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STUDY OF THE HOPF-HOPF BIFURCATION IN A SYSTEM OF COUPLED OSCILLATORS

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Abstract In this paper we study the double Hopf bifurcation in a 4D dynamical system modelling the dynamics of two coupled oscillators: a primary body (PB) subjected to vibrations induced by aerodynamic forces and a tuned mass damper (TMD) aiming to reduce the flow-induced vibrations of (PS). The stability of the trivial equilibrium of this model is studied in a four parameters space and the equation of the co-dimension 2 critical manifold of the double Hopf bifurcation is derived. The normal form method is then used in order to study the dynamics of the system when the parameters are slightly varied from their bifurcation values.

Keywords: oscillations, bifurcation, double Hopf, normal form.

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1. INTRODUCTION

The study of dynamical systems in dimension larger then two is a difficult task for mathematicians, especially if the system depends on more than two parameters. However, these systems are very common in mathematical modeling of complex phenomena. The dynamics of 3D systems is somehow easy to understand (not always to explain) because the trajectories can be represented in the 3D states' space and some complicated objects, strange attractors for example, can be visualized. But the dynamics of 4D systems can be understood only through a convenient reduction to systems in smaller dimension or using 2D, 3D sections in the phase space for representing the orbits (which provide incomplete knowledge about the dynamics).

In the last decades a great interest was given to the study of 4D systems, both for their practical importance but also because they represent the first step in eliminating geometrical intuition in the study of their dynamics. The bifurcation theory remains the main ingredient for a complete description of the dynamics. In this topic, the double Hopf bifurcation, corresponding to the case when an equilibrium point has two pairs of purely imaginary eigenvalues, is one of the most interesting cases. The study of Hopf-Hopf bifurcation was initiated by N. Gavrilov [3], and Guckenheimer [4] and normal form were used in[17] and [8] for understanding the complicate dynamics of the systems for parameters' values close to those for which the Hopf-Hopf bifurcation occurs. Explicit computational formulas for the normal form's coefficients were presented in [8].

There are not many papers studying the Hopf-Hopf bifurcation in systems modeling real life phenomena, due the difficulty in analyzing the complex dynamics of such systems. Double Hopf bifurcation and chaos of a nonlinear vibration system was studied in [12], the normal form of double Hopf bifurcation in forced oscillators was studied in [15], some discretization methods were used for the study of double Hopf bifurcation in heated rotating annulus [10], the existence of invariant tori as consequence of the double Hopf bifurcation was observed in [14], the existence of stable invariant cycles after a double Hopf bifurcation is observed in a corotating spiral Poiseuille flow [1], an efficient control technique of Hopf-Hopf interactions in as self-excited system with dry friction was proposed in [13], the Hopf-Hopf bifurcation and period double bifurcation in a four-species food web found practical significance in [16], theoretical and numerical results concerning the occurrence of the double Hopf bifurcation in a financial model are presented in [6].

In this paper we study the vibrations induced by a flow on a body. It is important both from practical and theoretical point of view. From practical point of view it must be used in the design of many structures subjected to wind and/or ocean waves when the primary body (PB) experiences a dynamical instability leading to oscillations of increasing amplitude in the direction normal to the flow. The reduction, even the elimination of these unwanted oscillations can be realized using additional devices. A particular attention received the tuned mass damper (TMD) method, i.e. the consideration of an additional device with small mass calibrated in such a way as to reduce/annihilate the oscillations of (PB) (see [2] and the references therein). From the theoretical point of view, the study gives new insights on the understanding of the mechanism of producing some bifurcations of codimension 2, which is a challenging subject in the theory of dynamical systems.

In [2] a four-dimensional mathematical model depending on four parameters (the ratio of the masses of the (TMD) and the (PB), the ratio of the frequencies of the (TMD) and the (PB), the damping of the (TMD) and the wind velocity) was derived. This model was studied using the perturbation multiple scale method introduced in [11], considering as parameters the ratio of the frequencies of (TMD and(PB) and the wind velocity .

The aim of the present paper is to study the same model in the situation when the aerodynamic force acting on the (PB) is constant, which means that the wind velocity is fixed. We consider as parameters the mass and the damping coefficient of (TMD). In this way we obtain some information about the useful calibration of the (TMD) in order to stabilize the dynamics of the (PB). We focus on the study of the double Hopf bifurcation which is interesting from practical point of view because for technical values of the parameters the eigenvalues are found to be complex conjugate in pairs [2]. We highlight the equation of the codimension 2 critical manifold and we apply the classical normal form theory [8] in order to identify the regions of the parameters' plane where the system has different dynamical properties, i.e. to construct the bifurcation diagram. The results are compared with numerical solutions obtained through direct integration of the equations of motion.

The paper is organized as follows: in Section 2 the model is presented and the stability of its (unique) equilibrium point is studied; in Section 3 the normal form of double Hopf bifurcation is derived and the bifurcation diagram is analysed; Section 4 is devoted to the numerical simulations. In Appendix is detailed the computation of the normal form.

2. THE MATHEMATICAL MODEL

In what follows we use the notations of [2]: q_1 and q_2, m_s and m_t, ξ_s and ξ_t respectively ω_s and ω_t are the absolute cross wind displacements, the masses, the damping coefficients, respectively the undamped frequencies of the isolated bodies (PB) and (TMD). The aerodynamic force is f_a .

The equations of motion governing the oscillations of an elastically supported body (PB) subjected to a steady flow and connected with a small added mass (TMD) are obtained writing the equilibrium of the forces acting on (PB) and (TMD). It is supposed that both (PB) and (TMD) are linear oscillators and the aerodynamic forces acting in the (TMD) are not important in comparison with those acting on (PS). The equations are ([2]):

$$\begin{array}{rcl} m_s q_1'' + 2m_s\omega_s\xi_s q_1' + 2m_t\omega_t\xi_t\left(q_1' - q_2'\right) + m_s\omega_s^2 q_1 + m_t\omega_t^2\left(q_1 - q_2\right) & = & f_a \\ m_t q_2'' + 2m_t\omega_t\xi_t\left(q_2' - q_1'\right) + m_t\omega_t^2\left(q_2 - q_1\right) & = & 0 \end{array} \tag{1}$$

The aerodynamic force, depending on q_1' and the wind velocity $\overline{U}$, can be expressed in a polynomial form $f_a = A_1\left(\frac{q_1'}{\overline{U}}\right) + A_3\left(\frac{q_1'}{\overline{U}}\right)^3$ by keeping a few terms from its expansion in Taylor series (see [2] for details).

Using the non-dimensional variables and parameters $x = \frac{q_1}{D}$, $z = \frac{q_2}{D}$, $\bar{t} = t\omega_s$, $m = \frac{m_t}{m_s}$, $\gamma = \frac{\omega_t}{\omega_s}$, $\delta = \frac{\rho_a D^2}{m_s}$, $U = \frac{\overline{U}}{\omega_s D}$ and considering $\xi = \xi_s - \frac{\delta A_1 U}{2}$, $C = \delta\frac{A_3}{\overline{U}}$, the system (1) can be transformed a 4D system of equations of first order:

$$\left\{\begin{array}{c} x' = y \\ y' = -\left(1 + m\gamma^2\right)x - 2\left(\xi + m\gamma\xi_t\right)y + m\gamma^2 z + 2m\gamma\xi_t u + Cy^3 \\ z' = u \\ u' = \gamma^2 x + 2\gamma\xi_t y - \gamma^2 z - 2\gamma\xi_t u \end{array}\right. \tag{2}$$

The system (2) depends on four parameters: the mass ratio $m > 0$, the frequency ratio $\gamma > 0$, the damping of (TMD) ξ_t and the apparent dumping ξ.

The analysis of the stability of the equilibrium point is made in the following.

Theorem 2.1. *The system (2) has the unique equilibrium point* $O(0,0,0,0)$. *The equilibrium point is asymptotically stable if and only if*

$$\begin{cases} \gamma(1+m)\xi_t + \xi > 0 \\ 4\gamma^2\xi(1+m)\xi_t^2 + \gamma\left(m^2\gamma^2 + 2m\gamma^2 + m + \gamma^2 + 4\xi^2\right)\xi_t + \\ +\xi\left(m\gamma^2+1\right) > 0 \\ 4\gamma^2\xi(1+m)\xi_t^3 + \gamma\left(4\xi^2 + 4\gamma^2\xi^2 + m + 4m\gamma^2\xi^2\right)\xi_t^2 + \\ +\xi\left(m^2\gamma^4 - 2\gamma^2 + \gamma^4 + 4\gamma^2\xi^2 + 2m\gamma^4 + 1\right)\xi_t + m\gamma^3\xi^2 > 0 \end{cases}.$$

Proof. The unique solution of the system

$$\begin{cases} y = 0 \\ -\left(1+m\gamma^2\right)x - 2\left(\xi + m\gamma\xi_t\right)y + m\gamma^2 z + 2m\gamma\xi_t u + Cy^3 = 0 \\ u = 0 \\ \gamma^2 x + 2\gamma\xi_t y - \gamma^2 z - 2\gamma\xi_t u = 0 \end{cases}$$

is $(0,0,0,0)$, which is the unique equilibrium point of the system (2).

The characteristic polynomial of the Jacobian matrix in $(0,0,0,0)$ is

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0 \tag{3}$$

where

$$\begin{aligned} a_1 &= 2\left(\xi + \gamma\xi_t + m\gamma\xi_t\right) \\ a_2 &= \left(\gamma^2 + 1 + 4\gamma\xi\xi_t + m\gamma^2\right) \\ a_3 &= 2\gamma\left(\xi_t + \gamma\xi\right) \\ a_4 &= \gamma^2. \end{aligned} \tag{4}$$

From the Routh-Hurvitz criterion it results that the polynomial (3) has four eigenvalues with strictly negative real part if and only if $a_1 > 0$, $a_1a_2 - a_3 > 0$, $a_3(a_1a_2 - a_3) - a_1^2 a_4 > 0$ and $a_4 > 0$. If we take into account the expressions of a_1, a_2, a_3, a_4, we make some simplifications and we group the terms with respect of the powers of ξ_t we obtain the inequalities presented in the conclusion of the Theorem. ∎

Theorem 2.2. *The characteristic polynomial* (3) *has a unique pair of pure imaginary eigenvalues if and only if*

a) $\gamma\xi_t(1+m) + \xi \neq 0$

b) $\gamma(1+m)\xi_t^2 + \xi\left(1+\gamma^2+m\gamma^2\right)\xi_t + \gamma\xi^2 > 0$

c) $$\begin{gathered} 4\gamma^2\xi(1+m)\xi_t^3 + \left(\gamma^3 - \gamma^5 + \gamma^3 m^2 - \gamma^5 m^2 + \gamma m + 2\gamma^3 m - 2\gamma^5 m + \right. \\ \left. +4\gamma\xi^2\left(\gamma^2 + \gamma^2 m + 1\right)\right)\xi_t^2 + \left(4\gamma^2\xi^3 + \xi\left(\gamma^4 m^2 - \gamma^4 + 2\gamma^2 m + 1\right)\right)\xi_t + \\ \gamma\xi^2\left(\gamma^2 m - \gamma^2 + 1\right) = 0 \end{gathered}$$

Proof. If $\lambda_{1,2} = \pm i\omega$ and $\lambda_{3,4}$ are not pure imaginary complex numbers the following relations must be fulfilled:

$\lambda_3 + \lambda_4 = -a_1; \omega^2 + \lambda_3\lambda_4 = a_2; \omega^2 (\lambda_3 + \lambda_4) = -a_3; \omega^2\lambda_3\lambda_4 = a_4$

If $a_1 \neq 0$, i.e. $\xi \neq -\gamma\xi_t (1 + m)$ they becomes

$\lambda_3 + \lambda_4 = -a_1; \quad \lambda_3\lambda_4 = a_4a_1/a_3; \quad \omega^2 = a_3/a_1 > 0; \quad \frac{a_3}{a_1} + \frac{a_4a_1}{a_3} = a_2$.

Replacing a_1, a_2, a_3, a_4 from (4) and collecting the terms in respect of the powers of ξ_t, the conditions $a_3/a_1 > 0$, respectively $\frac{a_3}{a_1} + \frac{a_4a_1}{a_3} = a_2$ one obtains the expressions form the conclusion of the Theorem.

If $a_1 = 0$ and we have also $a_3 = 0$. The eigenvalues λ_3 and $\lambda_4 = -\lambda_3$ may not be real numbers because in this case $\omega^2\lambda_3\lambda_4 = -\omega^2\lambda_3^2$ should be strictly negative and it is not possible because $\omega^2\lambda_3\lambda_4 = a_4 = \gamma^2 > 0$.

If $\lambda_{3,4} = \alpha \pm i\beta$ one obtains from $\lambda_3 + \lambda_4 = 0$ that $\alpha = 0$ and this is in contradiction with the Theorem's hypothesis. ∎

The following proposition give the expression of the critical manifold of the simple Hopf bifurcation.

Proposition 2.1. *The equation of the critical manifold of the Hopf bifurcation of (2) is*

$V_H : 4\gamma^2\xi (1 + m) \xi_t^3 + \left(\gamma^3 - \gamma^5 + \gamma^3m^2 - \gamma^5m^2 + \gamma m + 2\gamma^3m - 2\gamma^5m + 4\gamma\xi^2 \left(\gamma^2 + \gamma^2m + 1\right)\right) \xi_t^2 + \left(4\gamma^2\xi^3 + \xi \left(\gamma^4m^2 - \gamma^4 + 2\gamma^2m + 1\right)\right) \xi_t + \gamma\xi^2 \left(\gamma^2m - \gamma^2 + 1\right) = 0$

restricted to $\gamma (1 + m) \xi_t^2 + \xi \left(1 + \gamma^2 + m\gamma^2\right) \xi_t + \gamma\xi^2 > 0$ *and* $\gamma\xi_t (1 + m) + \xi \neq 0$

If we impose the existence of a pair of complex conjugate pure imaginary eigenvalues we obtain the following result.

Theorem 2.3. *The characteristic polynomial* (3) *has two pairs of pure imaginary eigenvalues if and only if*

a) $\gamma\xi_t (1 + m) + \xi = 0$;

b) $\xi_t + \xi\gamma = 0$;

c) $\gamma^2 + 1 + \gamma^2m + 4\gamma\xi\xi_t > 0$;

d) $|\gamma| < \frac{\left(\gamma^2+1+\gamma^2m+4\gamma\xi\xi_t\right)}{2}$.

Proof. The conditions $\lambda_{1,4} = \pm i\omega_1$, $\lambda_{2,3} = \pm i\omega_2$ leads to $a_1 = \gamma\xi_t (1 + m) + \xi = 0$; $\omega_1^2 + \omega_2^2 = a_2 > 0$; $a_3 = \gamma(\xi_t + \xi\gamma) = 0$; $\omega_1^2\omega_2^2 = a_4 > 0$.

The reduced polynomial $P (\lambda) = \lambda^4 + a_2\lambda^2 + a_4$ has the complex roots $\lambda_{1,4} = \pm i\omega_1$, $\lambda_{2,3} = \pm i\omega_2$ if and on if the equation $\omega^2 - a_2\omega + a_4 = 0$ have two strictly positive solutions $\omega_1^2 \neq \omega_2^2$. This leads to $a_2 > 0$ and $a_2^2 - 4a_4 > 0$. Using (4), one obtains the conditions c), d) in the Theorem. ∎

Proposition 2.2. *The equation of the critical manifold of the non-resonant Hopf-Hopf bifurcation of (2) with* $m > 0$ *is*

$$(V_{HH}) : \begin{cases} \gamma = \frac{1}{\sqrt{1+m}} \\ \xi = -\xi_t \sqrt{1+m} \end{cases} \tag{5}$$

with the restriction $|\xi_t| < \sqrt{\frac{\sqrt{m+1}-1}{2\sqrt{m+1}}}$.

Proof. The conditions a), b) in Theorem 2.3 give directly the equation of V_{HH}. The restriction occurs from the conditions c), d). ∎

In the following study we will consider as parameters of bifurcation m and ξ_t because they are related to the added mass (TMD).

3. THE NORMAL FORM OF DOUBLE HOPF BIFURCATION

In order to study the dynamics of (2) when a double Hopf bifurcation occurs, we will consider a generic point $\left(m_0, \gamma_0, \zeta_0, \xi_{t,0}\right) = \left(m_0, \frac{1}{\sqrt{1+m_0}}, -\xi_{t,0}\sqrt{1+m}, \xi_{t,0}\right)$ on the critical manifold, corresponding to fixed values $\xi_t = \xi_{t,0}$ and $m = m_0$ under the restriction $\xi_{t,0}^2 < \frac{\sqrt{m+1}-1}{2\sqrt{m+1}}$ and we will compute the associated normal form.

Let fix $\gamma_0 = \frac{1}{\sqrt{1+m_0}}$, $\xi_0 = -\xi_{t,0}\sqrt{m_0+1}$, consider the perturbation $m = m_0 + M$, $\xi_t = \xi_{t,0} + N$ and the parameter $\alpha = (M, N)$.

The system (2) becomes

$$\begin{cases} x' = y \\ y' = -\frac{1+2m_0+M}{1+m_0}x - 2\frac{\xi_{t,0}(M-1)+(m_0+M)N}{\sqrt{1+m_0}}y + \frac{m_0+M}{1+m_0}z + \\ \quad (m_0+M)\frac{1}{\sqrt{m_0+1}}\left(\xi_{t,0}+N\right)u + Cy^3 \\ z' = u \\ u' = \frac{1}{m_0+1}x + \frac{2}{\sqrt{m_0+1}}\left(\xi_{t,0}+N\right)y - \frac{1}{m_0+1}z - \\ \quad \frac{2}{\sqrt{m_0+1}}\left(\xi_{t,0}+N\right)u \end{cases} \tag{6}$$

The following theorem shows that the starting conditions for deriving the normal form are fulfilled.

Theorem 3.1. *a) For all* $\alpha = (M, N)$ *the system (6) has the unique equilibrium* $(0, 0, 0, 0)$.

b) for $\alpha = (0, 0)$ *The equilibrium* $(0, 0, 0, 0)$ *has two distinct pairs of purely imaginary eigenvalues* $\lambda_{1,4} = \pm i\omega_1$, $\lambda_{2,3} = \pm i\omega_2$ *with* $\omega_{1,2}^2 = \left(1 - 2\xi_{t,0}^2\right) \pm \sqrt{\left(1 - 2\xi_{t,0}^2\right)^2 - \frac{1}{m_0+1}}$.

Proof. a) results from simple algebraic computations.

b) For $\alpha = (0, 0)$ the condition (5) is fulfilled, so the eigenvalues of $(0, 0, 0, 0)$ are purely imaginary $\lambda_{1,4} = \pm i\omega_1$, $\lambda_{2,3} = \pm i\omega_2$. The values ω_1 and ω_2 are solutions of the equation

$$P\left(\omega^2\right) = \omega^4 - 2\left(1 - 2\xi_{t,0}^2\right)\omega^2 + \frac{1}{m_0 + 1} = 0$$

which are the expressions mentioned in the theorem, under the restriction imposed in the beginning of the Section. ∎

The normal form of the Hopf-Hopf bifurcation ([8] pp 349-369), is computed using the quantities p_{11}, p_{12}, p_{21}, p_{22}, s_1, s_2 defined in the Appendix of this paper.

In Appendix the computation of these quantities for the system (2) is presented with many details. The results obtained using the general algorithm for the system (2) in the point $M = \left(m_0, \gamma_0, \xi_0, \xi_{t,0}\right) = \Big(m_0, \frac{1}{\sqrt{1+m_0}}, -\xi_{t,0}\sqrt{1+m}, \xi_{t,0}\Big)$,are

$$\begin{aligned}
p_{11} &= \frac{3C\omega_1^4\left[(m_0+1)\omega_1^2-1\right]}{2\left[(m_0+1)\omega_1^4-1\right]}\\
p_{12} &= \frac{3C\omega_1^2\omega_2^2\left[(m_0+1)\omega_1^2-1\right]}{(m_0+1)\omega_1^4-1}\\
p_{21} &= \frac{3C\omega_1^2\omega_2^2\left[(m_0+1)\omega_2^2-1\right]}{(m_0+1)\omega_2^4-1}\\
p_{22} &= \frac{3C\omega_2^4\left[(m_0+1)\omega_2^2-1\right]}{2\left[(m_0+1)\omega_2^4-1\right]}\\
s_1 &= s_2 = 0
\end{aligned} \tag{7}$$

where ω_1, ω_2 are given in the previous Theorem.

Remark 3.1. *For all* $m_0 > 0$, *the inequality*

$$\frac{1}{4}\left(1 - \frac{1}{m_0 + 1}\right) < \frac{1}{2}\left(1 - \frac{1}{\sqrt{m_0 + 1}}\right)$$

is fulfilled.

Proof. $\frac{1}{4}\left(1 - \frac{1}{m_0+1}\right) = \frac{1}{2}\left(1 - \frac{1}{2}\left(\frac{1}{m_0+1} + 1\right)\right) < \frac{1}{2}\left(1 - \frac{1}{2} \cdot 2\frac{1}{\sqrt{m_0+1}}\right)$. ∎

Theorem 3.2. *Let consider* $m_0 > 0$.

a) If $\xi_{t,0}^2 \in \left(0, \frac{1}{2}\left(1 - \frac{1}{\sqrt{m_0+1}}\right)\right) \setminus \left\{\frac{1}{4}\left(1 - \frac{1}{m_0+1}\right)\right\}$, *then* $p_{11} > 0$, $p_{12} > 0$, $p_{21} > 0$, $p_{22} > 0$.

b) If $\xi_{t,0}^2 = \frac{1}{4}\left(1 - \frac{1}{m_0+1}\right)$, *then* $p_{11} > 0$, $p_{12} > 0$, $p_{21} = 0$, $p_{22} = 0$.

Proof. $P\left(\omega^2\right)$ is a second order polynomial in ω^2 with the roots ω_1^2 and ω_2^2.

$P\left(\frac{1}{\sqrt{m_0+1}}\right) = \frac{1}{m_0+1} - 2\left(1 - 2\xi_{t,0}^2\right)\frac{1}{\sqrt{m_0+1}} + \frac{1}{m_0+1} = \frac{2}{\sqrt{m_0+1}}(\frac{1}{\sqrt{m_0+1}} - 1 + 2\xi_{t,0}^2)$.

From the hypothesis of the theorem it results that $P\left(\frac{1}{\sqrt{m_0+1}}\right) < 0$, which means $\omega_1^2 < \frac{1}{\sqrt{m_0+1}} < \omega_2^2$.

It results that $(m_0+1)\,\omega_2^4 < 1 < (m_0+1)\,\omega_1^4$, hence

$$(m_0+1)\,\omega_2^4 - 1 < 0 \text{ and } (m_0+1)\,\omega_1^4 - 1 > 0. \tag{8}$$

$P\left(\frac{1}{m_0+1}\right) = \frac{1}{m_0+1}\left[4\xi_{t,0}^2 - \left(1 - \frac{1}{m_0+1}\right)\right]$.

If $\xi_{t,0}^2 < \frac{1}{4}\left(1 - \frac{1}{m_0+1}\right)$ then $P\left(\frac{1}{m_0+1}\right) < 0$, consequently $\omega_2^2 < \frac{1}{m_0+1} < \omega_1^2$, i.e.

$$(m_0+1)\omega_2^2 - 1 < 0 \text{ and } (m_0+1)\omega_1^2 - 1 > 0. \tag{9}$$

If $\frac{1}{4}\left(1 - \frac{1}{m_0+1}\right) < \xi_{t,0}^2 < \frac{1}{2}\left(1 - \frac{1}{\sqrt{m_0+1}}\right)$ then $P\left(\frac{1}{m_0+1}\right) > 0$. Because $\omega_1^2 + \omega_2^2 = 1 - 2\xi_{t,0}^2 > 1 - 1 + \frac{1}{\sqrt{m_0+1}} > \frac{1}{m_0+1}$ it results that $\frac{1}{m_0+1} < \omega_2^2 < \omega_1^2$. Consequently one has

$$(m_0+1)\,\omega_2^2 - 1 > 0 \text{ and } (m_0+1)\,\omega_1^2 - 1 > 0. \tag{10}$$

If $\xi_{t,0}^2 = \frac{1}{4}\left(1 - \frac{1}{m_0+1}\right)$ then $P\left(\frac{1}{m_0+1}\right) = 0$ and simple computation shows that $\omega_2^2 = \frac{1}{m_0+1}$. In this case

$$(m_0+1)\omega_2^2 - 1 = 0 \text{ and } (m_0+1)\omega_1^2 - 1 > 0. \tag{11}$$

The relations (8), (9), (11), (10), combined with (7) give the conclusion of the theorem. ∎

In the case $p_{11} \neq 0$, $p_{12} \neq 0$, $p_{21} \neq 0$, $p_{22} \neq 0$ the Hopf-Hopf bifurcation is non-degenerate and the Theorem ([8] pp 355-356) applied for the system (6) gives the following results:

Theorem 3.3. *Let consider* $m_0 > 0$, $\xi_{t,0}^2 \in \left(0, \frac{1}{2}\left(1 - \frac{1}{\sqrt{m_0+1}}\right)\right)$ $\setminus \left\{\frac{1}{4}\left(1 - \frac{1}{m_0+1}\right)\right\}$ *and the smooth system (6) with the parameter* $\alpha = (M, N)$

in $\mathbf{R}^2$*. Let denote* $\lambda_{1,2} = \mu_{1.2}(\alpha) + i\omega_{1,2}(\alpha)$ *its eigenvalues. The system is locally smoothly orbitally equivalent near the origin* $O(0,0,0,0)$ *to the system*

$$\begin{array}{c} r_1' = r_1\left(\mu_1 + p_{11}(\mu) r_1^2 + p_{12}(\mu) r_2^2 + s_1(\mu) r_2^4\right) + \Phi_1(r_1, r_2, \varphi_1, \varphi_2, \mu) \\ r_2' = r_2\left(\mu_2 + p_{21}(\mu) r_1^2 + p_{22}(\mu) r_2^2 + s_2(\mu) r_1^4\right) + \Phi_2(r_1, r_2, \varphi_1, \varphi_2, \mu) \\ \varphi_1' = \omega_1(\mu) + \Psi_1(r_1, r_2, \varphi_1, \varphi_2, \mu) \\ \varphi_2' = \omega_2(\mu) + \Psi_2(r_1, r_2, \varphi_1, \varphi_2, \mu) \end{array} \tag{12}$$

where $\Phi_k = O\left(\left(r_1^2 + r_2^2\right)^3\right)$ *and* $\Psi_k = O(1)$ *are* 2π*-periodic in* φ_k*.*

Remark 3.2. *The condition* $\xi_{t,0}^2 \in \left(0, \frac{1}{2}\left(1 - \frac{1}{\sqrt{m_0+1}}\right)\right) \setminus \left\{\frac{1}{4}\left(1 - \frac{1}{m_0+1}\right)\right\}$ *corresponds to the non-degenerate case* $p_{11} \neq 0$*,* $p_{12} \neq 0$*,* $p_{21} \neq 0$*,* $p_{22} \neq 0$*.*

The truncated form of (12), obtained from the expansion in Taylor series is

$$\begin{array}{rcl} r_1' &=& r_1\left(\mu_1 + p_{11} r_1^2 + p_{12} r_2^2\right) \\ r_2' &=& r_2\left(\mu_2 + p_{21} r_1^2 + p_{22} r_2^2\right) \\ \varphi_1' &=& \omega_1 \\ \varphi_2' &=& \omega_2 \end{array} \tag{13}$$

This form is not topologically or orbitally equivalent to (12), but it captures its main dynamical properties. The first pair of equations in (13) is independent of the second pair. The last two equations describe rotations in the planes $r_2 = 0$, respectively $r_1 = 0$ with the angular velocities ω_1, respectively ω_2. Therefore, the bifurcation diagram of (13) is determined by that of the planar system

$$\begin{cases} r_1' = r_1\left(\mu_1 + p_{11} r_1^2 + p_{12} r_2^2\right) \\ r_2' = r_2\left(\mu_2 + p_{21} r_1^2 + p_{22} r_2^2\right) \end{cases} \tag{14}$$

The equilibrium point $E_0 = (0,0)$ of (14) correspond to the equilibrium point $O(0,0,0,0)$ of the system (6). The equilibria on the axis $r_1 = 0$ or $r_2 = 0$ correspond to cycles of (6), while a nontrivial equilibrium with $r_1 > 0$, $r_2 > 0$ generates a two-dimensional torus of (6). If a limit cycle is present in (14), then (6) has a three dimensional torus. The stability of all invariant sets of (13) is detectable from that of the corresponding invariant sets of (14).

The study of (13) is simplified if one considers new variables $\rho_1 = r_1^2$ and $\rho_2 = r_2^2$. The system (14) reads

$$\begin{cases} \rho_1' = 2\rho_1\left(\mu_1 + p_{11}\rho_1 + p_{12}\rho_2\right) \\ \rho_2' = 2\rho_2\left(\mu_2 + p_{21}\rho_1 + p_{22}\rho_2\right) \end{cases} \tag{15}$$

The system (15) can be simplified using the new phase variables $\xi_1 = p_{11}\rho_1$, $\xi_2 = p_{22}\rho_2$ and rescaling the time $\tau = 2t$. The new system is

$$\begin{cases} \xi_1' = \xi_1\left(\mu_1 + \xi_1 + \theta\xi_2\right) \\ \xi_2' = \xi_2\left(\mu_2 + \delta\xi_1 + \xi_2\right) \end{cases}$$

where $\theta = \frac{p_{12}}{p_{22}} > 0$ and $\delta = \frac{p_{21}}{p_{11}} > 0$ and $\theta\delta = 4 > 1$

For $\alpha = (M, N)$ sufficiently close to $(0, 0)$ the system (14) has the equilibrium point $E_0 = (0, 0)$, other two equilibria $E_1 = (-\mu_1, 0)$, $E_2 = (0, -\mu_2)$ and a non-trivial one $E_3 = \left(\frac{\mu_1 - \theta\mu_2}{3}, \frac{\mu_2 - \delta\mu_1}{3}\right)$.

The equilibrium E_3 collides with E_1, respectively E_2 on the bifurcation curves

$$\begin{aligned} T_1 &= \{(\mu_1, \mu_2) : \ \mu_1 = \theta\mu_2, \ \mu_1 < 0\} \\ T_2 &= \{(\mu_1, \mu_2) : \ \mu_2 = \delta\mu_2, \ \mu_2 < 0\} . \end{aligned}$$

In order to obtain synchronized oscillations of the two coupled oscillators (PS) and (TMD), that is usually required, one has to fix the parameters m_0 and $\xi_{t,0}$, to compute γ_0 and ξ_0 and to find M and N such that the system (14) has two stable equilibria $E_1 = (-\mu_1, 0)$, $E_2 = (0, -\mu_2)$. Exemplification of this result is presented in the next section.

4. NUMERICAL RESULTS

The numerical results are in agreement with those derived from the study of the normal form associated to each point on the critical manifold.

As example we present some results obtained using Runge-Kutta method of order 4 for the integration of the system (2) for the following values of the parameters

$$m_0 = 0.21; \ \gamma_0 = \frac{1}{\sqrt{m_0+1}}; \ \xi_{t,0} = \frac{1}{20}\sqrt{\frac{1}{2}\left(1 - \frac{1}{\sqrt{m_0+1}}\right)}; \ \xi = -\xi_{t,0}\sqrt{m_0+1};$$
$$C = 0.1$$

that correspond to the point $M\,(0.21,\ 0.90909,\ 0.01066,\ -0.01172)$ on the critical manifold.

The eigenvalues of the Jacobian matrix are purely imaginary and two periodic orbits are formed, as shown in Figure 3. Because in the initial system (1) the quantities q_1 and q_2 describe the motion of (PB), respectively (TMD), we represent in Figure 3 their oscillations. The starting point of the orbits is $(x_0, y_0, z_0, u_0) = (0.01, \ \ 0.02, -0.03, 0.025)$. We remind that x_0 , z_0 are related to the initial value q_1 respectively q_2 and y_0, z_0 are related to the initial velocities of (PB), respectively (TMD).

It can be observed in Figure 3 that the oscillations are not uniform and there are short periods of stagnation of the motion followed by increasing amplitude of the oscillations.

In order to obtain regular oscillations we will consider values of the parameters that are close to the previous ones.

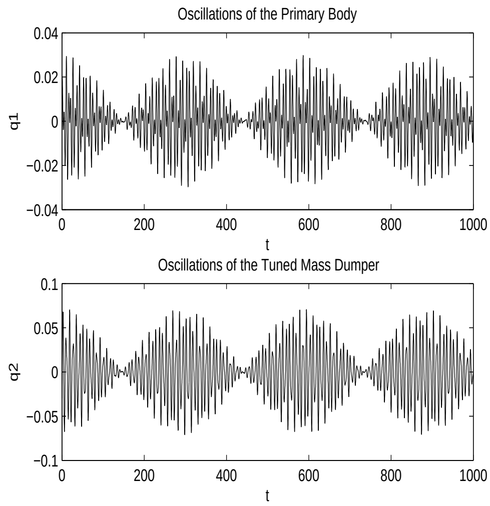

Fig. 1. Oscillations of the system of coupled oscillators for $m_0 = 0.21,\ \gamma_0 = 0.90909,$ $\xi_{t,0} = 0.01066,\ \xi_0 = -0.01172$

For $m = m_0 + 0.01$ and $\xi_t = \xi_{t,0} + 0,01$ the eigenvalues have negative real part $\mu_1\,(0.01, 0.01) = -0.00825$ and $\mu_2\,(0.01, 0.01) = -0.00294$. It results that the system (14) has the equilibria $E_1 = (0.00825, 0)$ and $E_2 = (0, 0.00294)$.

Technical computation shows that they are stable. Consequently, two stable limit cycles should appear in the four-dimensional system (2). This result is confirmed by the numerical simulation.

In Figure 1 one can observe the stabilization of the dynamics of the system and the formation of two stable limit cycles, as the theory predicted. In the simulation the same initial point

$$(x_0, y_0, z_0, u_0) = (0.01, \quad 0.02, -0.03, 0.025)$$

was used.

Similar results were obtained for all initial points used in simulations, under the restriction that the initial point must be close to $(0, 0, 0, 0)$.

The simulations show also that smaller $\xi_{t,0}$ is, more rapid stabilization of the oscillations is obtained.

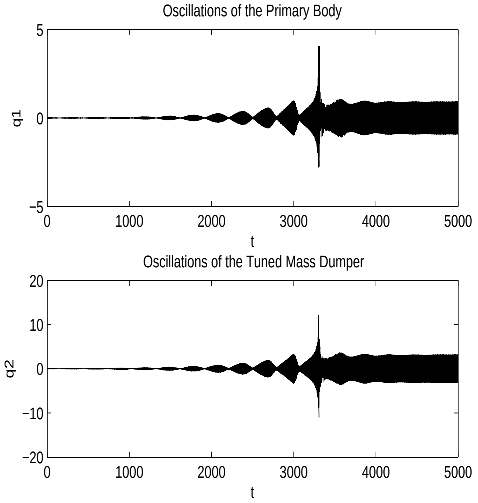

Fig. 2. Stabilized oscillations of the system of coupled oscillators for $m_0 = 0.22$, $\gamma_0 = 0.90909$, $\xi_{t,0} = 0.02066$, $\xi_0 = -0.01172$

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Appendix

For the computation of the coefficients involved in the normal form of (CD5) the algorithm presented in [8] is followed.

step1: the system is written in the form $x' = A(\alpha)x + F(x,\alpha)$. In our case $\alpha = (M, N)$. The matrix $A(\alpha)$ is:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{1+2m_0+M}{1+m_0} & -2\frac{\xi_{t,0}(M-1)+(m_0+M)N}{\sqrt{1+m_0}} & \frac{m_0+M}{1+m_0} & \frac{2(m_0+M)}{\sqrt{m_0+1}}\left(\xi_{t,0}+N\right) \\ 0 & 0 & 0 & 1 \\ \frac{1}{m_0+1} & \frac{2}{\sqrt{m_0+1}}\left(\xi_{t,0}+N\right) & -\frac{1}{m_0+1} & -\frac{2}{\sqrt{m_0+1}}\left(\xi_{t,0}+N\right) \end{pmatrix}$$

and $F(x,\alpha) = \begin{pmatrix} 0 \\ Cy^3 \\ 0 \\ 0 \end{pmatrix}$.

step 2: computation of the two complex eigenvectors $q_{1,2}$ of $A(0,0)$, corresponding to $\lambda_1(0) = i\omega_1$, $\lambda_2(0) = i\omega_2$

$$q_{1,2} = \begin{pmatrix} i \\ -\omega_{1,2} \\ \frac{2\sqrt{m_0+1}\xi_{t,0}}{m_0\omega_{1,2}} - i\frac{\omega_{1,2}^2-1}{m_0\omega_{1,2}^2} \\ \frac{\omega_{1,2}^2-1}{m_0\omega_{1,2}} + i\frac{2\sqrt{m_0+1}\xi_{t,0}}{m_0} \end{pmatrix}.$$

step 3: computation of the adjoint vectors $p_{1,2}$ and normalization (i.e. $\langle p_1, q_1\rangle = \langle p_2, q_2\rangle = 1$ and $\langle p_1, q_2\rangle = \langle p_2, q_1\rangle = 0$)

$$p_{1,2} = c_{1,2} \cdot \begin{pmatrix} \frac{-2m_0\omega_{1,2}^2+2\omega_{1,2}^2-1}{\omega_{1,2}^3(m_0+1)} + i\frac{2\xi_{t,0}}{\omega_{1,2}^2\sqrt{m_0+1}} \\ -i \\ \frac{-m_0\omega_{1,2}^2+\omega_{1,2}^2-1}{\omega_{1,2}^3(m_0+1)} - i\frac{2\xi_{t,0}}{\omega_{1,2}^2\sqrt{m_0+1}} \\ \frac{2\xi_{t,0}\sqrt{m_0+1}}{\omega_{1,2}} + i\frac{\omega_{1,2}^2-1}{\omega_{1,2}^2} \end{pmatrix}, \text{ where}$$

$$c_{1,2} = \frac{\omega_{1,2}}{\omega_{1,2}^4(m_0+1)-1}\left[\xi_{t,0}\cdot\omega_{1,2}\sqrt{m_0+1} - i\frac{\omega_{1,2}^2(m_0+1)-1}{2}\right].$$

step 4: consider $z_1 = \langle p_1, x\rangle$ and $z_2 = \langle p_2, x\rangle$ and compute

$$g(z_1, \overline{z}_1, z_2, \overline{z}_2) = \langle p_1, F(z_1q_1 + \overline{z}_1\overline{q}_1 + z_2q_2 + \overline{z}_2\overline{q}_2, (0,0))\rangle$$
$$h(z_1, \overline{z}_1, z_2, \overline{z}_2) = \langle p_2, F(z_1q_1 + \overline{z}_1\overline{q}_1 + z_2q_2 + \overline{z}_2\overline{q}_2, (0,0))\rangle.$$

For the system (6) we have

$$g(z_1, \overline{z}_1, z_2, \overline{z}_2) = -i\cdot C\cdot\overline{c}_1\left[\omega_1(z_1+\overline{z}_1) + \omega_2(z_2+\overline{z}_2)\right]^3$$
$$h(z_1, \overline{z}_1, z_2, \overline{z}_2) = -i\cdot C\cdot\overline{c}_2\left[\omega_1(z_1+\overline{z}_1) + \omega_2(z_2+\overline{z}_2)\right]^3$$

step 5: expand g, h in Taylor series

$$g(z_1, \overline{z}_1, z_2, \overline{z}_2) = \sum_{j+k+l+m\geq 2} g_{jklm} z_1^j \overline{z}_1^k z_2^l \overline{z}_2^m$$

respectively

$$h(z_1, \overline{z}_1, z_2, \overline{z}_2) = \sum_{j+k+l+m\geq 2} h_{jklm} z_1^j \overline{z}_1^k z_2^l \overline{z}_2^m$$

Because $g_{jklm} = h_{jklm} = 0$ for $j+k+l+m = 2$ it results (Lemma 8.14, [8], p 351) that

$$G_{2100}(0,0) = g_{2100} = -3iC\cdot\overline{c}_1\omega_1^3$$
$$G_{1011}(0,0) = g_{1011} = -6iC\cdot\overline{c}_1\omega_1\omega_2^2$$
$$H_{1110}(0,0) = h_{1110} = -6iC\cdot\overline{c}_2\omega_1^2\omega_2$$
$$H_{0021}(0,0) = h_{0021} = -3iC\cdot\overline{c}_2\omega_2^3$$
$$G_{1022}(0,0) = g_{1022} = 0;\ G_{3200}(0,0) = g_{3200} = 0;\ G_{2111}(0,0) = g_{2111} = 0$$
$$H_{2210}(0,0) = h_{2210} = 0;\ H_{1121}(0,0) = h_{1121} = 0; H_{0032}(0,0) = h_{0032} = 0$$

step 6: $p_{11} = Re\left(G_{2100}(0,0)\right) = \frac{3C\omega_1^4\left[(m_0+1)\omega_1^2-1\right]}{2\left[(m_0+1)\omega_1^4-1\right]}$

$p_{12} = Re\left(G_{1011}(0,0)\right) = \frac{3C\omega_1^2\omega_2^2\left[(m_0+1)\omega_1^2-1\right]}{(m_0+1)\omega_1^4-1}$

$p_{21} = Re\left(H_{1110}(0,0)\right) = \frac{3C\omega_1^2\omega_2^2\left[(m_0+1)\omega_2^2-1\right]}{(m_0+1)\omega_2^4-1}$

$p_{22} = Re\left(H_{0021}(0,0)\right) = \frac{3C\omega_2^4\left[(m_0+1)\omega_2^2-1\right]}{2\left[(m_0+1)\omega_2^4-1\right]}$

$s_1 = Re\left(G_{1022}(0,0)\right) + Re\left(G_{1011}(0,0)\right)\left[\frac{Re(H_{1121}(0,0))}{Re(H_{1110}(0,0))} - 2\frac{Re(H_{0032}(0,0))}{Re(H_{0021}(0,0))} - \frac{Re(G_{3200}(0,0))}{Re(G_{2100}(0,0))}\cdot\frac{Re(H_{0021}(0,0))}{Re(H_{1110}(0,0))}\right] = 0$

$s_2 = Re\left(H_{2210}(0,0)\right) + Re\left(H_{1110}(0,0)\right)\left[\frac{Re(G_{2111}(0,0))}{Re(G_{1011}(0,0))} - 2\frac{\varrho e(G_{3200}(0,0))}{Re(G_{2100}(0,0))} - \frac{Re(G_{1200}(0,0))}{Re(G_{1011}(0,0))}\cdot\frac{Re(H_{0032}(0,0))}{Re(H_{0021}(0,0))}\right] = 0$

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