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SOLUTION OF THE CENTER PROBLEM FOR CUBIC DIFFERENTIAL SYSTEMS WITH ONE OR TWO AFFINE INVARIANT STRAIGHT LINES OF TOTAL ALGEBRAIC MULTIPLICITY FOUR

Alexandru Şubă[1], Silvia Turuta[2]

[1] *Institute of Mathematics and Computer Science, Chişinău, Republic of Moldova*

[2] *Tiraspol State University, Chişinău, Republic of Moldova*

alexandru.suba@math.md, poderioghin_silvia@yahoo.com

Abstract In this article, we study the planar cubic differential systems with a non-degenerate monodromic critical point and one or two affine invariant straight lines of total multiplicity four. We classify these systems and prove that monodromic point is of the center type if and only if the first Lyapunov quantity vanishes.

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1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

We consider the real autonomous polynomial differential systems

$$\frac{dx}{dt} = P\left(x,y\right), \quad \frac{dy}{dt} = Q\left(x,y\right), \quad \gcd(P,Q)=1 \tag{1}$$

and the vector fields $\mathbb{X} = P\left(x,y\right)\frac{\partial}{\partial x} + Q\left(x,y\right)\frac{\partial}{\partial y}$ associated to systems (1).

Denote $n = \max\left\{\deg\left(P\right), \deg\left(Q\right)\right\}$. If $n=2$ $(n=3)$ then system (1) is called quadratic (cubic).

An algebraic curve $f(x,y)=0$, $f\in\mathbb{C}[x,y]$ (a function $f=\exp(g/h)$, $g,h\in\mathbb{C}[x,y]$) is called an invariant algebraic curve (exponential factor) of the system (1) if there exists a polynomial $K_f\in\mathbb{C}[x,y]$, $\deg(K_f)\le n-1$ such that the identity $\mathbb{X}(f)\equiv f(x,y)K_f(x,y)$ holds. In particular, a straight line $l\equiv\alpha x+\beta y+\gamma=0$, $\alpha,\beta,\gamma\in\mathbb{C}$ is invariant for (1) if there exists a polynomial $K_l\in\mathbb{C}[x,y]$ such that the identity $\alpha P(x,y)+\beta Q(x,y)\equiv(\alpha x+\beta y+\gamma)K_l(x,y)$, $(x,y)\in\mathbb{R}^2$, holds. If m is the greatest natural number such that l^m divides $E(\mathbb{X})=P\cdot\mathbb{X}(Q)-Q\cdot\mathbb{X}(P)$, then we say that the invariant straight line l has multiplicity m ([5]).

Denote by $m(l)$ the multiplicity of the invariant straight line l.

Let $f_1, \dots, f_r \quad (f_{r+1} = \exp(g_{r+1}/h_{r+1}), \dots, f_s = \exp(g_s/h_s))$ be invariant algebraic curves (exponential factors) of (1). The system (1) is called *Darboux integrable* if there exists a non-constant function of the form $f = f_1^{\lambda_1} \cdots f_s^{\lambda_s}$, $\lambda_j \in \mathbb{C}$, $j = \overline{1, s}$, such that either f is a first integral or f is an integrating factor for (1).

By present a great number of works have been dedicated to the investigation of polynomial differential systems with invariant straight lines.

In [2] it was proved that the cubic system (1) can have at most eight affine invariant straight lines. The cubic systems with invariant straight lines, including the line at infinity, of total multiplicity nine have been studied in [10]; with invariant straight lines of total geometric (parallel) multiplicity eight (seven) - in [3], [4] ([19]), and with six real invariant straight lines along two (three) directions - in [11] ([12]). The cubic systems with degenerate infinity and invariant straight lines of total parallel multiplicity six and total parallel multiplicity five were investigated in [13], [17], [18]. In [20] it was shown that in the class of cubic differential systems the maximal multiplicity of an affine real straight line (of the line at infinity) is seven. In [21] the cubic systems with two affine real non-parallel invariant straight lines of maximal multiplicity are classified.

In this work we consider the cubic systems of the form

$$\begin{cases} \dot{x} = y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3, \\ \dot{y} = -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3). \end{cases} \tag{2}$$

For (2) the critical point $(0,0)$ is non-degenerate and is either a focus or a center, i.e. is monodromic. The problem of distinguishing between a center and a focus is called *the center problem.*

A singular point $(0,0)$ is a center for (2) if and only if in a neighborhood of $(0,0)$ the system has a nonconstant analytic first integral $F(x,y)$ (an analytic integrating factor of the form $\mu(x,y) = 1 + \sum \mu_j(x,y)$) [1].

It is known there exists a formal power series $F(x,y) = x^2+y^2+\sum_{j\geq 3} F_j(x,y)$ such that the rate of change of $F(x,y)$ along trajectories of (2) is a linear combination of polynomials $\{(x^2+y^2)^j\}_{j=2}^{\infty}$, i.e. $\frac{dF}{dt} = \sum_{j=2}^{\infty} L_{j-1}(x^2+y^2)^j$. The quantities L_j, $j = \overline{1, \infty}$, are polynomials with respect to the coefficients of system (2) called to be *the Lyapunov quantities.* For example, the first Lyapunov quantity looks as

$$L_1 = (bd - ac + 2bf - 2ag + dg - cf + 3k - 3l + p - q)/4.$$

The origin $(0,0)$ is a center for (2) if and only if $L_j = 0$, $j = \overline{1, \infty}$.

The center problem is completely solved for quadratic systems ($k = l = m = n = p = q = r = s = 0$) [8] and for symmetric cubic systems ($a = b = c = d = f = g = 0$) [15]. For other polynomial differential systems necessary and sufficient conditions for a monodromic point to be a center were obtained in some particular cases (see, for example, [6], [14]).

The problem of coexistence in cubic systems of distinct invariant straight lines and critical points of center type was studied in [6], [7], [16]. In [7] (see also [6]) it was proved that if the cubic system (2) has four distinct invariant straight lines of the form $1+\alpha_j x+\beta_j y=0,\ j=1,2,3,4$ (respectively, $y\pm ix=0, 1+\alpha_j x+\beta_j y=0,\ j=1,2$) and the first Lyapunov quantity vanishes (respectively, the first two quantities vanish): $L_1=0$ (respectively, $L_1=L_2=0$), then the origin is a center.

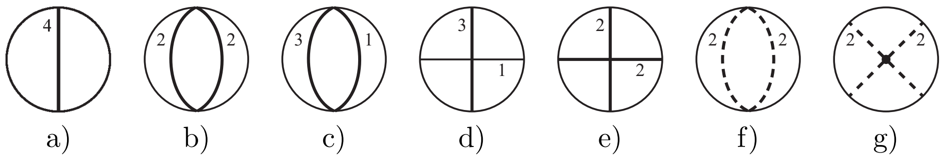

Fig. 1

In this paper we investigate the center problem for (2) with one or two affine invariant straight lines of total multiplicity four (see Fig. 1). Our main result is the following one:

Main Theorem *If a cubic system has a non-degenerate monodromic critical point $M_0(x_0,y_0)$ and one affine real invariant straight line of multiplicity four or two affine invariant straight lines of total multiplicity four, then the critical point M_0 is of the center type if and only if the first Lyapunov quantity vanishes.*

2. CUBIC SYSTEMS WITH A NON-DEGENERATE MONODROMIC CRITICAL POINT AND AN AFFINE REAL INVARIANT STRAIGHT LINE l_1, $m(l_1)\geq 4$;(Fig. 1, a)

Remark 2.1. *Via an affine transformation of coordinates and time rescaling any cubic system with a monodromic non-degenerate critical point can be writing in the form* (2).

Let the system (2) have an affine real invariant straight line l_1. By a transformation of the form

$$x\to\nu\cdot(x\cos\varphi+y\sin\varphi),\ y\to\nu\cdot(y\cos\varphi-x\sin\varphi),\ \nu\neq 0 \tag{3}$$

we can do l_1 to be described by the equation $x = 1$. Then, $k = -a, m = -c-1, p = -f, r = 0$ and (2) is reduced to the system

$$\begin{cases} \dot{x} = (1-x)(y + ax^2 + (c+1)xy + fy^2) \equiv P(x,y), \\ \dot{y} = -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x,y), \\ \quad \max\{deg(P),\, deg(Q)\} = 3,\ \gcd(P,Q) = 1. \end{cases} \tag{4}$$

2.1. Classification of cubic systems (4) with multiple invariant straight line $x - 1 = 0$.

Denote $\sigma(x,y) = E(\mathbb{X})/(x-1)$, $H_2(y) = \sigma(x,y)|_{x=1}$, $H_3(y) = \frac{\partial \sigma}{\partial x}\big|_{x=1}$ and $H_4(y) = \frac{1}{2}\frac{\partial^2 \sigma}{\partial x^2}\big|_{x=1}$.

The polynomial $H_2(y)$ looks as $H_2(y) = f_1 f_2$, where $f_1 = 1+g+s+(d+q)y+(b+n)y^2+ly^3$, $f_2 = a(a-d-q)+(c+2)(1+g+s)+2[a(2-b+c-n)+f(1+g+s)]y+[(2+c)(2-b+c-n)-3al+2af+df+fq]y^2+2(2+c)(f-l)y^3+f(f-l)y^4$. If $f_1 \equiv 0$ with respect to y, then $\deg(\gcd(P,Q)) > 1$. The identity $f_2 \equiv 0$ gives us the following lemma:

Lemma 2.1. *The invariant straight line $x - 1 = 0$ of the system* (4) *has multiplicity at least two if and only if one of the following four series of conditions holds:*

$$a = f = 0,\ c = -2; \tag{5}$$

$$a = f = l = 0,\ n = 2 - b + c,\ s = -g - 1,\ c + 2 \neq 0; \tag{6}$$

$$a = 0,\ l = f,\ q = ((c+2)(b+n-c-2) - df)/f,\ s = -g - 1; \tag{7}$$

$$\begin{aligned} &l = f,\ n = (2a + ac - ab + f + fg + fs)/a, \\ &q = ((c+2)(1+g+s) + a^2 - ad)/a. \end{aligned} \tag{8}$$

Remark 2.2. *The system $\{(4),(6),\ c+1 \neq 0\}$ has the parallel real invariant straight lines $x - 1 = 0$ and $(c+1)x + 1 = 0$.*

Solving in each of the conditions (5)-(8) the identities $H_3(y) \equiv 0$ and $\{H_3(y) \equiv 0,\ H_4(y) \equiv 0\}$ we obtain the following two lemmas, respectively.

Lemma 2.2. *The invariant straight line $x - 1 = 0$ of the system* (4) *has multiplicity at least three if and only if one of the following eight series of conditions holds:*

$$a = f = l = 0,\ c = -2,\ n = -b,\ s = -g - 1; \tag{9}$$

$$\begin{aligned} &a = f = l = 0,\ b = 1,\ d = -q(c+3), \\ &g = -2,\ n = c + 1,\ s = 1,\ q(c+2) \neq 0; \end{aligned} \tag{10}$$

$$\begin{aligned} &a = 0,\ b = 1,\ g = (2d + cd - 2f)/f,\ l = f, \\ &n = c + 1,\ q = -d,\ s = (f - 2d - cd)/f,\ c \neq -2; \end{aligned} \tag{11}$$

$$b = 1,\ c = -2,\ f = 0,\ g = ad - 2,\ l = 0,\ n = -1,\ q = a - d,\ s = 1; \qquad (12)$$

$$\begin{array}{l} b = 1,\ d = (c+3)(g+2)/a,\ f = 0,\ l = 0, \\ n = c+1,\ q = (a^2 - g - 2)/a,\ s = 1,\ c \neq -2; \end{array} \qquad (13)$$

$$\begin{array}{l} b = 1,\ d = a(c+3)/(c+2),\ f = l = 0,\ n = 1 + c, \\ q = a(c+1)/(c+2),\ s = (a^2 - (c+2)(g+1))/(c+2), \\ a^2 - (c+2)(g+2) \neq 0; \end{array} \qquad (14)$$

$$\begin{array}{l} d = (b-1)(c+3)/f,\ g = (ab - a - 2f)/f, \\ l = f,\ n = c+1,\ q = (1 - b + af)/f,\ s = 1; \end{array} \qquad (15)$$

$$\begin{array}{l} a = (b-1)(3 - b + c)/f,\ q = ((b-1)(5 - b + 2c) - df)/f, \\ g = ((3 - b + c)(4 - 5b + b^2 + c - bc + df) - 2f^2)/f^2,\ l = f, \\ n = c+1,\ s = (f^2 + (3 - b + c)(-3 + 3b - c + bc - df))/f^2. \end{array} \qquad (16)$$

Lemma 2.3. *The invariant straight line $x - 1 = 0$ of the system* (4) *has multiplicity at least four if and only if one of the following three series of conditions holds:*

$$a = f = l = 0,\ b = 2,\ c = g = n = -2,\ s = 1; \qquad (17)$$

$$a = d = 0,\ b = 1,\ c = g = -2,\ l = f, n = -1,\ q = 0,\ s = 1; \qquad (18)$$

$$\begin{array}{l} a = (b-1)^2/f,\ c = 2b - 4,\ d = 2(b-1)^2/f,\ l = f,\ q = (b-1)^2/f, \\ g = ((b-2)(b-1)^2 - 2f^2)/f^2,\ n = 2b - 3, s = ((b-1)^2 + f^2)/f^2. \end{array} \qquad (19)$$

Lemma 2.4. *In the class of cubic differential systems* (2) *the maximal multiplicity of an affine real invariant straight line is equal to 4. Via an affine transformation of coordinates and time rescaling each cubic system* (2) *which has an affine real invariant straight line of multiplicity 4 can be written in one of the following three forms:*

$$\dot{x} = (x-1)^2 y,\ \dot{y} = -x + 2x^2 - dxy - 2y^2 - x^3 - qx^2y + 2xy^2,\ d \neq 0; \quad (20)$$

$$\dot{x} = (x-1)y(x - fy - 1),\ \dot{y} = -x + 2x^2 - y^2 - x^3 + xy^2 - fy^3,\ f \neq 0; \quad (21)$$

$$\begin{array}{l} \dot{x} = -(x-1)((b-1)^2x^2 + (2b-3)fxy + fy(1 + fy))/f, \\ \dot{y} = -((b-1)^2x^2(b + 2 + x) + (b-1)^2 fx(2 + x)y + f^3y^3 \\ \qquad + f^2((x-1)^2x + (b - 3x + 2bx)y^2))/f^2,\ f(b-1) \neq 0. \end{array} \qquad (22)$$

Proof. Under conditions (17) (respectively, (18); (19)) the system (4) takes the form (20) (respectively, (21), (22)). For each of these systems the polynomial $E(\mathbb{X})$ looks as: $E(\mathbb{X}) = (x-1)^4(A_0(y) + A_1(y)(x-1))$. In the case (17) we have $A_0(y) = -y(d+q+(d+2q)y^2-2y^3)$; in the case (18): $A_0(y) = -f^2y^4$ and in the case (19): $A_0(y) = (1-b-fy)((b-1)^2(8b-5b^2+b^3-3f^2+bf^2-4)+(b-1)f(12b-11b^2+3b^3-5f^2+3bf^2-4)y+3(b-1)f^2((b-1)^2+f^2)y^2+f^3((b-1)^2+f^2)y^3)/f^4$. In all the cases $A_0(y) \not\equiv 0$. ∎

2.2. Integrability of systems (20), (21), (22).

For system (20) the first Lyapunov quantity is $L_1 = -2q$. If $q = 0$, then at $(0,0)$ the system (20) has the analytic integrating factor

$$\mu(x,y) = \frac{1}{(x-1)^6}\exp\Big[\frac{d(-d+3dx+6y-6xy)}{6(x-1)^6}\Big].$$

The systems (21) and (22) are Darboux integrable and have the first integrals, respectively,

$$F(x,y) = (x-1)^6\exp\Big[\frac{4-6x+2x^3+3y^2-3xy^2+2fy^3}{(1-x)^3}\Big],$$

$$F(x,y) = f_1^{\alpha_1} f_2^{\alpha_2} f_3^{\alpha_3} f_4^{\alpha_4}, \qquad (23)$$

where $f_1 = x-1$,

$$f_2 = \exp\Big[\tfrac{b-1+fy}{f(x-1)}\Big],\ f_3 = \exp\Big[\tfrac{1-2b+b^2+2bfy-2fxy+f^2y^2}{(x-1)^2}\Big],$$
$$f_4 = \exp\Big[\tfrac{1}{3(x-1)^3}\cdot(3x-3b^2x-6f^2x-12bx^2+12b^2x^2+12f^2x^2-x^3$$
$$+6bx^3-3b^2x^3-2b^3x^3-6f^2x^3-3fy+3bfy-3fx^2y+9bfx^2y$$
$$-6b^2fx^2y+6f^2xy^2-6bf^2xy^2-2f^3y^3)\Big],\ \alpha_1 = 2((b-1)^2+f^2),$$
$$\alpha_2 = f(b+1),\ \alpha_3 = \alpha_4 = 1.$$

Therefore, in the cases of cubic systems with a non-degenerate monodromic critical point and a real invariant straight lines l_1, $m(l_1) \geq 4$, the Main Theorem holds.

3. CUBIC SYSTEMS (2) WITH TWO INVARIANT STRAIGHT LINES l_1, l_2, $m(l_1) + m(l_2) \geq 4$

3.1. The cases when l_1 and l_2 are real and parallel (Fig. 1, b), c)).

Assume that $l_1, l_2 \in \mathbb{R}[x,y]$ and let $l_1 \parallel l_2$. Without loss of generality we can consider $l_1 = x-1$. Then $l_2 = x - \alpha$, when $\alpha \in \mathbb{R}$ and $\alpha \notin \{0;1\}$. The straight line l_2 is invariant for system (4) if

$$a = f = 0,\ \alpha = -1/(c+1),\ c \notin \{-2,-1\}. \qquad (24)$$

Taking into account (24) and Lemmas 2.1 and 2.2 we obtain the following two lemmas, respectively.

Lemma 3.1. *Any cubic system with a non-degenerate monodromic critical point and two distinct real and parallel invariant straight lines* l_1, l_2, $m(l_1) \geq$

2, $m(l_2) \geq 2$, *can be written via an affine transformation and time rescaling in the following form:*

$$\begin{array}{l} \dot{x} = -(x-1)(1+x+cx)y, c \notin \{-2;-1\}, \\ \dot{y} = -x - cx^2 + x^3 + cx^3 - dxy - qx^2y + cy^2 - 2xy^2 - 2cxy^2. \end{array} \tag{25}$$

Lemma 3.2. *Via an affine transformation and time rescaling any cubic system with a non-degenerate monodromic critical point and two distinct real and parallel invariant straight lines* l_1, l_2, $m(l_1) \geq 3$, *can be written in one of the following two forms:*

$$\begin{array}{l} \dot{x} = -(x-1)(1+x+cx)y, \ c \notin \{-2;-1\}, \\ \dot{y} = -x - gx^2 + x^3 + gx^3 - y^2 - xy^2 - cxy^2, \ g \neq -2; \end{array} \tag{26}$$

$$\begin{array}{l} \dot{x} = -(x-1)(1+x+cx)y, \ c \notin \{-2;-1\}, \\ \dot{y} = -x + 2x^2 - x^3 + 3qxy + cqxy - qx^2y \\ -y^2 - xy^2 - cxy^2, \ q \neq 0. \end{array} \tag{27}$$

3.2. The cases when l_1 and l_2 are real and nonparallel (Fig. 1, d), e)).

We consider $l_1 \equiv x - 1 = 0$. The straight line $l_2 \equiv y - Ax - B = 0, \ A, B \in \mathbb{R}$, $B \neq 0$ is invariant for system (4) if $\varphi(x) \equiv 0$, where $\varphi(x) = \Big(A \cdot P(x,y) + B \cdot Q(x,y)\Big)\Big|_{y=Ax+B}$.

Lemma 3.3. *Via an affine transformation and time rescaling any cubic system with a non-degenerate monodromic critical point and two distinct real and nonparallel invariant straight lines* l_1, l_2, $m(l_1) \geq 3$, *can be written in one of the following four forms:*

$$\begin{array}{l} \dot{x} = (x-1)^2y, \ d \neq 0, \\ \dot{y} = -x - gx^2 - dxy - y^2 + (g+1)x^3 - d(g+1)x^2y + xy^2; \end{array} \tag{28}$$

$$\begin{array}{l} \dot{x} = \frac{1}{2f(b-1)}(x-1)(2(1-b)fy - (b-1)(b^2 - b \pm \Delta - f^2)x^2 \\ \quad +f(f^2 - 4 + 7b - 3b^2 \mp \Delta)xy - 2f^2(b-1)y^2), \\ \dot{y} = \frac{1}{2f^2(b-1)}\Big(2f^2(1-b)x + (1-b)((b-1)(b^2 - b \pm \Delta) \\ \quad -f^2(b+3))x^2 + f(1-b)(3b(b-1) \pm \Delta - f^2)xy \\ \quad +2bf^2(1-b)y^2 + 2f^2(1-b)x^3 + f(b-1)(f^2 \mp \Delta \\ \quad -(b-1)(b-2))x^2y + f^2(f^2 \mp \Delta - (b-1)(3b-4))xy^2 \\ \quad +2f^3(1-b)y^3\Big), \end{array} \tag{29}$$

where $\Delta = \sqrt{4f^2(1-b) + (f^2 + b(b-1))^2}$;

$$\begin{array}{l} \dot{x} = (1-x)y(1-x+fy), \ df \neq 0, \\ \dot{y} = -x + 2x^2 - dxy - y^2 - x^3 + dx^2y + xy^2 - fy^3; \end{array} \tag{30}$$

$$\begin{array}{l}\dot{x} = (1-x)(fy + (1-b)(b-c-3)x^2 + f(c+1)xy + f^2y^2)/f,\\ \dot{y} = (f^2(c-b+3)x + (c-b+3)(cf^2 + (b-1)((b-2)(b-3)+5c\\ \quad -2bc + c^2 - 2f^2))x^2 - f(2(1-b)(b-c-3)^2 + f^2(2b-c-4))xy\\ \quad -bf^2(b-c-3)y^2 + (b-c-3)(2b-c-3)((b-1)(b-c-3)\\ \quad -f^2)x^3 + f(f^2(2b-c-4) - (b-1)^2(b-c-3))x^2y\\ \quad -f^2(c+1)(b-c-3)xy^2 - f^3(b-c-3)y^3)/(f^2(b-c-3)).\end{array} \tag{31}$$

Proof. In the case of conditions (9) we have the implications:

$$\begin{array}{l}\varphi(x) = -B(A+bB) + (-1 - A^2 + 2AB - 2AbB + bB^2 - Bd)x + (2A^2 -\\ A^2b - AB + 2AbB - Ad - g - Bq)x^2 + (1 - A^2 + A^2b + g - Aq)x^3 \equiv 0 \Rightarrow\\ \{b=1,\ q = d(1+g)\} \Rightarrow (28) \text{ and } l_2 = 1 - x + dy.\end{array}$$

In each of conditions (10), (12)-(14) the polynomial $\varphi(x)$ has the form $\varphi(x) = -B(B+A) + \psi(x),\ \psi(0) = 0$. Taking into account that $B \neq 0$, the identity $\varphi(x) \equiv 0$ gives us $B = -A$. Thus, in conditions (10), (12)-(14) we have, respectively,

$$\begin{array}{l}\psi(x) = -(1 + 3Aq + Acq)x + (2 + 4Aq + Acq)x^2 - (1 + Aq)x^3;\\ \psi(x) = (Ad-1)x + (2 - ad - 2Ad)x^2 + (Ad-1)x^3;\\ \psi(x) = -((a - 6A - 2Ac - 3Ag - Acg)x + (8A + 2Ac + ag + 4Ag + Acg)x^2\\ \qquad +(a - 2A - Ag)x^3)/a;\\ \psi(x) = (-2 + 3aA - c + aAc)x - (4aA + aAc + 2g + cg)x^2\\ \qquad +(2 - a^2 + aA + c + 2g + cg)x^3/(c+2).\end{array}$$

In the cases of conditions (10) and (12) the identity $\psi(x) \equiv 0$ gives $\deg(\gcd(P,Q)) > 0$, and in the cases of conditions (13) and (14) it is easy to show that $\psi(x) \not\equiv 0$.

Under the conditions (11) we have

$$\begin{array}{l}\varphi(x) = -B(A+B)(1+Bf) - ((A+B)(A+B+Bc+2ABf) + 1 + Bd)x\\ \qquad -(d(c+2) + df(A-B) - 2f + A(A+B)f(1+c+Af))x^2/f\\ \qquad +(d(c+2) - f + Adf)x^3/f.\end{array}$$

If $B = -A$, then $\varphi(x) \not\equiv 0$ and if $B = -1/f$, then $\deg(\gcd(P,Q)) > 0$.

Let the conditions (15) hold. Then

$$\begin{aligned}\varphi(x) = & -B(A+bB+fB(A+B)-((b-1)(c+3)B \\ & +B(A+B)f(c+2Af)+f(1+A^2+2AbB+B^2))x/f \\ & -((b-1)(a+3A-B+Ac)+f(A+B)(a+Ac+A^2f) \\ & +f(A^2b+AB-2))x^2/f+((b-1)A-f)x^3/f \equiv 0 \Rightarrow \\ & A = f/(b-1) \Rightarrow \\ \varphi(x) = & B((b-1)(b+Bf)B+f(1+Bf))/(1-b)-((b-1)^3(c+3)B \\ & +f(b-1)^2(1+c)B^2+2f^3(b-1)B^2+f(f^2+(b-1)^2) \\ & +f^2(b-1)(2b+c)B+2f^4B)x/(f(b-1)^2)-(f^5+(b-1)^4(a-B) \\ & +f(b-1)^3(1+aB+c)+f^2(b-1)^2(a+B+Bc) \\ & +f^3(b-1)(b+c+Bf))x^2/(f(b-1)^3) \equiv 0 \Rightarrow \\ & \{a = -(b-b^2+f^2 \pm \Delta)/(2f),\ c = (6-9b+3b^2-f^2 \mp \Delta)/(2(b-1)), \\ & B = (b-b^2-f^2 \pm \Delta)/(2f(b-1))\} \Rightarrow (29).\end{aligned}$$

In conditions (16) the polynomial $\varphi(x)$ has the form $\varphi(x) = -BEq_0 - Eq_1x - Eq_2x^2/f^2 - Eq_3x^3/f^3$, where

$$\begin{aligned} Eq_0 = & A(1+Bf)+B(b+Bf); \\ Eq_1 = & 1+A^2+2AbB+B^2+Bd+B(A+B)(c+2Af); \\ Eq_2 = & (3-b+c)((b-1)(b-c-4)+df)+Af((b-1)(3-b+c)+df) \\ & -2f^2+Bf((b-1)(5-b+2c)-df)+A^2(b+c)f^2+AB(1+c)f^2 \\ & +A^2(A+B)f^3; \\ Eq_3 = & f^2+(3-b+c)((b-1)(c+3)-df)+Af(b-1)(c+2)-Adf^2. \end{aligned}$$

If $B = -1/f$, then $\{Eq_0 = 0,\ Eq_3 = 0\} \Rightarrow \{b = 1, A = -(d(c+2)-f)/(df)\} \Rightarrow \{Eq_1 = 0,\ Eq_2 = 0\} \Rightarrow \{d = f\} \Rightarrow \deg(\gcd(P,Q)) > 0$.

Let $B \neq -1/f$. Then $\{Eq_0 = 0,\ Eq_1 = 0\} \Rightarrow \{A = -B(b+Bf)/(1+Bf),\ d = (-1-2fB+((b-1)(1+b+c)-f^2)B^2+f(b-1)(c+2)B^3)/(B(1+Bf)^2)\} \Rightarrow \{Eq_2 = -Eq_{21}Eq_{22}/(B(1+Bf)^2),\ Eq_3 = Eq_{21}Eq_{31}/(B(1+Bf)^3),$
where
$Eq_{21} = b-c-3+Bf(2b-c-4),\ Eq_{22} = B((b-1)(b-c-4)-f^2)-f,$ $Eq_{31} = (b-1)B(1-b+(b-c-4)(1+Bf))-f(1+Bf)^2$.

Let $\{Eq_{22} = 0,\ Eq_{31} = 0\}$. From the equation $Eq_{22} = 0$ we express B and substituting it in Eq_{31} we obtain $Eq_{31} = f(b-1)^2/((b-1)(4-b+c)+f^2) \neq 0$. Therefore, in condition $f(1+Bf) \neq 0$ the system $\{Eq_{22} = 0,\ Eq_{31} = 0\}$ is not compatible.

Let now $Eq_{21} = 0$. If $c = 2b-4$, then $\{b = 1,\ d = -1/B\}$ or $\{b = 1,\ B = -1/d\} \Rightarrow$ system (30) which has the invariant straight line $1-x+dy = 0$. Suppose that $c-2b+4 \neq 0$. Then $Eq_{21} = 0 \Rightarrow \{B = (c-b+3)/((2b-c-4)f),\ d = f(2b-c-4)/(b-c-3)\} \Rightarrow$ system (31) which has the invariant straight line $(b-c-3)(-1+(2b-c-3)x)-f(2b-c-4)y = 0$. ∎

The following lemma can be proved in the same way:

Lemma 3.4. *Via an affine transformation and time rescaling any cubic system with a non-degenerate monodromic critical point and two distinct real and nonparallel invariant straight lines each of multiplicity at least two can be written in one of the following seven forms:*

$$\begin{aligned}&\dot{x} = y(x-1)^2,\\ &\dot{y} = -(B^2x + AB(2+(A+B)^2)x^2 - 2Bxy + A^2(1+(A+B)^2)x^3\\ &\qquad -ABy^2 - 2A(1+(A+B)^2)x^2y + (1+A^2+2AB)xy^2)/B^2;\end{aligned} \tag{32}$$

$$\begin{aligned}&\dot{x} = y(x-1)(x-1-fy),\\ &\dot{y} = -(x-2x^2+x^3+y^2-xy^2+fy^3) - xy(2Bx+y-2B)/B^2;\end{aligned} \tag{33}$$

$$\begin{aligned}&\dot{x} = (1-x)(y+ax^2+(2aA-1)xy+aA^2y^2),\\ &\dot{y} = -x-((a-A+aA^2-2A^3)x^2 + Ay(a+A+aA^2)(2x+Ay)\\ &\qquad +A^3x^3 + A(a-2A+aA^2)x^2y + A^2(2a-A+2aA^2)xy^2\\ &\qquad +aA^3(1+A^2)y^3)/(A(1+A^2));\end{aligned} \tag{34}$$

$$\begin{aligned}&\dot{x} = (x-1)^2y,\\ &\dot{y} = -(B^2x + 2ABx^2 + B((A+B)(2A+B)-2)xy - B(2A+B)y^2\\ &\qquad +A^2x^3 + A(A(A+B)-2)x^2y - (A^2+(A+B)^2-1)xy^2\\ &\qquad +(A+B)y^3)/B^2, \;\; A+B \neq 0;\end{aligned} \tag{35}$$

$$\begin{aligned}&\dot{x} = (1-x)y(1-x+f^2x+fy), \;\; f \neq 0,\\ &\dot{y} = -(x-2x^2+fxy+y^2+x^3-fx^2y+(f^2-1)xy^2+fy^3);\end{aligned} \tag{36}$$

$$\begin{aligned}&\dot{x} = (x-1)(-B^2y + B(1+Bf)x^2 + (B^2+B^3f+B^4f^2-1)xy\\ &\qquad -B^2fy^2)/B^2,\\ &\dot{y} = -x-(B^3f(3+Bf)x^2 + B(2+Bf)(-1+B^3f+B^4f^2)xy\\ &\qquad -B^3f(2+Bf)y^2 - B^3fx^3 - B^2f(1+B^2+B^3f)x^2y\\ &\qquad +(1-B^4f^2)xy^2 + B^2fy^3)/B^2;\end{aligned} \tag{37}$$

$$\begin{aligned}&\dot{x} = y(x-1)^2,\\ &\dot{y} = -(B^2x + 2ABx^2 + 2B(A^2+AB-1)xy - 2ABy^2 + A^2x^3\\ &\qquad -A(2-A^2+B^2)x^2y + (1-2A^2)xy^2 + Ay^3)/B^2, \; A \neq 0.\end{aligned} \tag{38}$$

3.3. The cases when l_1 and l_2, $l_2 = \overline{l_1}$, are complex and parallel (Fig. 1, f)).

Via a transformation (3) we can consider that invariant straight line l_1 is described by equation $l_1 = x - \alpha - i$ and system (4) has the form

$$\begin{aligned}&\dot{x} = y[(x-\alpha)^2+1]/(\alpha^2+1),\\ &\dot{y} = -(x+gx^2+dxy+by^2+sx^3+qx^2y+nxy^2+ly^3).\end{aligned} \tag{39}$$

The line l_1 has for (39) multiplicity two if

$$\left((1+\alpha^2)^2 E_1(\mathbb{X})/(1+(x-\alpha)^2)\right)\Big|_{x=\alpha+i} = f_1(y)f_2(y) \equiv 0,$$

where

$$\begin{aligned} f_1(y) = & -g+\alpha-3s\alpha+g\alpha^2+s\alpha^3+(-q+d\alpha+q\alpha^2)y+(b+n\alpha)y^2+ly^3 \\ & +i(1-s+2g\alpha+3s\alpha^2+(d+2q\alpha)y+ny^2), \\ f_2(y) = & -(1+\alpha^2)(-g+\alpha-3s\alpha+g\alpha^2+s\alpha^3) \\ & +(b+n\alpha)(1+\alpha^2)y^2+2l(1+\alpha^2)y^3 \\ & +i(-(1+\alpha^2)(1-s+2g\alpha+3s\alpha^2)+(2+n+n\alpha^2)y^2). \end{aligned}$$

Taking into account that the system (39) is real we have

$$\begin{aligned} f_1(y) \equiv 0 \Rightarrow \{Re(f_1(y)) \equiv 0,\, Im(f_1(y)) \equiv 0\} \Rightarrow \{b=d=l=n=q=0, \\ g=-2\alpha s=-2\alpha/(1+\alpha^2)\} \Rightarrow \deg(\gcd(P,Q))>0. \end{aligned}$$

The identity $f_2(y) \equiv 0$ holds if

$$l=0,\ b=-g=(2\alpha)/(1+\alpha^2),\ s=-n/2=1/(1+\alpha^2).$$

In these conditions the system (39) looks as

$$\begin{aligned} \dot{x} &= \tfrac{y(1+(x-\alpha)^2)}{1+\alpha^2}, \\ \dot{y} &= -\tfrac{(1+\alpha^2)x-2\alpha x^2+d(1+\alpha^2)xy+2\alpha y^2+x^3+q(1+\alpha^2)x^2y-2xy^2}{1+\alpha^2}. \end{aligned} \tag{40}$$

The straight line $l_2=\overline{l_1}=x-\alpha+i$ is also invariant for (40) and it has the multiplicity two.

3.4. The cases when l_1 and l_2, $l_2=\overline{l_1}$, are homogeneous complex invariant straight lines.

In these cases the straight lines $l_{1,2}$ are invariant for (2) if they are described by equations $y \mp ix = 0$ and if the conditions

$$f=a+d,\ g=b+c,\ q=l-k+p,\ s=m+n-r \tag{41}$$

hold. In these conditions the polynomial $E(\mathbb{X})$ for system (2) has the form $E(\mathbb{X}) = -l_1l_2(1+\nu(x,y))$, where $\nu(0,0)=0$. It is clear that l_1 (l_2), i.e. $y-ix$ $(y+ix)$, does not divide $1+\nu(x,y)$ and therefore the multiplicity of the invariant straight line l_1 (l_2) is not greater than one.

3.5. The cases when l_1 and l_2, $l_2=\overline{l_1}$, are nonparallel nonhomogeneous complex invariant straight lines (Fig. 1, g)).

Let $l_{1,2}\in\mathbb{C}[x,y]\setminus\mathbb{R}[x,y]$, $l_2=\overline{l_1}$ and $l_1\nparallel l_2$. Via a transformation of the form (3) we can do the line l_1 (l_2) to be described by the equation $y-(\alpha+\beta i)x-1=0$, $\beta\neq 0$ $(y-(\alpha-\beta i)x-1=0,\ \beta\neq 0)$, i.e. we can do the straight lines l_1 and l_2 to pass through the point $(0,1)$. The lines $l_{1,2}$ are invariant for (2) if and only if the following conditions

$$\begin{array}{l} k=g-2\alpha+2a\alpha,\ l=-b,\ m=2-a+d+2c\alpha-3\alpha^2+\beta^2,\\ n=-1-d-2\alpha^2-f\alpha^2-2\beta^2-f\beta^2,\ p=b-c+4\alpha+2f\alpha,\\ q=-g-c\alpha^2+2\alpha^3-c\beta^2+2\alpha\beta^2,\ r=-1-f,\\ s=-(-1+a)(\alpha^2+\beta^2) \end{array}\tag{42}$$

hold, i.e. if (2) has the form

$$\begin{array}{l} \dot{x}=y+ax^2+cxy+fy^2+(2-a+d+2c\alpha-3\alpha^2+\beta^2)x^2y\\ \qquad +(g-2\alpha+2a\alpha)x^3+(b-c+4\alpha+2f\alpha)xy^2-(f+1)y^3,\\ \dot{y}=-x-gx^2-dxy-by^2+(g+(c-2\alpha)(\alpha^2+\beta^2))x^2y\\ \qquad +(a-1)(\alpha^2+\beta^2)x^3+(1+d+(f+2)(\alpha^2+\beta^2))xy^2\\ \qquad +by^3,\ \beta\neq 0. \end{array}\tag{43}$$

For system (43) we have $\left(E(\mathbb{X})/(l_1l_2)\right)\Big|_{y=(\alpha+i\beta)x+1}=g_1(x)g_2(x)$, where

$$\begin{array}{l} g_1(x)=b+(f+2)(\alpha-i\beta)+(2+d+2b\alpha+c\alpha+2\alpha^2+2f\alpha^2+4\beta^2\\ \qquad +2f\beta^2+i\beta(2b-c+2\alpha))x+(g+a\alpha+d\alpha+b\alpha^2-b\beta^2\\ \qquad +(c+f\alpha)(\alpha^2+\beta^2)+i\beta(2-a+d+2b\alpha+(2+f)(\alpha^2+\beta^2)))x^2,\\ g_2(x)=(f+2)(b+\alpha(f+2)+i\beta(f+2))\\ \qquad +2(b+(f+2)(\alpha+i\beta))(c+\alpha+2f\alpha+i\beta(3+2f))x\\ \qquad -((f+2)(g-(\alpha+i\beta)(1+2a+d+(\alpha+i\beta)(6b+6c-17\alpha-7i\beta))\\ \qquad -6(\alpha+i\beta)^3(f+2))-b(3(a-1)+5(\alpha+i\beta)(c-3\alpha-i\beta))\\ \qquad -(c-3\alpha-i\beta)(2+d+c\alpha-2\alpha^2+2\beta^2+i\beta(c-4\alpha)))x^2\\ \qquad +x^3(a-1+c\alpha+\alpha^2(f-1)-\beta^2(f+1)+i\beta(c+2f\alpha))\\ \qquad \times(2(2+d+c\alpha+2b\alpha+2(f+1)(\alpha^2-\beta^2)+i\beta(2b+c+4\alpha+4f\alpha))\\ \qquad +x(g+\alpha(a+d)+(b+c+f\alpha)(\alpha^2-\beta^2)-2f\alpha\beta^2\\ \qquad +i\beta(a+d+2b\alpha+2c\alpha+3f\alpha^2-f\beta^2))). \end{array}$$

If $g_1(x)\equiv 0$, then $\deg(\gcd(P,Q))>0$, i.e. the system (43) is degenerate. The identity $g_2(x)\equiv 0$ yields two series of equalities: $\{a-2=b=f+2=g=0,\ d=-(16+c^2+16\beta^2)/8,\ \alpha=c/4\}$ and $\{a=(16-c^2-16\beta^2)/16,\ b=0,\ d=-(16+c^2+16\beta^2)/8,\ f=-2,\ \alpha=c/4\}$. From these and (42) we obtain the following lemma:

Lemma 3.5. *Via an affine transformation and time rescaling any cubic system with a non-degenerate monodromic critical point and two distinct complex and nonparallel invariant straight lines* $l_1, l_2, \ l_2 = \overline{l_1}, \ m(l_{1,2}) \geq 2$, *can be written in one of the following two forms:*

$$\begin{array}{l} \dot{x} = (16y + 32x^2 + 16cxy - 32y^2 + 8cx^3 + (-32 + 3c^2 - 16\beta^2)x^2y \\ \quad -16cxy^2 + 16y^3)/16, \\ \dot{y} = -x(32 - 4(16 + c^2 + 16\beta^2)y - 2(c^2 + 16\beta^2)x^2 \\ \quad -c(c^2 + 16\beta^2)xy + 4(8 + c^2 + 16\beta^2)y^2)/32, \ \beta \neq 0; \end{array} \qquad (44)$$

$$\begin{array}{l} \dot{x} = (32y + 2(16 - c^2 - 16\beta^2)x^2 + 32cxy - 64y^2 - c(c^2 + 16\beta^2)x^3 \\ \quad +8(c^2 - 4)x^2y - 32cxy^2 + 32y^3)/32, \\ \dot{y} = -x(256 + 256gx - 32(16 + c^2 + 16\beta^2)y + (c^2 + 16\beta^2)^2x^2 \\ \quad -8c(c^2 + 16\beta^2)xy + 32(8 + c^2 + 16\beta^2)y^2)/256, \ \beta \neq 0. \end{array} \qquad (45)$$

3.6. Integrability of systems (25)-(38), (40), (44), (45).

System (25) has the integrating factor

$$\mu(x,y) = (x-1)^{\frac{cd^2}{(c+2)^3}-3}(1+x+cx)^{-3-\frac{cd^2}{(c+2)^3}} \exp\Big[\frac{d(2d+cdx+(c+2)^2y)}{(2+c)^2(x-1)(1+x+cx)}\Big]$$

and *System* (32) (respectively, (33), (34)) has the integrating factor of the form $\mu(x,y) = 1/(l_1^2 l_2^2)$, where $l_1 = x - 1$ and $l_2 = Ax - y + B$ (respectively, $l_2 = Bx + y - B$, $l_2 = A^2x - Ay - A^2 - 1$).

Systems (26), (28), (30) have the first integrals, respectively:

$$\begin{array}{l} F(x,y) = \frac{(x-1)^{2(1+c)(4+c+3g+cg)}}{(1+x+cx)^{2(c-g)}} \exp\Big[\frac{(1+c)(2+c)(-2(2+g)(x-1)+(2+c)y^2)}{(x-1)^2}\Big]; \\ F = \frac{(x-1)^{2(1-d^2-d^2g)}}{(x-dy-1)^{-2}} \exp\Big[\frac{(2d^2(3+2g)-2dy)(x-1)+d^2(2+g)}{(x-1)^2}\Big]; \\ F = (x-1)^{d+d^3-f}(1-x+dy)^{f-d} \exp\frac{d(2(1-x)(d^2+(d-f)y)+dfy^2)}{2(x-1)^2}. \end{array}$$

For *system* (27) we obtain $L_1 \equiv (c+2)q/4 = 0 \Rightarrow q = 0$. The system $\{(27), q = 0\}$ has the first integral

$$F(x,y) = (x-1)^{2(1+c)}(1+x+cx)^2 \exp\Big[\frac{-(c+1)(2+c)y^2}{(x-1)^2}\Big].$$

Systems (29) *and* (31) have the first integrals of the form (23). In the case (29):

$$\begin{array}{l} f_1 = x - 1, \ f_2 = b - b^2 - f^2 \mp \Delta + 2f^2x + 2fy - 2bfy, \\ f_3 = \exp((b - 1 + fy)/(x - 1)), \ f_4 = \exp((b - 1 + fy)^2/(x - 1)^2), \end{array}$$

$$\begin{array}{l} \alpha_1 = 2(b-1)(1 - 2b + b^2 + f^2)(-b + b^2 + f^2 \mp \Delta), \ \alpha_2 = -4(-1+b)^2f^2, \\ \alpha_3 = 2(b-1)((b-2)(b-1)(b^2 - b \mp \Delta + 2f^2) + f^4 \mp f^2\Delta), \\ \alpha_4 = (b-1)^2(b(b-1) \mp \Delta) + f^2((b-1)(2b-3) + f^2 \mp \Delta) \end{array}$$

and in the case (31):

$f_1 = x-1,\ f_2 = (b-c-3)(-1+(2b-c-3)x) - f(2b-c-4)y,$
$f_3 = \exp((1-b-fy)/(x-1)),\ f_4 = \exp[((1-b)(3b-2c-5)(b-c-3)$
$+2f^2(2b-c-4) + 2((b-1)(b-c-3)(2b-c-3) - f^2(2b-c-4))x$
$+2f(b-1)(b-c-3)y + f^2(b-c-3)y^2)/(x-1)^2],$
$\alpha_1 = -2((b-1)(b-c-3)(-61+95b-48b^2+8b^3-47c+48bc-12b^2c$
$-12c^2+6bc^2-c^3) - f^2(2b-c-4)^3),$
$\alpha_2 = 2(1-b)(b-c-3)^2,\ \alpha_3 = 2(1-b)(b-c-3)(2b-c-5)(2b-c-4),$
$\alpha_4 = (2b-c-4)^2.$

For *Systems* (35) and (36) we have respectively $L_1 = -2(A+B)(1+(A+B)^2)/B^2 \neq 0,\ L_1 = -2f^3 \neq 0$, and therefore, the critical point $(0,0)$ is of the focus type.

Systems (37)*,* (38) *and* (44). If for each of these systems the first Lyapunov quantity L_1 vanishes, then we have respectively: $1+B^3f+B^4f^2=0$, $1+A^2-B^2=0$, $c=0$ and the following integrating factors:

$$\mu(x,y) = (x-1)^{-2}(B^2fx-y+B)^{-2},$$
$$\mu(x,y) = \frac{1}{(x-1)^3(y-Ax-B)^3}\exp\Big[(A+B)^2((x-1)^{-1}+B(y-Ax-B)^{-1})\Big],$$
$$\mu(x,y) = (16-32y+16y^2+16x^2\beta^2)^{-3}\exp\Big[\frac{2(1-y)\beta^2}{1-2y+y^2+x^2\beta^2}\Big].$$

For *system* (40) the equality $L_1=0$ yields $q=0$ and the system $\{(40),\ q=0\}$ has the integrating factor of the Darboux form

$$\mu(x,y) = f_1^{\alpha_1}f_2^{\alpha_2}f_3^{\alpha_3}f_4^{\alpha_4}, \tag{46}$$

where $f_{1,2} = x-\alpha\mp i$, $f_{3,4} = \exp\Big[\frac{d(1+\alpha^2)^2-d\alpha(1+\alpha^2)x-2y}{x-\alpha\mp i}\Big]$, $\alpha_{1,2} = -3\pm id^2\alpha(1+\alpha^2)^2/4$, $\alpha_{3,4} = \mp id(1+\alpha^2)/4$ or

$$\mu(x,y) = \frac{1}{(1+(x-\alpha)^2)^3}\exp\Big[-\frac{d\ (1+\alpha^2)(-d(1+\alpha^2)^2+d\alpha(1+\alpha^2)x+2y)}{2(1+(x-\alpha)^2)} + \frac{1}{4}d^2\alpha\ (1+\alpha^2)^2\arctan\frac{2(x-\alpha)}{1+(x-\ \alpha)^2}\Big].$$

System (45) has Darboux first integral

$$F(x,y) = f_1^{\alpha_1}f_2^{\alpha_2}\exp\Big[\frac{8\beta h}{f_1f_2}\Big]$$

or

$$F(x,y) = (16+8cx+c^2x^2-32y-8cxy+16y^2+16x^2\beta^2)^{128\beta^3} \times\exp\Big[2(16c+c^3-64g+48c\beta^2)\arctan\big[\frac{4x\beta}{4y-cx-4}\big]+\frac{8\beta h}{f_1f_2}\Big],$$

where

$$f_1 = 4y - (c+4i\beta)x - 4,\ f_2 = \overline{f_1},\ \alpha_{1,2} = -(4+cx-4y \pm 4\beta ix),$$
$$h = 512\beta^2 + 4(16c + c^3 + 80c\beta^2 - 64g)x - 512\beta^2 y +$$
$$((c^2+16\beta^2)(16+c^2+16\beta^2) - 64cg)x^2 - 4(16c + c^3 - 64g + 16c\beta^2)xy.$$

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