = I^{\gamma}y(1)$ *and there exists* $r(.) \in L^1(I, \mathbf{R}_+)$ *verifying* $d(D^{\alpha}y(t) + \lambda D^{\alpha-1}y(t), F(t, y(t), V(y)(t))) \leq r(t)$ *a.e.* (I).

Then there exists $x(.)$ *a solution of problem (1.7)-(1.8) satisfying for all* $t \in I$

$$|x(t) - y(t)| \leq \frac{M_3}{1 - M_3 L_0} \int_0^1 r(t)dt.$$

Proof. The proof use the same pattern as in the proof of Theorem 3.1. ∎

If in Theorem 3.3 we take $y(.) = 0$ and $r(.) = l(.)$ then we deduce the following consequence of Theorem 3.3.

Corollary 3.5. *Assume that Hypothesis H1 is satisfied,* $d(0, F(t, 0, 0)) \leq l(t)$ *a.e.* (I) *and* $M_3 L_0 < 1$. *Then there exists* $x(.)$ *a solution of problem (1.7)-(1.8) satisfying for all* $t \in I$

$$|x(t)| \leq \frac{M_3}{1 - M_3 L_0} \int_0^1 l(t)dt. \tag{3.10}$$

Remark 3.3. *In the case when* $V(x)(t) \equiv D^{\alpha-1}x(t)$ *a similar existence result as in Corollary 3.5 may be found in Theorem 3.1 in [4]. The result in [4] is obtained using Covitz-Nadler fixed point theorem and does not contain a priori bounds for solution as in (3.10).*

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