ROMAI J., v.15, no.2(2019), 27–41

PROPERTIES OF RELATIVE ASYMMETRIC NEAR-METRIC SPACES AND APPLICATIONS

Julian Dimitrov

University of Mining and Geology, Sofia, Bulgaria

juldim@abv.bg

Abstract We study the some properties and length structures of not full metric spaces which are with generalized relative distance as component - wise M - relative space. Two cases of described properties are given: i) asymmetric relative metric space and ii) near-metric space.

Two applications for estimations in engineering practice of physical parameters and probabilities are described: i) estimation of sensiti-vity coefficients with relative distance and ii) estimation the variations in Pearsons test procedure by using the Pearson distance.

Keywords: relative distance, Hadamard product, semimetric spaces, M distance, Pearson distance, length structure.

2010 MSC: Primary 26B12, 54E25, 54E70; Secondary 49Q12, 68N30.

1. INTRODUCTION

The applied digital models in the technique are characterized by large parameter errors. However, these models are represented by continuous dependencies. Sensitivity analysis is practically applied to solve this problem as a local measure of the effect of an input on an output [2], [14]. The estimation is made by computing the sensitivity coefficients as partial derivatives by the input parameters. Directly using the partial derivatives, as it is done in [2], [3], [8], [14] and others, causes some disadvantages. For optimal evaluation of the dependencies in [6] is used *relative distance* in space of input parameters and calculations with semi-logarithmic derivatives.

There are many definitions of relative distance as the p - relative distance as it is done in [5] and [11]. In [7] was introduced the main results of M - relative distances $\rho_M(x,y) = |x-y|/M(x,y)$, where $M(x,y)$ is symmetric positive function. We study the asymmetric relative distance and introduce a component - wise generalization of M - relative distance (Section 2 and Section 3). Moreover, it turns out that relative distance is *near metric distance*.

In the Section 4 we study the some properties and length structures of not full metric spaces which are with generalized relative distance as compo- nent - wise M - relative distance. To motivate the paper study in Section 5 we make two significant applications for estimations in engineering practice of physical parameters and probabilities: i) estimation of sensitivity coefficients with relative distance and ii) estimation the variations in Pearsons test procedure by using the Pearson distance.

Some denotations and abbreviations used in the paper:

Denotations:

$x \circ y$	Hadamart product of vectors x and y
$x \prec y$ $(x \preceq y)$	Vector x is less than (less than or equal) to y
$x \succ y$ $(x \succeq y)$	Vector x is greater than (greater than or equal) to y
$\| \ \|$	Euclidean distance
$minh(x, y)$ and $maxh(x, y)$	Component wise comparison of vectors x and y - the result is a vector with the smallest (greatest) values of the components
$min(x)$ and $max(x)$	Is equal to the smallest (greatest) component of vector $x = (x_1, ..., x_m)$
e	Vector $e = (1, ..., 1)$ - identity element in R^m

Abbreviations:

MI	Moderately increasing
SMI	Strictly moderately increasing
MIV	Moderately increasing vector
SMIV	Strictly moderately increasing vector
GR	Generalized relative
IV	Increasing vector
sHV	s - homogeneous vector
GsHV	Generalized s - homogeneous vector
SU and MIV	Sub and upper component - wise homogeneous and Moderately increasing vector

1.1. HADAMARD PRODUCT AND HADAMARD NOTATION

Let $x_0 = (x_1^0, ..., x_m^0), x = (x_1, ..., x_m) \in R^m$. Following R.A. Horn, C. R. Johnson [9] we denote the *Hadamard product* of x_0 and x as vector $\widehat{x} = x_0 \circ x = (x_1^0 \cdot x_1, ..., x_m^0 \cdot x_1) \in R^m$.

Some properties: Hadamard product is associative, distributive and commutative; Vector $e = (1, ..., 1) \in R^m$ is identity element - $e \circ x = = x \circ e = x$; For vector $x = (x_1, ..., x_m)$ with nonzero components $(x_i \neq 0, \ i = 1, ..., m)$ there is unique element $x^{-1} = 1/x = (1/x_1, ..., 1/x_m)$ such that $x \circ x^{-1} = e$ and $(x^{-1})^{-1} = x$; With the Hadamard product the positive cone $K_m^+ = \{(x_1, ..., x_m) \in R^m, x_j > 0\}$ is multiplicative group. We will say that we have Hadamard notation when we denote component-wise operations in vector space. For example let $x_0 = (x_1^0, ..., x_m^0), x = = (x_1, ..., x_m) \in K_m^+ \subset R^m$. Then we will use the notation: $\sqrt{x} = (\sqrt{x}_1, ... \ ..., \sqrt{x}_m)$; $x^2 = (x_1^2, ..., x_m^2)$;

$f(x) = (f(x_1), ..., f(x_m))$ - for every function $f : (0, \infty) \to R$; For inequality between two vectors $x_0, x \in R^m$ we have denotation $x_0 \prec x$ ($x_0 \preceq x$) when $x_i^0 < x_i$ (or $x_i^0 \leqq x_i$) for every $i = 1, ..., m$. Let we denote with $\| \ \|$ the Euclidean norm. Then from $x_0, x \in K_m^+$ and $x_0 \prec x \Rightarrow \|x_0\| < \|x\|$; In addition we introduce the denotations $minh(x_0, x) = (min(x_1^0, x_1), ..., min(x_m^0, x_m))$, $maxh(x_0, x) = = (max(x_1^0, x_1), ..., max(x_m^0, x_m))$, $min(x_0) = min(x_1^0, ..., x_m^0)$ and $max(x_0) = max(x_1^0, ..., x_m^0)$.

1.2. TRANSFORMATION MODEL AND RELATIVE DISTANCE

Similarly of the operating in the area of numerical algorithms, the applied numerical models (and so technical models) transforms from input parameters (data) to output. Let $f : X \subset K_m^+ \to Y \subset R^n$ is a continuous mapping. Then we will say that domain X is *space of input parameters*, range Y - *space of output parameters* and mapping f - *transformation model* [6]. Moreover, we use *estimation of relative errors* in space of input parameters for unify these parameters about physical dimensions of parameters and limits of numerical values.

Let $f(x) : X \subset K_m^+ \to Y \subset R$ is a smoothly function and then $D_{ln} f = = (\partial f(x_0)/\partial ln x_1, ..., \partial f(x_0)/\partial ln x_m)$ - the gradient of $f(x)$ in *semilogarithmic derivatives*, ensuring accordance with the relative variation of argument x. The deviation of $f(x)$ is given by Taylors formula [6]: $\Delta f = Df \cdot \Delta x + + O(\|\Delta x\|^2) = D_{ln} f \cdot \Delta_{ln} x + O(\|\Delta x\|^2)$, where $\Delta x = x - x_0$, $\Delta_{ln} x = = (x - x_0) \circ x_0^{-1}$, Df is the functional gradient and $D_{ln} f$ - semi-logarithmic gradient. This brings us to the definition of relative distance.

Definition 1.1. ***(Relative Distance).*** *Let $x_0, x \in K_m^+$. We define relative pseudo-distance (relative distance) as $b^r(x_0, x) = \|(x - x_0) \circ x_0^{-1}\|$.*

Remark 1.1. ***(Some properties of relative distance).*** *Let $x_0, x \in K_m^+$ then:*

1. $b^r(x_0, x) \geqq 0$, $b^r(x_0, x) = 0 \Leftrightarrow x_0 = x$.
2. $b^r(\lambda x_0, \lambda x) = b^r(x_0, x)$ *for* $\lambda \neq 0$.
3. $b^r(x_0^{-1}, x^{-1}) = \|x^{-1} \circ (x_0^{-1})^{-1} - e\| = \|x_0 \circ x^{-1} - e\| = b^r(x, x_0)$.
4. *if* $x_0 \prec x$ *then* $b^r(x_0, x) = \|(x - x_0) \circ x_0^{-1}\| \succeq \|(x_0 - x) \circ x^{-1}\| = b^r(x, x_0)$.

The defined here relative distance corresponds to relative error estimations.

An approach for Functional Dependence Estimation. Let we have a smoothly function $f(x) : X \subset K_m^+ \to Y$ defined in the ball $B_\delta^r(x_0) = = \{x : b^r(x_0, x) < \delta\}$, where δ is small enough so that f is linear with sufficient precision [6]. Than $\Delta f = Df \cdot \Delta x = D_{ln} f \cdot \Delta_{ln} x$ and from Cauchy Schwarz inequality $|\Delta f| = |D_{ln} f \cdot \Delta_{ln} x| \leqq \|D_{ln} f\| \cdot \|\Delta_{ln} x\|$, where $\|\Delta_{ln} x\| = = b^r(x_0, x)$. So we obtain *condition for recoverability of functional dependence*

f from discrete presentation

$$k(1-p)^2\|D_{ln}f\| \leqq \frac{\varepsilon}{\delta}, \tag{1}$$

where $k = cos\psi$, $\psi \in [0, \pi/2]$ and ψ is the angle of segment in the ball $B^r_\delta(x_0)$ [6]. The condition is with *confidential probability* $1 - \bar{p} = 1 - p^3$ and the coefficient $k(1-p)^2$ is determined by the input and output data of the model.

1.3. ASYMMETRIC AND NEAR METRIC DISTANCE

Definition 1.2. *Let $X \subset K^+_m \subset R^m$ and is nonempty set. Then a function $b : X \times X \to R$ is called distance function if for any $x, y, z \in X$, we have:*

1. *Non negativity $b(x, y) \geqq 0$;*
2. *Identity of indiscernible $b(x, y) = 0 \Leftrightarrow x = y$;*
3. *Symmetry $b(x, y) = b(y, x)$;*
4. *Triangle inequality $b(x, y) \leqq b(x, z) + b(z, y)$.*

We have *asymmetric metric* as in [12] if function b is not fall in the point 3. Where point 4. is not satisfied and triangle inequality is replaced by *relaxed triangle inequality* $b(x, y) \leqq \sigma[b(x, z) + b(z, y)]$ we have *near metric* [15].

In addition we consider such distance function b that satisfied properties:

Addition to Definition 1.2:

5. *Regularity $b(x, y) \geqq b(x + z, y + z)$;*
6. *Homogeneity of order $s \in R$ if $b(cx, cy) = c^s \circ b(x, y)$ where $c \in K^+_m$.*

The regularity and homogeneity properties are characteristic of distances in probability theory [13].

2. MODERATELY INCREASING FUNCTION AND GENERALIZED RELATIVE DISTANCE

Definition 2.1. *By following [7] function $\mu(x) : K^+_1 \to K^+_1$ is moderately increasing (MI) if is increasing and $\mu(x)/x$ is decreasing may be not strictly. If $\mu(x)$ is strictly increasing function and $\mu(x)/x$ - strictly decreasing, then will say that $\mu(x)$ is strictly moderately increasing (SMI) function. We define a function of two variables $\mu_i(x^0_i, x_i) : K^+_2 \to K^+_1$ as MI (or SMI) function if $\mu^1_i(x^0_i) = \mu_i(x^0_i, x_i)$, $x_i = const$ and $\mu^2_i(x) = \mu_i(x^0_i, x_i)$, $x^0_i = const$ are moderately increasing.*

Let $x_0, x \in K^+_m$ and $\mu_i(x^0_i, x_i) : K^+_2 \to K^+_1$, $i = 1, ..., m$ are MI functions. Then we will say that $M(x_0, x) = (\mu_1(x^0_1, x_1), ..., \mu_m(x^0_m, x_m))$ is moderately increasing vector function (MIV or SMIV).

Proposition 2.1. *Let $M(x_0, x) : K^+_m \times K^+_m \to K^+_m$ is MIV function and $x, y, z \in K^+_m$. Then*

(i) $minh(e, z/y) \preceq M(x,z)/M(x,y) \preceq maxh(e, z/y)$ *and*
(ii) $minh(e, x/y) \preceq M(x,z)/M(y,z) \preceq maxh(e, x/y)$.

Proof. We will start by proving for the i component $\mu_i : K_2^+ \to K_1^+$. Let $y_i \leqq z_i$ then $1 \leqq \mu_i(x_i, z_i)/\mu_i(x_i, y_i) \leqq z_i/y_i$. In opposite case $y_i > z_i$ and $z_i/y_i \leqq \mu_i(x_i, z_i)/\mu_i(x_i, y_i) \leqq 1$. $\Rightarrow$ (i). By analogy we receive (ii). ∎

Proposition 2.2. *If function* $M(x_0, x) : K_m^+ \times K_m^+ \to K_m^+$ *is MIV then it is continuous.*

Proof. Let we fix point $(x, y) \in K_m^+ \times K_m^+$ and choose another one $(u, v) \in K_m^+ \times K_m^+$. Let $L_1 = min(minh(x, u))$, $L_2 = min(minh(y, v))$, $V_1 = max(maxh(x, u))$, $V_2 = max(maxh(y, v))$. From Proposition 2.1 $\Rightarrow$
$(L_2/V_2)e \preceq minh(e, y/v) \preceq M(x,y)/M(x,v) \preceq maxh(e, y/v) \preceq (V_2/L_2)e$.
Then by analogy $(L_1/V_1)e \preceq M(x,v)/M(u,v) \preceq (V_1/L_1)e$. $\Rightarrow$
$(L_1/V_1) \cdot (L_2/V_2)e \preceq M(x,y)/M(u,v) \preceq (V_1/L_1) \cdot (V_2/L_2)e$. Let point (u, v) convergent to (x, y) $((u, v) \to (x.y))$. Then $L_1/V_1 \to 1$ and $L_2/V_2 \to 1$. $\Rightarrow$
$M(u, v) \to M(x.y)$. ∎

Here we define generalized relative distance as component - wise M relative distance and in addition, so as to be satisfied 4. in Remark 1.1.

Definition 2.2. ***(Generalized Relative Distance).*** *Let* $x_0, x \in K_m^+$ *and* $M(x_0, x)$ *is a MIV function that satisfies the inequality* $M(x_0, x) \preceq M(x, x_0)$ *whenever* $x_0 \prec x$. *Then we define componentwise* M *relative distance - generalized relative (GR) distance in* X *as* $b^M(x_0, x) = \|(x - x_0) \circ M^{-1}(x_0, x)\|$.

The property of $M(x_0, x)$ (corresponding to point 5. in Definition 1.2) as component - wise *increasing vector (IV)* function is a *generalized form of regularity*, because this fact is equivalent to inequality

$$M(x_0, x) \preceq M(x_0 + u, x + v), \tag{2}$$

where $u, v \in [0, \infty)$. We will define a generalized form of homogeneity by analogy with point 6. in Deffinition 1.2.

Definition 2.3. *We will say that* $M(x_0, x)$ *is generalized s - homogeneous vector (GsHV) function when*

$$M(u \circ x_0, v \circ x) = u^\alpha \circ v^\beta \circ M(x_0, x), \tag{3}$$

and $x_0, x, u, v \in K_m^+$, $\alpha \in [0, \infty)$, $\beta \in [0, \infty)$, $\alpha \geqq \beta$ *and* $\alpha + \beta = s$.
In special case when $u = v$ *we have s - homogeneous vector (sHV) function defined with equation* $M(u \circ x_0, u \circ x) = u^s \circ M(x_0, x)$.

Remark 2.1. *Let* $x, y \in K_m^+$. *For the general type of GsHV function* $M(x, y)$ *we get the from* $M(x, y) = x^\alpha \circ y^\beta \circ M(e, e)$, *for* $\alpha \in [0, \infty)$, $\beta \in [0, \infty)$, $\alpha \geqq \beta$

and $\alpha + \beta = s$. In addition, if $M(x, y)$ is GsHV function then it is MIV function.

Proposition 2.3. *[7] Let $M(x_0, x) : K_m^+ \times K_m^+ \to K_m^+$ be IV and sHV function with $s \in [0, 1]$. Then function M is MIV. If function M is MIV and sHV then $s \in [0, 1]$.*

Proof. (i) Let M be increasing and s - homogeneous. Let $x, y, z \in K_m^+$, $s \in [0, 1]$ and $x \succeq y \succeq z$. Then function $\varphi(z) = M(z, y)/z = z^s \circ M(e, y/z)/z = z^{s-1} \circ M(e, y/z)$ is decreasing because M is IV function and $s \in [0, 1]$. $\Rightarrow$ $\varphi(z) \succeq \varphi(x) = M(x, y)/x$. By analogy $M(z, y)/y \succeq M(z, x)/x$. $\Rightarrow$ function M is MIV.

(ii) Let $M(x_0, x) : K_m^+ \times K_m^+ \to K_m^+$ is MIV and sHV. Let's fix the points $x_0, z_0 \in K_m^+$ and $x = cx_0$, $z = cz_0$, $c \in R$, $c > 0$. Then: **first** the function $\psi(x) = M(x, z) = c^s M(x_0, z_0) = x^s \circ M(x_0, z_0)/(x_0)^s$ is increasing $\Rightarrow s \geqq 0$; **Second** the function $\chi(x) = M(x, z)/x = c^s M(x_0, z_0)/x = x^{s-1} M(x_0, z_0)/(x_0)^{s+1}$ is decreasing $\Rightarrow s \leqq 1 \Rightarrow$ as final $s \in [0, 1]$. ∎

Remark 2.2. *Let $M(x_0, x) : K_m^+ \times K_m^+ \to K_m^+$ is GsHV function. Then as a consequence of Proposition 2.3 we get that the function M is MIV if and only if $s \in [0, 1]$.*

Proposition 2.4. *Let $M : K_m^+ \times K_m^+ \to K_m^+$ is SMIV function, $x, y, z \in K_m^+$ and $x \prec y \prec z$. Then*

(i) There is $\beta \in [0, 1)$ such that $M(x, z)/M(x, y) \succ (z/y)^\beta$ and

(ii) There is $\alpha \in (0, 1]$, $\alpha > \beta$ such that $M(x, z)/M(y, z) \succ (x/y)^\alpha$.

Proof. Let μ_i is i - component of SMIV fuction $M : K_m^+ \times K_m^+ \to K_m^+$. We chose $x, y, z \in K_m^+$ such that $x \prec y \prec z$.

(i) Function $f_i(\beta) = [\mu_i(x_i, z_i)/\mu_i(x_i, y_i)] \cdot (y_i/z_i)^\beta$ is strictly decreasing continuous function, $f_i(0) > 1$ and $f_i(1) < 1$. $\Rightarrow$ There is unique $\beta_i^0 \in (0, 1)$ such that $f_i(\beta_i^0) = 1$. For every $\beta \in [0, \beta_i^0)$ we have $f_i(\beta) \geqq 1$. $\Rightarrow$ $\mu_i(x_i, z_i)/\mu_i(x_i, y_i) > (z_i/y_i)^\beta$. Let $\beta_0 = \min_i\{\beta_i^0\}$. Then for $\beta \in [0, \beta^0)$ is fulfilled $M(x, z)/M(x, y) \succ (z/y)^\beta$.

(ii) Function $\varphi_i(\alpha) = [\mu_i(x_i, z_i)/\mu_i(y_i, z_i)] \cdot (y_i/x_i)^\alpha$ is strictly increasing continuous function, $\varphi_i(0) < 1$ and $\varphi_i(1) > 1$. $\Rightarrow$ There is unique $\alpha_i^0 \in (0, 1)$ such that $\varphi_i(\alpha_i^0) = 1$. $\Rightarrow$ $\mu_i(x_i, z_i)/\mu_i(y_i, z_i) > (x_i/y_i)^\alpha$ for $\alpha \in (\alpha_i^0, 1] \Rightarrow$ $M(x, z)/M(y, z) \succ (x/y)^\alpha$ for $\alpha \in (\alpha_0, 1]$ with $\alpha_0 = \max_i\{\alpha_i^0\}$. ∎

We will consider the special case of MIV function where we have opposite inequalities to those of points (i) and (ii) from Proposition 2.4 satisfied for suitable parameters $\alpha \in (0, 1]$, $\beta \in [0, 1)$ such that $\alpha \geqq \beta$.

Definition 2.4. *We will say that a MIV function is sub and upper componentwise homogeneous (SU and MIV) function if for every $u, v, c \in K_m^+$, $c \succ e$ and* *some* $\alpha \in (0,1]$, $\beta \in [0,1)$, $\alpha \geqq \beta$ *the inequalities* $c^{\alpha} \circ M(u,v) \preceq \preceq M(c \circ u, v)$ *and* $M(u, c \circ v) \preceq c^{\beta} \circ M(u,v)$ *are fulfilled.*

Proposition 2.5. *MIV function* $M : K_m^+ \times K_m^+ \to K_m^+$ *is SU if and only if for* $x,y,z \in K_m^+$, $x \prec y \prec z$ *there is* $\alpha \in (0,1]$, $\beta \in [0,1)$, $\alpha \geqq \beta$ *such that* $M(x,z)/M(y,z) \preceq (x/y)^{\alpha}$ *and* $M(x,z)/M(x,y) \preceq (z/y)^{\beta}$.

Proof. Let $x,y,z \in K_m^+$ and $x \prec y \prec z$

(i) Let $c = y/x$, $u = x$ and $v = y$ then $M(x,z)/M(y,z) \preceq (x/y)^{\alpha}$ if and only if $c^{\alpha} \circ M(u,v) \preceq M(c \circ u, v)$.

(ii) If $c = z/y$, $u = x$ and $v = y$ then $M(x,z)/M(x,y) \preceq (z/y)^{\beta}$ if and only if $M(u, c \circ v) \preceq c^{\beta} \circ M(u,v)$. ∎

Remark 2.3. *In both cases of Proposition 2.5 the equalities are reached when function M is GsHV.*

Other cases of SU and MIV function in GR distance are:
(i) Relative distance (Definition 2.1) - $M(x,y) = x$, $\alpha = 1$ *and* $\beta = 0$*;*
(ii) Pearson distance (Definition 4.2) - $M(x,y) = \sqrt{x}$, $\alpha \in [0.5,1]$, $\beta = 0$*;*
(iii) Special case of M - relative distance [5] (Barlow (at all) relative distance [11]) - $M(x,y) = \sqrt{x \circ y}$, $\alpha \in [0.5,1]$ *and* $\beta \in [0.5,1]$.

3. GENERALIZED RELATIVE METRICS AS ASYMMETRIC NEAR-METRICS

GR distance $b^M(x,y) = \|(y-x) \circ M^{-1}(x,y)\|$, where $x,y \in K_m^+$ is symmetric or asymmetric depending on whether the vector function $M(x,y)$ is symmetric or asymmetric, respectively. In this section we discuss both cases to make the necessary conclusions. Let $M(x,y)$ is asymmetric, GsHV and MIV then we get the general form $M(x,y) = x^{\alpha} \circ y^{\beta} \circ M(e,e)$ with $\alpha \in (0,\infty)$, $\beta \in [0,1)$, $\alpha + \beta \in [0,1)$ and $\alpha > \beta$.

We will discuss the GR metrics in special case of bounded set X_{LV} that is subset in positive cone. We denote $X_{LV} = \{x : x \in K_m^+, Le \preceq x \preceq Le\}$, where L and V are two positive constants $0 < L < V < \infty$. This set is bounded from the side of origin and ∞. Such denotation corresponds to the applications in practice, where technical parameters have numerical values limited from the side of the zero and ∞. In this section we will prove that GR distance with function M that is SU and SMIV induced a near metrics everywhere dense in K_m^+. The GR distance satisfied the relaxed inequality in X_{LV} with relaxed constant $\sigma = V/L$.

Proposition 3.1. *Let* $M : X_{LV} \times X_{LV} \to K_m^+$ *is IV function,* $x,y,z \in X_{LV}$ *and* $\sigma = V/L$*. Then M is a MIV function if and only if the inequalities*

$$\sigma M^{-1}(x,y) \succeq M^{-1}(x,z) \tag{4}$$

$$\sigma M^{-1}(y,x) \succeq M^{-1}(z,x). \tag{5}$$

are filfulled.

Proof. Let $M : X_{LV} \times X_{LV} \to K_m^+$ is MIV function, $x, y, z \in X_{LV}$ and $\sigma = V/L$. Then from Proposition 2.1

(i) $(1/\sigma)e = (L/V)e \preceq M(x,z)/M(x,y) \preceq (V/L)e \Rightarrow (4)$;

(ii) $(1/\sigma)e = (L/V)e \preceq M(z,x)/M(y,x) \preceq (V/L)e \Rightarrow (5)$. ∎

Theorem 3.1. *Let $M : X_{LV} \times X_{LV} \to K_m^+$ is MIV function and $x, y, z \in X_{LV}$. Then the GR distance $b^M : X_{LV} \times X_{LV} \to [0, \infty)$ satisfied the relaxed inequality $[b^M(x,y) + b^M(y,z)]\sigma \geqq b^M(x,z)$.*

Proof. From Proposition 3.1. $\Rightarrow [b^M(x,y) + b^M(y,z)]\sigma =$
$= \|(y-x) \circ \sigma M^{-1}(x,y)\| + \|(z-y) \circ \sigma M^{-1}(y,z)\| \geqq \|(y-x) \circ M^{-1}(x,z)\| +$
$+ \|(z-y) \circ M^{-1}(x,z)\| \geqq \|(z-x) \circ M^{-1}(x,z)\| = b^M(x,z)$. ∎

Theorem 3.2. *Let $b^M(x_0,x) = \|(x-x_0) \circ M^{-1}(x_0,x)\|$ is GR distance in K_m^+ with SU and SMIV function $M(x_0,x)$. Then for every $y \in K_m^+$ and neighborhood $U(y)$ of y, there is two others points $x, y \in U(y)$, such that the triangle inequality $b^M(x,y) + b^M(y,z) \geqq b^M(x,z)$ does not fulfilled.*

Proof. Let $y \in K_m^+$, $x = (1-\varepsilon)y$, $z = (1+\delta)y$, $\varepsilon > 0$ and $\delta > 0$. Here ε and δ are small enough.

Then $x \prec y \prec z$. We denote $q = b^M(x,y) + b^M(y,z) - b^M(x,z) = \|(y-x) \circ M^{-1}(x,y)\| + \|(z-y) \circ M^{-1}(y,z)\| - \|(z-x) \circ M^{-1}(x,z)\| = = \varepsilon\|y \circ M^{-1}(x,y)\| + \delta\|y \circ M^{-1}(y,z)\| - (\varepsilon+\delta)\|y \circ M^{-1}(x,z)\|$. From Proposition 2.5 $\Rightarrow M^{-1}(x,y) \preceq (z/y)^\beta \circ M^{-1}(x,z) = (1+\delta)^\beta M^{-1}(x,z)$
and $M^{-1}(y,z) \preceq (x/y)^\alpha \circ M^{-1}(x,z) = (1-\varepsilon)^\alpha M^{-1}(x,z)$, where $\alpha \in (0,1]$, $\beta \in [0,1)$ and $\alpha \geqq \beta$. $\Rightarrow q \leqq \|y \circ M^{-1}(x,z)\| \cdot [\varepsilon(1+\delta)^\beta + \delta(1-\varepsilon)^\alpha - (\varepsilon+\delta)]$.
Let $\chi(\varepsilon,\delta) = \varepsilon(1+\delta)^\beta + \delta(1-\varepsilon)^\alpha - (\varepsilon+\delta)$. We have three cases:

(i) $\alpha = 1 \Rightarrow \chi(\varepsilon,\delta) = \varepsilon(1+\delta)^\beta - \varepsilon(1+\delta) < 0$;

(ii) $\beta = 0 \Rightarrow \chi(\varepsilon,\delta) = \delta(1-\varepsilon)^\alpha - \delta < 0$ and

(iii) $\alpha \in (0,1)$ and $\beta \in (0,1)$. Then **first** $\chi(0,\delta) = 0$ and $\chi'_\varepsilon(\varepsilon,\delta) = = (1+\delta)^\beta + \delta\alpha(1-\varepsilon)^{\alpha-1}(-1) - 1$. **Second** we have $\chi'_\varepsilon(\varepsilon,0) = 0$ and $\chi''_{\varepsilon\delta}(\varepsilon,\delta) = \beta(1+\delta)^{\beta-1} - \alpha(1-\varepsilon)^{\alpha-1} < 0$ when $\alpha \geqq \beta$, $\varepsilon \to 0$ and $\delta \to 0$. $\Rightarrow \chi'_\varepsilon(\varepsilon,\delta) < 0$ for small enough $\varepsilon > 0$ and $\delta > 0$. $\Rightarrow \chi(\varepsilon,\delta) < 0$. $\Rightarrow$
For all three cases $q < 0$. ∎

4. LENGTH STRUCTURES

4.1. RECTIFIABLE PATH INDUCED BY GR METRICS

Definition 4.1. ***(Length of path, rectifiable path).*** *Let $(X, b^r), X \subset K_m^+$ is generalized relative metric space and $\gamma(t) : [a,b] \to X$ - a path from $\gamma(a)$ to*

$\gamma(b)$, where $\gamma(t)$ is a smooth function. Let $T = \{a = t_0 <, ..., < t_N = b\}$ is a partition of $[a, b]$. Then we say that

$$l^M(\gamma) = \sup_T \sum_{i=0}^{N-1} b^M\Big(\gamma(t_i), \gamma(t_{i+1})\Big) \tag{6}$$

is the length of path $\gamma(t)$.

We denote as l^r the length induced by relative metric (Definition 2.1) and l^e length in Euclidean space.

Theorem 4.1. *Let $\gamma(t) : [a, b] \to X \subset K_m^+$ is a smooth vector function with components $\gamma_j(t) \geqq C > 0$ for $j = 1, ..., m$. Then the length of path $l^M(\gamma)$ induced by $b^M(x_0, x) = \|(x - x_0) \circ M^{-1}(x_0, x)\| : K_m^+ \times K_m^+ \to [0, \infty)$ is finite and $l^M(\gamma) = \int_a^b \|\gamma'(t) \circ M^{-1}(\gamma(t), \gamma(t))\| dt$.*

Proof. For $\varepsilon > 0$ there is partitions $T = \{a = t_0 <, ..., < t_N = b\}$ such that

$$\Big|\frac{|\gamma_j(t_{i+1}) - \gamma_j(t_i)|}{\Delta t_i} - |\gamma_j'(t_i)|\Big| < \varepsilon \tag{7}$$

for every $j = 1, ..., m$, $i = 0, ..., N-1$ and small enough $\Delta t_i = t_{i+1} - t_i$. This means that there is $\delta > 0$ such that if $\max_i \Delta t_i < \delta$ then (7) is fulfilled.
Let $M : X \times X \to K_m^+$ is a MIV function. Then for j - component

$$\Big|\frac{|\gamma_j(t_{i+1}) - \gamma_j(t_i)|}{\Delta t_i \cdot \mu_j\Big(\gamma_j(t_i), \gamma_j(t_{i+1})\Big)} \cdot \Delta t_i - \frac{|\gamma_j'(t_i)|}{\mu_j\Big(\gamma_j(t_i), \gamma_j(t_i)\Big)} \cdot \Delta t_i\Big| <$$

$$< \varepsilon \cdot \Delta t_i \cdot \frac{1}{\mu_j(Ce, Ce)}, \tag{8}$$

where $\mu_j(Ce, Ce) \leqq \min_i(\mu_j(\gamma_j(t_i), \gamma_j(t_{i+1})), \mu_j(\gamma_j(t_i), \gamma_j(t_i)))$.

(i) If $\gamma_j'(t) \neq 0$ for $t \in (a, b)$ then the expressions $\gamma_j(t_{i+1}) - \gamma_j(t_i)$ and $\gamma_j'(t_i)$ have the same sign. In this case, inequality (8) is equivalent to (9)

$$\Big|\frac{\gamma_j(t_{i+1}) - \gamma_j(t_i)}{\Delta t_i \cdot \mu_j\Big(\gamma_j(t_i), \gamma_j(t_{i+1})\Big)} \cdot \Delta t_i - \frac{\gamma_j'(t_i)}{\mu_j\Big(\gamma_j(t_i), \gamma_j(t_i)\Big)} \cdot \Delta t_i\Big| <$$

$$< \varepsilon \cdot \Delta t_i \cdot \frac{1}{\mu_j(Ce, Ce)}. \tag{9}$$

Due to the continuity of $\gamma_j'(t)$ we have only other two cases: (ii) when $\gamma_j'(t) = 0$ at intervals or (iii) at isolated points of $[a, b]$.

In the case (ii) the inequalities (8) and (9) are equivalent when Δt_i is small enough. Case (iii) can be ignored because the properties at individual isolated points, which form a set with measure zero, do not affect the values of the integrals that we calculate. These remarks replace the study of *absolutely continuity* and *local Lipschitz properties* in [1].

We denote a vector function $\Phi(x) = \int_{\gamma(a)}^{x} M^{-1}(v,v)dv : K_m^+ \to R^m$. Then

$S_N^M = \sum_{i=0}^{N-1} b^M\Big(\gamma(t_i),\gamma(t_i)\Big)\cdot \Delta t_i = \sum_{i=0}^{N-1} \|\gamma^{'}(t_i)\circ M^{-1}\Big(\gamma(t_i),\gamma(t_i)\Big)\|\cdot \Delta t_i =$

$= \sum_{i=0}^{N-1} \|\Phi_t^{'}(\gamma(t_i))\|\cdot\Delta t_i$ is a partial sum of $\int_a^b \|\Phi_t^{'}(\gamma(t))\|dt$ and $\|\Phi_t^{'}(\gamma(t))\|$ is metric derivative of $\Phi(\gamma(t))$ [1]. Let $L_N^M(\gamma) = \sum_{i=0}^{N-1} b^M\Big(\gamma(t_i),\gamma(t_{i+1})\Big) =$

$= \sum_{i=0}^{N-1} \|\frac{\Big(\gamma(t_{i+1})-\gamma(t_i)\Big)\circ M^{-1}\Big(\gamma(t_i),\gamma(t_{i+1})\Big)}{\Delta t_i}\cdot\Delta t_i\|$ is the total length of all inscribed rectilinear segments corresponding to a partition T. Then

$$|L_N^M(\gamma) - S_N^M| = \Big|\sum_{i=0}^{N-1}\Big[\frac{b^M\Big(\gamma(t_i),\gamma(t_{i+1})\Big)}{\Delta t_i}\cdot\Delta t_i - \|\Phi_t^{'}(\gamma(t))\|\cdot\Delta t_i\Big]\Big| \leqq$$

$$\leqq \sum_{i=0}^{N-1}\|\frac{\gamma(t_{i+1})-\gamma(t_i)}{\Delta t_i}\circ M^{-1}\Big(\gamma(t_i),\gamma(t_{i+1})\Big) - \gamma^{'}(t_i)\circ M^{-1}\Big(\gamma(t_i),\gamma(t_i)\Big)\|\cdot\Delta t_i <$$

$< \varepsilon\cdot(b-a)\cdot\|M^{-1}(Ce,Ce)\|$. Because $\|\Phi_t^{'}(\gamma(t))\|$ is an integrable function

$$\Rightarrow \lim_{\max_i(\Delta t_i)\to 0} S_N^M = \int_a^b \|\Phi_t^{'}(\gamma(t))\|dt = \lim_{\max_i(\Delta t_i)\to 0} L_N^M = l^M(\gamma).$$

$$\Rightarrow l^M(\gamma) = \int_a^b \|\gamma^{'}(t)\circ M^{-1}\Big(\gamma(t),\gamma(t)\Big)\|dt$$

∎

4.2. RELATION BETWEEN PATHS IN GR METRICS

Eexpression by Euclidean Length. It follows from Theorem 4.1:

$$l^M(\gamma(t)) = \int_a^t \|\Phi_t'(\gamma(v))\| \cdot dv = l^e\Big(\Phi(\gamma(t))\Big), \tag{10}$$

where $\Phi(x) = \int_{\gamma(a)}^x M^{-1}(v,v)dv$. Euclidean distance is special case of GR distance $b^M(x,x_0) = \|(x-x_0) \circ M^{-1}(x_0,x)\|$ with constant function $M(x_0,x) = e = (1,...,1)$.

Length in Relative Metric Space. Introduced from Definition 2.1 relative distance is GR distance with MI function $M(x_0,x) = x_0 = (x_1^0,...,x_m^0)$ and $\Phi(x) = ln(x) = (ln(x_1),...,ln(x_m))$. Then

$$l^r(\gamma(t)) = \int_a^t \|\frac{\partial ln\gamma(v)}{\partial v}\| \cdot dv = l^e\Big(ln(\gamma(t))\Big). \tag{11}$$

A case where the triangle inequality is not satisfied for relative distance: Let $x = e$, $y = 3e$ and $z = 2e$. Then $b^r(x,y) = 2 \cdot \sqrt{m}$, $b^r(x,z) = \sqrt{m}$ and $b^r(z,y) = 0.5 \cdot \sqrt{m}$. $\Rightarrow b^r(x,z) + b^r(z,y) < b^r(x,y)$.

Definition 4.2. ***(Pearson Distance).*** *Let $x_0, x \in K_m^+$. We define Pearson distance as $b^p(x_0,x) = \|(x-x_0) \circ (x_0)^{-1/2}\|$. Here the corresponding MI function is $M(x_0,x) = \sqrt{x_0} = (\sqrt{x_1^0},...,\sqrt{x_m^0})$ and $\Phi(x) = 2\sqrt{x}$.*

Pearson distance corresponds to Pearson's criterion for chi-square test of events sets [4], [13]. A case where for Pearson distance the triangle inequality is not satisfied: Let $x = e$, $y = 3e$ and $z = 2e$. Then $b^p(x,y) = 2 \cdot \sqrt{m}$, $b^p(x,z) = \sqrt{m}$ and $b^p(z,y) = (1/\sqrt{2}) \cdot \sqrt{m}$. $\Rightarrow b^p(x,z) + b^p(z,y) < b^p(x,y)$. The eexpression for length l^p induced by Pearson distance is

$$l^p(\gamma(t)) = \int_a^t \|\frac{\partial\Big(2\sqrt{\gamma(v)}\Big)}{\partial v}\| \cdot dv = l^e\Big(2\sqrt{\gamma(t)}\Big). \tag{12}$$

Remark 4.1. ***(Interpretation the Relative and Pearson Metric.)*** *We present an interesting interpretation of Theorem 4.1. Let $X \subset R^m$ is smooth manifold and $\gamma(t) : [\alpha,\beta]$ is a smooth path on X. Then $G = (X, l^M)$ is a set of paths and $T_G = (X, l^e)$ is tangential space on start point $\gamma(\alpha)$. On Figure 1 and Figure 2 are made illustrations in two cases of GR metrics.*

5. EXAMPLES

Here we present two practical examples with applications of the relative distance and the Pearson's distance.

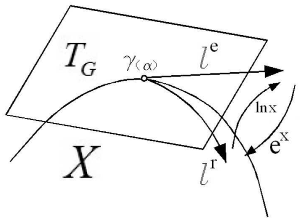

Figure 1: Length in vector space R^m with Euclidean distance and relative distance.

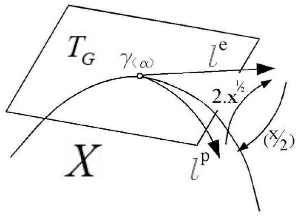

Figure 2: Length in vector space R^m with Euclidean distance and Pearson distance.

5.1. RELATIVE DISTANCE

We are looking a technical task at opencast mining for determination of the height h of resistance vertical slope at bed triming [10]. Height h in meters is given by

$$h = \frac{c}{\gamma} \cdot \frac{cos\varphi}{cos\beta sin(\beta - \varphi)}, \tag{13}$$

where γ - volume weight of rock kN/m^3; β - angle of layer inclination to the horizon; c - cohesion of rocke kN/m^2; φ - angle of internal friction. We use the following samples of data for the parameters of the model: $\gamma = 22kN/m^3$, $\beta = 30^o$, $c = 22kN/m^2$, $\nu = tg\varphi = 0.32$. Parameters are presented with their absolute and relative errors respectively: $c =$
$= \{20, 22, 24\} \pm 1 = \{22\} \pm 4.1\%\ kN/m^2$ and $\nu = \{0.29, 0.32, 0.35\} \pm 0.015 =$
$= \{0.32\} \pm 4.7\%$.

We consider two groups of data presented in Table1 and Table 2. Let's calculate the gradient: $\frac{\partial h}{\partial lnc} = h$ and $\frac{\partial h}{\partial ln\nu} = h \cdot q$, where $q = \frac{\nu}{tg\beta - \nu}$.
Then $\|D_{ln}h\| = h\sqrt{1 + Q^2}$. The permissible error of input data is $\varepsilon =$
$= \sqrt{4.1^2 + 4.7^2}/100 = 6.2\%$.

For function $f = h$, with two stochastic arguments c and ν, we receive $k = cos(\pi/2)$, coefficient values $k(1-p)^2$ and confidence probability $1-\bar{p} = = 1-p^3$ (Table 2). Assuming that for the parameter h admissible error is $0.25m$. Then for the experimental data of group 1 condition of recovery is filled with confidence 98% and for group 2 with confidence 95%.

5.2. PEARSON DISTANCE

In this example we have another technical task at opencast mining - determining the statistical homogeneity of the solid inclusions percentage in the

Table 1 Parameters for two groups of data

N^o	$c\ kN/m^2$	$\nu = tg\varphi$	q	$h[m]$	$\|D_{ln}h\|[m]$	$\|D_{ln} \cdot \delta[m]\|$
1	22	0.27	0.878	6.51	7.85	0.49
2	24	0.32	1.243	6.28	10.02	0.62

Table 2 Two groups of data - continuation

N^o	ε	$k(1-p)^2$	p	$1-\bar{p}$
1	0.25	0.510	0.29	98%
2	0.25	0.403	0.36	95%

clayey overburden. The number of points that is fall in separate area is the frequency of Poisons process.

Table 3 Surfaces of sections and frequency values

N^o	1	2	3	4	5	6	7	Σ
S_{wi}	1.89	0.25	0.79	2.71	6.97	1.45	0.50	14.59
n_{js}	7.6	1	3.2	11	27.9	5.8	2	58.5

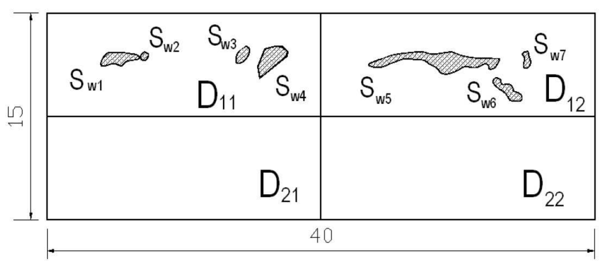

Figure 3: Schematic representation of solid inclusions plane

Following [4] we have the Pearson's criterion and can express it by the Pearson distance

$$\chi^2 = \sum_{j,s} \frac{(n_{js} - n^{'}_{js})^2}{n^{'}_{js}} = [b^p(x_0, x)]^2, \tag{14}$$

where we have vectors $x_0 = \{n^{'}_{js}\}$ and $x = \{n_{js}\}$.

We received that the distribution of solid inclusions is the same in the areas D_{11} and D_{12} with a level of agreement 0.16. The received result is with confidence 84%.

6. CONCLUSIONS

Generalized relative (GR) distance is introduced and studied in paper as component - wise M - relative distance. We use a special notation - Hadamard notation for component wise operation in subspace of positive cone.

It's proved that introduced here generalized relative (GR) metrics is near metrics in significant cases. The length structures in GR metric spaces were investigated and was received formula for the relationship between the paths in such spaces

We made two applications: i) estimation of sensitivity coefficients with relative distance and ii) estimation the variations in Pearsons test procedure by using the Pearson distance.

References

[1] L. Ambrosio, N. Gigli, G. Savarr, *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, Birkhuser Verlag, Basel, 2005.

[2] B.F. Blackwell, K. J. Dowding, *Sensitivity Analysis and Uncertainty Propagation of Computational Models*, Handbook of Numerical Heat Transfer, ch14(2000), 443469, doi:10.1002/9780470172599.ch14.

[3] D. Cacuci, *Sensitivity and Uncertainty Analysis. Volume 1*, Chapman and Hall/CRC, 2003, doi:10.1201/9780203498798.

[4] N. A. C. Cressie, *Statistics for spatial data*, Wiley, New York, 2005, doi:10.21236/ada441157.

[5] M. M. Deza, E. Deza, *Encyclopedia of Distances*, Springer Berlin Heidelberg, 2009, doi:10.1007/978-3-642-00234-2-1.

[6] J. L. Dimitrov, *Mathematical Formalization of Local Sensitivity Analysis in Computational Models*, In Proceeding of the Fifth Conference on Mathematical Foundations of Informatics, Iasi 3-6 July, Daniela Gifu, Bogdan Aman, Adrian Iftene, Diana Trandabat, Editors, Published by University Alexandru Ioan Cuza, Romania, 2019, 215 228.

[7] P. Hasto, *A new weighted metric: the relative metric I*, J. Math. anal. appl. 274 (2002) 3858, doi:10.1016/s0022-247x(02)00219-6.

[8] N. J. Higham, *Accuracy and Stability of Numerical Algorithms, Society for Industrial and Applied Mathematics*, Philadelphia, USA, 2002, doi:10.1137/1.9780898718027.

[9] R.A. Horn, C. R. Johnson, *Matrix Analysis*, Cambridge etc., Cambridge University Press, 2009, doi:10.1017/cbo9781139020411.

[10] S.G. Hristov, *Slope Stability and Drainage Exercise Guide in Open Mines and Quarries*, Sofia, 1999 (in Bulgarian).

[11] R.C. Li, *Relative perturbation theory I: Eigenvalue and Singular Value Variations*, SIAM J. Matrix Anal. Appl. 19(4), (1998), doi:10.21236/ada636847.

[12] A. C. Mennucci, *On Asymmetric Distances. Analysis and Geometry in Metric Spaces*, 1, (2013), 200231, doi:10.2478/agms-2013-0004.

[13] S. T. Rachev, L. B. Klebanov, S. V. Stoyanov, F. Fabozzi, *The Methods of Distances in the Theory of Probability and Statistics*, Springer, New York, 2013, doi:10.1007/978-1-4614-4869-3.

[14] A. Saltelli, S. Tarantola, F. Campolongo, M. Ratto, *Sensitivity Analysis in Practice*, John Wiley and Sons, Ltd, 2002, doi:10.1002/0470870958.

[15] Q. Xia, *The Geodesic Problem in Quasimetric Spaces*, Journal of Geom. Analysis, 19(2), (2009), 452479, doi:10.1007/s12220-008-9065-4.