ROMAI J., v.15, no.2(2019), 117–137

PARTITIONED CONSERVATIVE, VARIABLE STEP, SECOND-ORDER METHOD FOR MAGNETO-HYDRODYNAMICS IN ELSÄSSER VARIABLES

Cătălin Trenchea
Department of Mathematics, University of Pittsburgh, PA, USA
trenchea@pitt.edu

Abstract The magnetohydrodynamics flows are governed by the Navier-Stokes equations coupled with the Maxwell equations. We propose a partitioned, variable step, second-order in time, method for the evolutionary full MHD equations, at *high* magnetic Reynolds number. The method is based on the refactorization of the midpoint rule. We prove the convergence of the subiterates, the energy equality at the discrete time levels, and the conservation of energy, cross-helicity and magnetic helicity.

Keywords: magnetohydrodynamics, Elsässer variables, partitioned, symplectic, second-order methods, variable steps, time adaptivity.
2010 MSC: 76W05, 76D05, 65M12, 65P10, 35Q30, 35Q61.

1. INTRODUCTION

The equations of magnetohydrodynamics (MHD) describe the motion of electrically conducting, incompressible flows in the presence of a magnetic field. If an electrically conducting fluid moves in a magnetic field, the magnetic field exerts forces which may substantially modify the flow. Conversely, the flow itself gives rise to a second, induced field and thus modifies the magnetic field. Initiated by Alfvén in 1942 [2], MHD is widely exploited in numerous branches of science including astrophysics and geophysics [34, 47, 25, 21, 20, 6, 11, 24], as well as engineering. Understanding MHD flows is central to many important applications, e.g., liquid metal cooling of nuclear reactors [5, 32, 49], process metallurgy [17], sea water propulsion [42].

The MHD flows involve different physical processes: the motion of fluid is governed by hydrodynamics equations and the magnetic field is governed by Maxwell equations. One approach to coupled problem is by monolithic methods. In these methods, the globally coupled problem is assembled at each time step and then solved iteratively. Partitioned methods, which solve the coupled problem by successively solving the sub-physics problems, are another attractive and promising approach for solving MHD system.

Most terrestrial applications, in particular most industrial and laboratory flows, involve *small* magnetic Reynolds number. In this cases, while the magnetic field considerably alters the fluid motion, the induced field is usually found to be negligible by comparison with the imposed field [17]. Neglecting the induced magnetic field one can reduce the MHD systems to the significantly simpler Reduced MHD (RMHD), for which several implicit-explicit (IMEX) schemes were studied in [39, 38].

The equations of magnetohydrodynamics describing the motion of an incompressible fluid flow in presence of a magnetic field are the following (see, e.g. [37, 9, 8])

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{B}\cdot\nabla)\boldsymbol{B} - \nu\Delta\boldsymbol{u} + \nabla p = 0, \qquad \nabla\cdot\boldsymbol{u} = 0, \qquad (1)$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} - (\boldsymbol{B}\cdot\nabla)\boldsymbol{u} - \nu_m\Delta\boldsymbol{B} = 0, \qquad \nabla\cdot\boldsymbol{B} = 0, \qquad (2)$$

in $\Omega\times(0,T)$, where Ω is the fluid domain, $\boldsymbol{u} = (u_1(\boldsymbol{x},t), u_2(\boldsymbol{x},t), u_3(\boldsymbol{x},t))$ is the fluid velocity, $p(\boldsymbol{x},t)$ is the pressure, $\boldsymbol{B} = (B_1(\boldsymbol{x},t), B_2(\boldsymbol{x},t), B_3(\boldsymbol{x},t))$ is the magnetic field, ν is the kinematic viscosity and ν_m is the magnetic resistivity. The total magnetic field can be split in two parts $\boldsymbol{B} = \boldsymbol{B}_\circ + \boldsymbol{b}$ (mean and fluctuations). We prescribe homogeneous Dirichlet boundary conditions for $\boldsymbol{u}$, and $\boldsymbol{B} = \boldsymbol{B}_\circ$ on the boundary (see [28] for typical magnetic boundary conditions). Then the Elsässer fields [23]

$$\boldsymbol{z}^+ = \boldsymbol{u} + \boldsymbol{b}, \quad \boldsymbol{z}^- = \boldsymbol{u} - \boldsymbol{b}, \qquad (3)$$

merging the physical properties of the Navier-Stokes and Maxwell equations, suggest stable time-splitting schemes for the full MHD equations. The momentum equations, in the Elsässer variables, are

$$\frac{\partial \boldsymbol{z}^\pm}{\partial t} \mp (\boldsymbol{B}_\circ\cdot\nabla)\boldsymbol{z}^\pm + (\boldsymbol{z}^\mp\cdot\nabla)\boldsymbol{z}^\pm - \frac{\nu+\nu_m}{2}\Delta\boldsymbol{z}^\pm - \frac{\nu-\nu_m}{2}\Delta\boldsymbol{z}^\mp + \nabla p = 0, \quad (4)$$

while the continuity equations are $\nabla\cdot\boldsymbol{z}^\pm = 0$. We note that the nonlinear interactions occur between the Alfvénic fluctuations $\boldsymbol{z}^\pm$. The mean magnetic field plays an important role in MHD turbulence, for example it can make the turbulence anisotropic; suppress the turbulence by decreasing energy cascade, etc. In the presence of a strong mean magnetic field, $\boldsymbol{z}^+$ and $\boldsymbol{z}^-$ wavepackets travel in opposite directions with the phase velocity of $\boldsymbol{B}_\circ$, and interact weakly. For Kolmogorov's and Iroshnikov/Kraichnan's phenomenological theories of MHD isotropic and anisotropic turbulence, see [35, 36, 19, 43, 53, 48, 27, 29, 55].

One approach to coupled MHD problem is monolithic methods, or implicit (fully coupled) algorithms, that are robust and stable, but quite demanding in computational time and resources. In [52] we proposed and analyzed an original partitioned first-order method, in which the coupling terms are lagged, thus the system uncouples into two subproblem solves. More precisely, the momentum equations (4) and continuity equations in the Elsässer variables are approximated by the following first-order IMEX scheme (backward-Euler forward-Euler)

$$\frac{\boldsymbol{z}^{\pm}_{n+1} - \boldsymbol{z}^{\pm}_{n}}{\tau_n} \mp (\boldsymbol{B}_{\circ} \cdot \nabla)\boldsymbol{z}^{\pm}_{n+1} + (\boldsymbol{z}^{\mp}_{n} \cdot \nabla)\boldsymbol{z}^{\pm}_{n+1} - \frac{\nu + \nu_m}{2}\Delta \boldsymbol{z}^{\pm}_{n+1} - \frac{\nu - \nu_m}{2}\Delta \boldsymbol{z}^{\mp}_{n} + \nabla p^{\pm}_{n+1} = 0, \tag{5}$$

$$\nabla \cdot \boldsymbol{z}^{\pm}_{n+1} = 0, \tag{6}$$

where $\tau_n > 0$ denotes the timestep, and time instances are $t_{n+1} = t_n + \tau_n$. The scheme (5)-(6) is modular, in the sense that it decouples the computations for the variables $\boldsymbol{z}^+$ and $\boldsymbol{z}^-$, and is unconditionally stable, in the sense that possess an energy equality independent of the time step τ_n. This allows decoupled calculations of the fluctuations $\boldsymbol{z}^+_{n+1}$ and $\boldsymbol{z}^-_{n+1}$ using two similar copies of a Navier-Stokes solver ([28, section 3.6.4]), cutting in half the storage cost versus the monolithic methods. The existence and uniqueness of the solution $\boldsymbol{z}^+_{n+1}, \boldsymbol{z}^-_{n+1}$ of (5)-(6) is a consequence of a classical projection theorem [51, Theorem 2.2]. The pressure is recovered via classical DeRham theorem (see [40]). Let us briefly mention that, in the original velocity and magnetic field variables $\boldsymbol{u}$ and $\boldsymbol{B}$, the method (5)-(6) writes:

$$\frac{\boldsymbol{u}_{n+1} - \boldsymbol{u}_n}{\tau_n} + (\boldsymbol{u}_n \cdot \nabla)\boldsymbol{u}_{n+1} - (\boldsymbol{B}_n \cdot \nabla)\boldsymbol{B}_{n+1} - \tfrac{\nu+\nu_m}{2}\Delta \boldsymbol{u}_{n+1} - \tfrac{\nu-\nu_m}{2}\Delta \boldsymbol{u}_n + \nabla \tfrac{p^+_{n+1} + p^-_{n+1}}{2} = 0, \tag{7}$$

$$\frac{\boldsymbol{B}_{n+1} - \boldsymbol{B}_n}{\tau_n} + (\boldsymbol{u}_n \cdot \nabla)\boldsymbol{B}_{n+1} - (\boldsymbol{B}_n \cdot \nabla)\boldsymbol{u}_{n+1} - \tfrac{\nu+\nu_m}{2}\Delta \boldsymbol{B}_{n+1} + \tfrac{\nu-\nu_m}{2}\Delta \boldsymbol{B}_n + \nabla \tfrac{p^+_{n+1} - p^-_{n+1}}{2} = 0, \tag{8}$$

and $\nabla \cdot \boldsymbol{u}_n = 0, \nabla \cdot \boldsymbol{B}_n = 0$. In the continuous case it is well-known (see e.g. [46]) that the Lagrange multiplier corresponding to the momentum equation (2) in $\boldsymbol{B}$ is a harmonic function and therefore zero, equivalently, the momentum and the continuity equations in $\boldsymbol{B}$ are linearly dependent. As a consequence, both momentum equations (4) have the same Lagrange multiplier, the fluid pressure term p. In the semi-discrete method (5)-(6) there are two distinct Lagrange multipliers p^+_{n+1}, p^-_{n+1}. The homologous form (7)-(8) indicates that the fluid pressure $p(t_{n+1})$ is approximated by $\frac{1}{2}(p^+_{n+1} + p^-_{n+1})$, and the momentum equation in $\boldsymbol{B}$ has as Lagrange multiplier $\frac{1}{2}(p^+_{n+1} - p^-_{n+1})$, which is not a harmonic function:

$$\Delta(p^+_{n+1} - p^-_{n+1}) = -2\big((\boldsymbol{u}_n\cdot\nabla)\boldsymbol{B}_{n+1} - (\boldsymbol{B}_n\cdot\nabla)\boldsymbol{u}_{n+1}\big).$$

Employing the spectral deferred correction method [22, 45, 44, 12], the initial algorithm in the Elsässer variables [52] has been improved to second order accuracy in [56]. In [41] and [33], the authors extended the first-order IMEX algorithm [52] to second-order accuracy using a BDF2 approximation of the time derivative. For coupled unconditionally stable schemes, in the velocity and magnetic field variables, see e.g., [3, 18, 7, 15] and also [28, chapter 3]. A component splitting scheme for the Maxwell equation has been proposed in [54].

Remark 1.1. *Using the defect-correction method [**?**], a second order scheme can be constructed from* (5)-(6)*, see e.g. [56].*

In the remainder we denote by $|\cdot|$ the usual $L^2(\Omega)$ norm.

Theorem 1.1 (Unconditional stability of Algorithm (5)-(6))**.** *Let* $\boldsymbol{z}^+_n, \boldsymbol{z}^-_n, p_n$ *satisfy* (5)-(6) *for each* $n \in \{1, 2, \dots, \frac{T}{\Delta t}\}$. *Then the following energy estimate holds*

$$\begin{aligned}
&\frac{|\boldsymbol{z}^+_N|^2 + |\boldsymbol{z}^-_N|^2}{2\Delta t} + \frac{1}{2\Delta t}\sum_{n=1}^{N}\big(|\boldsymbol{z}^+_n - \boldsymbol{z}^+_{n-1}|^2 + |\boldsymbol{z}^-_n - \boldsymbol{z}^-_{n-1}|^2\big) \qquad (9)\\
&\quad + \frac{(\nu-\nu_m)^2}{4(\nu+\nu_m)}\big(|\nabla\boldsymbol{z}^+_N|^2 + |\nabla\boldsymbol{z}^-_N|^2\big) + \frac{\nu\nu_m}{\nu+\nu_m}\sum_{n=1}^{N}\big(|\nabla\boldsymbol{z}^-_n|^2 + |\nabla\boldsymbol{z}^+_n|^2\big)\\
&\quad + \tfrac{|\nu-\nu_m|}{4}\sum_{n=1}^{N}\Big(\sqrt{\tfrac{\nu+\nu_m}{|\nu-\nu_m|}}|\nabla\boldsymbol{z}^+_n| + \sqrt{\tfrac{|\nu-\nu_m|}{\nu+\nu_m}}|\nabla\boldsymbol{z}^-_{n-1}|\Big)^2\\
&\quad + \tfrac{|\nu-\nu_m|}{4}\sum_{n=1}^{N}\Big(\sqrt{\tfrac{\nu+\nu_m}{|\nu-\nu_m|}}|\nabla\boldsymbol{z}^-_n| + \sqrt{\tfrac{|\nu-\nu_m|}{\nu+\nu_m}}|\nabla\boldsymbol{z}^+_{n-1}|\Big)^2\\
&\le \frac{|\boldsymbol{z}^+_0|^2 + |\boldsymbol{z}^-_0|^2}{2\Delta t} + \frac{(\nu-\nu_m)^2}{4(\nu+\nu_m)}\big(|\nabla\boldsymbol{z}^+_0|^2 + |\nabla\boldsymbol{z}^-_0|^2\big).
\end{aligned}$$

Proof. First we multiply the momentum equations (5) with $\boldsymbol{z}^+_{n+1}, \boldsymbol{z}^-_{n+1}$, respectively, use the continuity equations and the polarized identity to obtain

$$\begin{aligned}
&\frac{|\boldsymbol{z}^+_{n+1}|^2 - |\boldsymbol{z}^+_n|^2 + |\boldsymbol{z}^+_{n+1} - \boldsymbol{z}^+_n|^2}{2\Delta t} + \frac{\nu+\nu_m}{2}|\nabla\boldsymbol{z}^+_{n+1}|^2 + \frac{\nu-\nu_m}{2}\langle\nabla\boldsymbol{z}^-_n, \nabla\boldsymbol{z}^+_{n+1}\rangle = 0,\\
&\frac{|\boldsymbol{z}^-_{n+1}|^2 - |\boldsymbol{z}^-_n|^2 + |\boldsymbol{z}^-_{n+1} - \boldsymbol{z}^-_n|^2}{2\Delta t} + \frac{\nu+\nu_m}{2}|\nabla\boldsymbol{z}^-_{n+1}|^2 + \frac{\nu-\nu_m}{2}\langle\nabla\boldsymbol{z}^+_n, \nabla\boldsymbol{z}^-_{n+1}\rangle = 0.
\end{aligned}$$

Then add up the two relations to get

$$\frac{|\boldsymbol{z}^+_{n+1}|^2+|\boldsymbol{z}^-_{n+1}|^2-|\boldsymbol{z}^+_n|^2-|\boldsymbol{z}^-_n|^2}{2\Delta t}+\frac{|\boldsymbol{z}^+_{n+1}-\boldsymbol{z}^+_n|^2+|\boldsymbol{z}^-_{n+1}-\boldsymbol{z}^-_n|^2}{2\Delta t} \tag{10}$$
$$+\frac{\nu+\nu_m}{2}\left(|\nabla\boldsymbol{z}^+_{n+1}|^2+|\nabla\boldsymbol{z}^-_{n+1}|^2\right)+\frac{\nu-\nu_m}{2}\left(\langle\nabla\boldsymbol{z}^-_n,\nabla\boldsymbol{z}^+_{n+1}\rangle+\langle\nabla\boldsymbol{z}^+_n,\nabla\boldsymbol{z}^-_{n+1}\rangle\right)=0.$$

Secondly, the dissipation terms can be estimated, using the Cauchy-Schwarz inequality and the polarized identity, as follows

$$\frac{\nu+\nu_m}{2}|\nabla\boldsymbol{z}^+_{n+1}|^2+\frac{\nu+\nu_m}{2}|\nabla\boldsymbol{z}^-_{n+1}|^2+\frac{\nu-\nu_m}{2}\langle\nabla\boldsymbol{z}^-_n,\nabla\boldsymbol{z}^+_{n+1}\rangle+\frac{\nu-\nu_m}{2}\langle\nabla\boldsymbol{z}^+_n,\nabla\boldsymbol{z}^-_{n+1}\rangle$$
$$\geq\frac{(\nu-\nu_m)^2}{4(\nu+\nu_m)}\left(|\nabla\boldsymbol{z}^+_{n+1}|^2+|\nabla\boldsymbol{z}^-_{n+1}|^2-|\nabla\boldsymbol{z}^-_n|^2-|\nabla\boldsymbol{z}^+_n|^2\right)+\frac{\nu\nu_m}{\nu+\nu_m}\left(|\nabla\boldsymbol{z}^-_{n+1}|^2+|\nabla\boldsymbol{z}^+_{n+1}|^2\right)$$
$$+\tfrac{|\nu-\nu_m|}{4}\left(\sqrt{\tfrac{|\nu-\nu_m|}{\nu+\nu_m}}|\nabla\boldsymbol{z}^-_n|+\sqrt{\tfrac{\nu+\nu_m}{|\nu-\nu_m|}}|\nabla\boldsymbol{z}^+_{n+1}|\right)^2+\tfrac{|\nu-\nu_m|}{4}\left(\sqrt{\tfrac{|\nu-\nu_m|}{\nu+\nu_m}}|\nabla\boldsymbol{z}^+_n|+\sqrt{\tfrac{\nu+\nu_m}{|\nu-\nu_m|}}|\nabla\boldsymbol{z}^-_{n+1}|\right)^2.$$

The substitution of the above estimate in (10) implies

$$\frac{|\boldsymbol{z}^+_{n+1}|^2+|\boldsymbol{z}^-_{n+1}|^2-|\boldsymbol{z}^+_n|^2-|\boldsymbol{z}^-_n|^2}{2\Delta t}+\frac{|\boldsymbol{z}^+_{n+1}-\boldsymbol{z}^+_n|^2+|\boldsymbol{z}^-_{n+1}-\boldsymbol{z}^-_n|^2}{2\Delta t}$$
$$+\frac{(\nu-\nu_m)^2}{4(\nu+\nu_m)}\left(|\nabla\boldsymbol{z}^+_{n+1}|^2+|\nabla\boldsymbol{z}^-_{n+1}|^2-|\nabla\boldsymbol{z}^-_n|^2-|\nabla\boldsymbol{z}^+_n|^2\right)+\frac{\nu\nu_m}{\nu+\nu_m}\left(|\nabla\boldsymbol{z}^-_{n+1}|^2+|\nabla\boldsymbol{z}^+_{n+1}|^2\right)$$
$$+\tfrac{|\nu-\nu_m|}{4}\left(\sqrt{\tfrac{|\nu-\nu_m|}{\nu+\nu_m}}|\nabla\boldsymbol{z}^-_n|+\sqrt{\tfrac{\nu+\nu_m}{|\nu-\nu_m|}}|\nabla\boldsymbol{z}^+_{n+1}|\right)^2$$
$$+\tfrac{|\nu-\nu_m|}{4}\left(\sqrt{\tfrac{|\nu-\nu_m|}{\nu+\nu_m}}|\nabla\boldsymbol{z}^+_n|+\sqrt{\tfrac{\nu+\nu_m}{|\nu-\nu_m|}}|\nabla\boldsymbol{z}^-_{n+1}|\right)^2\leq 0.$$

Finally, summation from $n=0$ to $N-1$ gives the energy estimate (9), which yields the unconditional stability of scheme (5)-(6). ∎

In this report we propose is a variable step, second-order, symplectic, partitioned method for the evolutionary full MHD equations, at *high* magnetic Reynolds number, in the Elsässer variables. The method generalizes the first-order method proposed in [52], based on the refactorization of the midpoint rule [13]. Also, the method conserves, at discrete level, all three invariants of the MHD equations (the energy $\mathcal{E}(t)=\frac{1}{2}\int_\Omega(\boldsymbol{u}^2+\boldsymbol{B}^2)\,d\boldsymbol{x}$, cross-helicity $\mathcal{H}_C(t)=\frac{1}{2}\int_\Omega\boldsymbol{u}\cdot\boldsymbol{B}\,d\boldsymbol{x}$, and magnetic helicity $\mathcal{H}_M(t)=\frac{1}{2}\int_\Omega\mathbb{A}\cdot\boldsymbol{B}\,d\boldsymbol{x}$, where $\mathbb{A}_n$ is the vector potential, $\boldsymbol{B}=\nabla\times\mathbb{A}$).

2. VARIABLE STEP, SECOND-ORDER CONSERVATIVE PARTITIONED SCHEME

Let us denote by $\tau_n>0$ the timestep, and time instances $t_{n+1}=t_n+\tau_n$, $t_{n+1/2}=t_n+\tau_n/2$. Then for all $n\geq 1$, define

$$\widehat{z}^{\pm}_{n+1/2}=z^{\pm}_{n-1/2}+\frac{\tau_n+\tau_{n-1}}{\tau_{n-1}}\left(z^{\pm}_n-z^{\pm}_{n-1/2}\right), \tag{11}$$

also the initial iterates, and the limit values

$$z^{\pm}_{(0)} = \widehat{z}^{\pm}_{n+1/2}, \qquad z^{\pm}_{n+1/2} = \lim_{\kappa\to\infty} z^{\pm}_{(\kappa)}, \tag{12}$$

and compute iteratively, for $\kappa \geq 1$

$$\begin{aligned}\frac{\boldsymbol{z}^{+}_{(\kappa)} - \boldsymbol{z}^{+}_{n}}{\tau_n/2} &- (\boldsymbol{B}_{\circ}\cdot\nabla)\boldsymbol{z}^{+}_{(\kappa)} + (\widehat{\boldsymbol{z}}^{-}_{n+1/2}\cdot\nabla)\boldsymbol{z}^{+}_{(\kappa)} - \tfrac{\nu+\nu_m}{2}\Delta\boldsymbol{z}^{+}_{(\kappa)} - \\ &- \tfrac{\nu-\nu_m}{2}\Delta\boldsymbol{z}^{-}_{(\kappa-1)} + \nabla p^{+}_{(\kappa)} = 0.\end{aligned}$$

$$\begin{aligned}\frac{\boldsymbol{z}^{-}_{(\kappa)} - \boldsymbol{z}^{-}_{n}}{\tau_n/2} &+ (\boldsymbol{B}_{\circ}\cdot\nabla)\boldsymbol{z}^{-}_{(\kappa)} + (\widehat{\boldsymbol{z}}^{+}_{n+1/2}\cdot\nabla)\boldsymbol{z}^{-}_{(\kappa)} - \tfrac{\nu+\nu_m}{2}\Delta\boldsymbol{z}^{-}_{(\kappa)} - \\ &- \tfrac{\nu-\nu_m}{2}\Delta\boldsymbol{z}^{+}_{(\kappa-1)} + \nabla p^{-}_{(\kappa)} = 0.\end{aligned}$$

$$\begin{aligned}&\frac{\boldsymbol{z}^{\pm}_{(\kappa)} - \boldsymbol{z}^{\pm}_{n}}{\tau_n/2} \mp (\boldsymbol{B}_{\circ}\cdot\nabla)\boldsymbol{z}^{\pm}_{(\kappa)} + (\widehat{\boldsymbol{z}}^{\mp}_{n+1/2}\cdot\nabla)\boldsymbol{z}^{\pm}_{(\kappa)} \\ &\qquad - \frac{\nu+\nu_m}{2}\Delta\boldsymbol{z}^{\pm}_{(\kappa)} - \frac{\nu-\nu_m}{2}\Delta\boldsymbol{z}^{\mp}_{(\kappa-1)} + \nabla p^{\pm}_{(\kappa)} = 0,\end{aligned} \tag{13}$$

$$\nabla\cdot\boldsymbol{z}^{\pm}_{(\kappa)} = 0. \tag{14}$$

Assuming the limit (12) exists, in the limit (13)-(14) implies the backward-Euler IMEX scheme

$$\begin{aligned}&\frac{\boldsymbol{z}^{\pm}_{n+1/2} - \boldsymbol{z}^{\pm}_{n}}{\tau_n/2} \mp (\boldsymbol{B}_{\circ}\cdot\nabla)\boldsymbol{z}^{\pm}_{n+1/2} + (\widehat{\boldsymbol{z}}^{\mp}_{n+1/2}\cdot\nabla)\boldsymbol{z}^{\pm}_{n+1/2} \\ &\qquad - \frac{\nu+\nu_m}{2}\Delta\boldsymbol{z}^{\pm}_{n+1/2} - \frac{\nu-\nu_m}{2}\Delta\boldsymbol{z}^{\mp}_{n+1/2} + \nabla p^{\pm}_{n+1/2} = 0,\end{aligned} \tag{15}$$

$$\nabla\cdot\boldsymbol{z}^{\pm}_{n+1/2} = 0. \tag{16}$$

$$\begin{aligned}&\frac{\boldsymbol{z}^{+}_{n+1/2} - \boldsymbol{z}^{+}_{n}}{\tau_n/2} - (\boldsymbol{B}_{\circ}\cdot\nabla)\boldsymbol{z}^{+}_{n+1/2} + (\widehat{\boldsymbol{z}}^{-}_{n+1/2}\cdot\nabla)\boldsymbol{z}^{+}_{n+1/2} \\ &\qquad -\frac{\nu+\nu_m}{2}\Delta\boldsymbol{z}^{+}_{n+1/2} - \frac{\nu-\nu_m}{2}\Delta\boldsymbol{z}^{-}_{n+1/2} + \nabla p^{+}_{n+1/2} = 0. \\ &\frac{\boldsymbol{z}^{-}_{n+1/2} - \boldsymbol{z}^{-}_{n}}{\tau_n/2} + (\boldsymbol{B}_{\circ}\cdot\nabla)\boldsymbol{z}^{-}_{n+1/2} + (\widehat{\boldsymbol{z}}^{+}_{n+1/2}\cdot\nabla)\boldsymbol{z}^{-}_{n+1/2} \\ &\qquad -\frac{\nu+\nu_m}{2}\Delta\boldsymbol{z}^{-}_{n+1/2} - \frac{\nu-\nu_m}{2}\Delta\boldsymbol{z}^{+}_{n+1/2} + \nabla p^{-}_{n+1/2} = 0.\end{aligned} \tag{17}$$

To evaluate $z^{\pm}_{n+1}$, use the linear extrapolation

$$\boldsymbol{z}^{\pm}_{n+1} = 2\boldsymbol{z}^{\pm}_{n+1/2} - \boldsymbol{z}^{\pm}_{n}. \tag{18}$$

This is equivalent to the forward-Euler IMEX counterpart of (15)

$$\frac{\boldsymbol{z}^{\pm}_{n+1} - \boldsymbol{z}^{\pm}_{n+1/2}}{\tau_n/2} \mp (\boldsymbol{B}_\circ \cdot \nabla)\boldsymbol{z}^{\pm}_{n+1/2} + (\widehat{\boldsymbol{z}}^{\mp}_{n+1/2} \cdot \nabla)\boldsymbol{z}^{\pm}_{n+1/2} - \frac{\nu + \nu_m}{2}\Delta \boldsymbol{z}^{\pm}_{n+1/2} - \frac{\nu - \nu_m}{2}\Delta \boldsymbol{z}^{\mp}_{n+1/2} + \nabla p^{\pm}_{n+1/2} = 0. \tag{19}$$

$$\begin{aligned}
&\frac{\boldsymbol{z}^{+}_{n+1} - \boldsymbol{z}^{+}_{n+1/2}}{\tau_n/2} - (\boldsymbol{B}_\circ \cdot \nabla)\boldsymbol{z}^{+}_{n+1/2} + (\widehat{\boldsymbol{z}}^{-}_{n+1/2} \cdot \nabla)\boldsymbol{z}^{+}_{n+1/2} \\
&\quad -\frac{\nu + \nu_m}{2}\Delta \boldsymbol{z}^{+}_{n+1/2} - \frac{\nu - \nu_m}{2}\Delta \boldsymbol{z}^{-}_{n+1/2} + \nabla p^{+}_{n+1/2} = 0. \\
&\frac{\boldsymbol{z}^{-}_{n+1} - \boldsymbol{z}^{-}_{n+1/2}}{\tau_n/2} + (\boldsymbol{B}_\circ \cdot \nabla)\boldsymbol{z}^{-}_{n+1/2} + (\widehat{\boldsymbol{z}}^{+}_{n+1/2} \cdot \nabla)\boldsymbol{z}^{-}_{n+1/2} \\
&\quad -\frac{\nu + \nu_m}{2}\Delta \boldsymbol{z}^{-}_{n+1/2} - \frac{\nu - \nu_m}{2}\Delta \boldsymbol{z}^{+}_{n+1/2} + \nabla p^{-}_{n+1/2} = 0.
\end{aligned} \tag{20}$$

2.1. CONVERGENCE OF THE SUBITERATES

The first result of this section proves the existence of the limit in (12), namely the convergence of the iterates to the midpoint values satisfying (15)-(16).

Theorem 2.1. *At every $n \geq 1$, the sequence of iterates $\{\boldsymbol{z}^{\pm}_{(\kappa)}\}_{\kappa \geq 0}$ defined by (11)-(14) converges to $\boldsymbol{z}^{\pm}_{n+1/2}$ in $H^1(\Omega)$:*

$$\boldsymbol{z}^{\pm}_{(\kappa)} \xrightarrow{\kappa \nearrow \infty} \boldsymbol{z}^{\pm}_{n+1/2}. \tag{21}$$

Proof. First we subtract (13)-(14) at iteration κ from the same equations, at the next iteration $\kappa + 1$ to obtain

$$\begin{aligned}
&\frac{\boldsymbol{z}^{+}_{(\kappa+1)} - \boldsymbol{z}^{+}_{(\kappa)}}{\tau_n/2} - (\boldsymbol{B}_\circ \cdot \nabla)\big(\boldsymbol{z}^{+}_{(\kappa+1)} - \boldsymbol{z}^{+}_{(\kappa)}\big) + (\widehat{\boldsymbol{z}}^{-}_{n+1/2} \cdot \nabla)\big(\boldsymbol{z}^{+}_{(\kappa+1)} - \boldsymbol{z}^{+}_{(\kappa)}\big) \\
&\quad -\frac{\nu + \nu_m}{2}\Delta\big(\boldsymbol{z}^{+}_{(\kappa+1)} - \boldsymbol{z}^{+}_{(\kappa)}\big) - \frac{\nu - \nu_m}{2}\Delta\big(\boldsymbol{z}^{-}_{(\kappa)} - \boldsymbol{z}^{-}_{(\kappa-1)}\big) \\
&\quad +\nabla\big(p^{+}_{(\kappa+1)} - p^{+}_{(\kappa)}\big) = 0, \\
&\frac{\boldsymbol{z}^{-}_{(\kappa+1)} - \boldsymbol{z}^{-}_{(\kappa)}}{\tau_n/2} + (\boldsymbol{B}_\circ \cdot \nabla)\big(\boldsymbol{z}^{-}_{(\kappa+1)} - \boldsymbol{z}^{-}_{(\kappa)}\big) + (\widehat{\boldsymbol{z}}^{+}_{n+1/2} \cdot \nabla)\big(\boldsymbol{z}^{-}_{(\kappa+1)} - \boldsymbol{z}^{-}_{(\kappa)}\big) \\
&\quad -\frac{\nu + \nu_m}{2}\Delta\big(\boldsymbol{z}^{-}_{(\kappa+1)} - \boldsymbol{z}^{-}_{(\kappa)}\big) - \frac{\nu - \nu_m}{2}\Delta\big(\boldsymbol{z}^{+}_{(\kappa)} - \boldsymbol{z}^{+}_{(\kappa-1)}\big) \\
&\quad +\nabla\big(p^{-}_{(\kappa+1)} - p^{-}_{(\kappa)}\big) = 0.
\end{aligned} \tag{22}$$

$$\begin{aligned}
&\frac{\boldsymbol{z}^{\pm}_{(\kappa+1)}-\boldsymbol{z}^{\pm}_{(\kappa)}}{\tau_n/2} \mp (\boldsymbol{B}_\circ\cdot\nabla)\big(\boldsymbol{z}^{\pm}_{(\kappa+1)}-\boldsymbol{z}^{\pm}_{(\kappa)}\big) + (\widehat{\boldsymbol{z}}^{\mp}_{n+1/2}\cdot\nabla)\big(\boldsymbol{z}^{\pm}_{(\kappa+1)}-\boldsymbol{z}^{\pm}_{(\kappa)}\big)\\
&\quad -\frac{\nu+\nu_m}{2}\Delta\big(\boldsymbol{z}^{\pm}_{(\kappa+1)}-\boldsymbol{z}^{\pm}_{(\kappa)}\big) - \frac{\nu-\nu_m}{2}\Delta\big(\boldsymbol{z}^{\mp}_{(\kappa)}-\boldsymbol{z}^{\mp}_{(\kappa-1)}\big)\\
&\quad +\nabla\big(p^{\pm}_{(\kappa+1)}-p^{\pm}_{(\kappa)}\big)=0,
\end{aligned} \tag{23}$$

$$\nabla\cdot\big(\boldsymbol{z}^{\pm}_{(\kappa+1)}-\boldsymbol{z}^{\pm}_{(\kappa)}\big)=0. \tag{24}$$

Then we multiply (23) by $\boldsymbol{z}^{\pm}_{(\kappa+1)}-\boldsymbol{z}^{\pm}_{(\kappa)}$, respectively, use the divergence-free condition (24), and add to obtain

$$\begin{aligned}
0 = {}& \frac{1}{\tau_n}\Big(\|\boldsymbol{z}^{+}_{(\kappa+1)}-\boldsymbol{z}^{+}_{(\kappa)}\|^2 + \|\boldsymbol{z}^{-}_{(\kappa+1)}-\boldsymbol{z}^{-}_{(\kappa)}\|^2\Big)\\
&+\frac{\nu+\nu_m}{2}\Big(\big\|\nabla\big(\boldsymbol{z}^{+}_{(\kappa+1)}-\boldsymbol{z}^{+}_{(\kappa)}\big)\big\|^2 + \big\|\nabla\big(\boldsymbol{z}^{-}_{(\kappa+1)}-\boldsymbol{z}^{-}_{(\kappa)}\big)\big\|^2\Big)\\
&+\frac{\nu-\nu_m}{2}\Big(\big\langle\nabla\big(\boldsymbol{z}^{-}_{(\kappa)}-\boldsymbol{z}^{-}_{(\kappa-1)}\big),\nabla\big(\boldsymbol{z}^{+}_{(\kappa+1)}-\boldsymbol{z}^{+}_{(\kappa)}\big)\big\rangle\\
&\quad+\big\langle\nabla\big(\boldsymbol{z}^{+}_{(\kappa)}-\boldsymbol{z}^{+}_{(\kappa-1)}\big),\nabla\big(\boldsymbol{z}^{-}_{(\kappa+1)}-\boldsymbol{z}^{-}_{(\kappa)}\big)\big\rangle\Big),
\end{aligned}$$

where here and in the remainder $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the usual $L^2(\Omega)$ inner product and norm. Let us notice that the last term in the parenthesis above writes

$$\begin{aligned}
&\frac{\nu-\nu_m}{2}\langle\nabla\big(\boldsymbol{z}^{\mp}_{(\kappa)}-\boldsymbol{z}^{\mp}_{(\kappa-1)}\big),\nabla\big(\boldsymbol{z}^{\pm}_{(\kappa+1)}-\boldsymbol{z}^{\pm}_{(\kappa)}\big)\rangle\\
&=-\frac{|\sqrt{\nu}+\sqrt{\nu_m}|^2}{4}\big\|\nabla\big(\boldsymbol{z}^{\pm}_{(\kappa+1)}-\boldsymbol{z}^{\pm}_{(\kappa)}\big)\big\|^2 - \frac{|\nu-\nu_m|^2}{4|\sqrt{\nu}+\sqrt{\nu_m}|^2}\big\|\nabla\big(\boldsymbol{z}^{\mp}_{(\kappa)}-\boldsymbol{z}^{\mp}_{(\kappa-1)}\big)\big\|^2\\
&+\frac{|\nu-\nu_m|}{4}\left\|\frac{\sqrt{\nu}+\sqrt{\nu_m}}{\sqrt{|\nu-\nu_m|}}\nabla\big(\boldsymbol{z}^{\pm}_{(\kappa+1)}-\boldsymbol{z}^{\pm}_{(\kappa)}\big)+\operatorname{sign}(\nu-\nu_m)\frac{\sqrt{|\nu-\nu_m|}}{\sqrt{\nu}+\sqrt{\nu_m}}\nabla\big(\boldsymbol{z}^{\mp}_{(\kappa)}-\boldsymbol{z}^{\mp}_{(\kappa-1)}\big)\right\|^2
\end{aligned}$$

and therefore

$$\begin{aligned}
0 = {}& \frac{1}{\tau_n}\Big(\|\boldsymbol{z}^{+}_{(\kappa+1)}-\boldsymbol{z}^{+}_{(\kappa)}\|^2 + \|\boldsymbol{z}^{-}_{(\kappa+1)}-\boldsymbol{z}^{-}_{(\kappa)}\|^2\Big)\\
&+\Big(\frac{\nu+\nu_m}{2}-\frac{|\sqrt{\nu}+\sqrt{\nu_m}|^2}{4}\Big)\Big(\big\|\nabla\big(\boldsymbol{z}^{+}_{(\kappa+1)}-\boldsymbol{z}^{+}_{(\kappa)}\big)\big\|^2 + \big\|\nabla\big(\boldsymbol{z}^{-}_{(\kappa+1)}-\boldsymbol{z}^{-}_{(\kappa)}\big)\big\|^2\Big)\\
&-\frac{|\nu-\nu_m|^2}{4|\sqrt{\nu}+\sqrt{\nu_m}|^2}\Big(\big\|\nabla\big(\boldsymbol{z}^{+}_{(\kappa)}-\boldsymbol{z}^{+}_{(\kappa-1)}\big)\big\|^2 + \big\|\nabla\big(\boldsymbol{z}^{-}_{(\kappa)}-\boldsymbol{z}^{-}_{(\kappa-1)}\big)\big\|^2\Big)\\
&+\frac{|\nu-\nu_m|}{4}\Big[\Big\|\frac{\sqrt{\nu}+\sqrt{\nu_m}}{\sqrt{|\nu-\nu_m|}}\nabla\big(\boldsymbol{z}^{+}_{(\kappa+1)}-\boldsymbol{z}^{+}_{(\kappa)}\big)+\operatorname{sign}(\nu-\nu_m)\frac{\sqrt{|\nu-\nu_m|}}{\sqrt{\nu}+\sqrt{\nu_m}}\nabla\big(\boldsymbol{z}^{-}_{(\kappa)}-\boldsymbol{z}^{-}_{(\kappa-1)}\big)\Big\|^2\\
&+\Big\|\frac{\sqrt{\nu}+\sqrt{\nu_m}}{\sqrt{|\nu-\nu_m|}}\nabla\big(\boldsymbol{z}^{-}_{(\kappa+1)}-\boldsymbol{z}^{-}_{(\kappa)}\big)+\operatorname{sign}(\nu-\nu_m)\frac{\sqrt{|\nu-\nu_m|}}{\sqrt{\nu}+\sqrt{\nu_m}}\nabla\big(\boldsymbol{z}^{+}_{(\kappa)}-\boldsymbol{z}^{+}_{(\kappa-1)}\big)\Big\|^2\Big].
\end{aligned}$$

Since

$$\frac{\nu+\nu_m}{2}-\frac{|\sqrt{\nu}+\sqrt{\nu_m}|^2}{4}=\frac{|\sqrt{\nu}-\sqrt{\nu_m}|^2}{4},\qquad \frac{|\nu-\nu_m|^2}{|\sqrt{\nu}+\sqrt{\nu_m}|^2}=|\sqrt{\nu}-\sqrt{\nu_m}|^2,$$

multiplying by τ_n, we have

$$\begin{aligned}0=&\|\boldsymbol{z}^+_{(\kappa+1)}-\boldsymbol{z}^+_{(\kappa)}\|^2+\|\boldsymbol{z}^-_{(\kappa+1)}-\boldsymbol{z}^-_{(\kappa)}\|^2\\&+\frac{\tau_n}{4}|\sqrt{\nu}-\sqrt{\nu_m}|^2\Big(\big\|\nabla\big(\boldsymbol{z}^+_{(\kappa+1)}-\boldsymbol{z}^+_{(\kappa)}\big)\big\|^2+\big\|\nabla\big(\boldsymbol{z}^-_{(\kappa+1)}-\boldsymbol{z}^-_{(\kappa)}\big)\big\|^2\Big)\\&-\frac{\tau_n}{4}|\sqrt{\nu}-\sqrt{\nu_m}|^2\Big(\big\|\nabla\big(\boldsymbol{z}^+_{(\kappa)}-\boldsymbol{z}^+_{(\kappa-1)}\big)\big\|^2+\big\|\nabla\big(\boldsymbol{z}^-_{(\kappa)}-\boldsymbol{z}^-_{(\kappa-1)}\big)\big\|^2\Big)\\&+\tau_n\mathcal{A}_\kappa,\end{aligned}$$

where $\mathcal{A}_\kappa$ denotes the non-negative expression

$$\begin{aligned}\mathcal{A}_\kappa=&\frac{1}{4}|\nu-\nu_m|\Big[\Big\|\frac{\sqrt{\nu}+\sqrt{\nu_m}}{\sqrt{|\nu-\nu_m|}}\nabla\big(\boldsymbol{z}^+_{(\kappa+1)}-\boldsymbol{z}^+_{(\kappa)}\big)+\operatorname{sign}(\nu-\nu_m)\frac{\sqrt{|\nu-\nu_m|}}{\sqrt{\nu}+\sqrt{\nu_m}}\nabla\big(\boldsymbol{z}^-_{(\kappa)}-\boldsymbol{z}^-_{(\kappa-1)}\big)\Big\|^2\\&+\Big\|\frac{\sqrt{\nu}+\sqrt{\nu_m}}{\sqrt{|\nu-\nu_m|}}\nabla\big(\boldsymbol{z}^-_{(\kappa+1)}-\boldsymbol{z}^-_{(\kappa)}\big)+\operatorname{sign}(\nu-\nu_m)\frac{\sqrt{|\nu-\nu_m|}}{\sqrt{\nu}+\sqrt{\nu_m}}\nabla\big(\boldsymbol{z}^+_{(\kappa)}-\boldsymbol{z}^+_{(\kappa-1)}\big)\Big\|^2\Big].\end{aligned}$$

Summation over $\kappa\geq 1$ yields

$$\begin{aligned}&\frac{\tau_n}{4}|\sqrt{\nu}-\sqrt{\nu_m}|^2\Big(\big\|\nabla\big(\boldsymbol{z}^+_{(\ell)}-\boldsymbol{z}^+_{(\ell-1)}\big)\big\|^2+\big\|\nabla\big(\boldsymbol{z}^-_{(\ell)}-\boldsymbol{z}^-_{(\ell-1)}\big)\big\|^2\Big)\\&\quad+\sum_{\kappa=1}^{\ell-1}\Big(\|\boldsymbol{z}^+_{(\kappa+1)}-\boldsymbol{z}^+_{(\kappa)}\|^2+\|\boldsymbol{z}^-_{(\kappa+1)}-\boldsymbol{z}^-_{(\kappa)}\|^2\Big)+\tau_n\sum_{\kappa=1}^{\ell-1}\mathcal{A}_\kappa\\&=\frac{\tau_n}{4}|\sqrt{\nu}-\sqrt{\nu_m}|^2\Big(\big\|\nabla\big(\boldsymbol{z}^+_{(1)}-\boldsymbol{z}^+_{(0)}\big)\big\|^2+\big\|\nabla\big(\boldsymbol{z}^-_{(1)}-\boldsymbol{z}^-_{(0)}\big)\big\|^2\Big),\end{aligned}$$

hence, the sequence $\{\boldsymbol{z}^\pm_{(\kappa)}\}_{\kappa\geq 1}$ are Cauchy sequences in $H^1(\Omega)$, which by completeness concludes the proof. ∎

We note that, since the convergence of the iterates above and the energy equality in section 2.2 hold independent of the time step, time-adaptivity can be implemented with non-intrusive minimal algorithmic changes, mitigating the fact that the midpoint rule is not a Poisson map [30]. This entails that the number of the κ iterates necessary for convergence can be minimized (see e.g., [13]). Most linear multistep methods (LMMs) [31, 16], when considered with variable time steps, do not preserve the zero-stability or unconditional A-stability properties of the constant step versions. For example, the variable step version of the trapezoidal method (Crank-Nicolson) is unstable [16],[50, pp. 181-182]; similarly, BDF2 loses zero-stability and A-stability when used with a variable stepsize. The trapezoidal method, even in the constant step case, is A-stable but not B-stable [1]. Also, “it is not known which of the LMMs preserve quadratic invariants” [10].

2.2. ENERGY EQUALITY

The second result of this section establishes the unconditional stability of the midpoint algorithm (15)-(18). We denote by $\mathcal{E}^N$ the discrete kinetic energy of the system, by $\mathcal{D}^N$ the viscous dissipation rate, and by $\mathcal{N}^N$ the numerical dissipation rate:

$$\mathcal{E}^N = \frac{1}{2}\Big(\|\boldsymbol{z}_N^+\|^2 + \|\boldsymbol{z}_N^-\|^2\Big),$$

$$\mathcal{D}^N = \min\{\nu, \nu_m\} \sum_{n=1}^{N-1} \tau_n \Big(\|\nabla \boldsymbol{z}_{n+1/2}^+\|^2 + \|\nabla \boldsymbol{z}_{n+1/2}^-\|^2\Big),$$

$$\mathcal{N}^N = \frac{|\nu - \nu_m|}{2} \sum_{n=1}^{N-1} \tau_n \big\|\nabla \boldsymbol{z}_{n+1/2}^+ + \operatorname{sign}(\nu - \nu_m)\nabla \boldsymbol{z}_{n+1/2}^-\big\|^2,$$

Then the stability of the scheme is given in the following theorem.

Theorem 2.2. *The Algorithm* (15)-(18) *is unconditionally stable and satisfies the energy equality*

$$\mathcal{E}^N + \mathcal{D}^N + \mathcal{N}^N = \mathcal{E}^1.$$

Proof. Multiply (15) by $\boldsymbol{z}_{n+1/2}^\pm$, respectively, use the continuity equation (16), and add to obtain

$$\begin{aligned} 0 = {} & \frac{1}{\tau_n}\Big(\|\boldsymbol{z}_{n+1/2}^+\|^2 + \|\boldsymbol{z}_{n+1/2}^-\|^2 - \|\boldsymbol{z}_n^+\|^2 - \|\boldsymbol{z}_n^-\|^2\Big) \\ & + \frac{1}{\tau_n}\Big(\|\boldsymbol{z}_{n+1/2}^+ - \boldsymbol{z}_n^+\|^2 + \|\boldsymbol{z}_{n+1/2}^- - \boldsymbol{z}_n^-\|^2\Big) \\ & + \frac{\nu + \nu_m}{2}\Big(\|\nabla \boldsymbol{z}_{n+1/2}^+\|^2 + \|\nabla \boldsymbol{z}_{n+1/2}^-\|^2\Big) + (\nu - \nu_m)\langle \nabla \boldsymbol{z}_{n+1/2}^+, \nabla \boldsymbol{z}_{n+1/2}^-\rangle, \end{aligned} \tag{25}$$

which by the polarized identity

$$\begin{aligned} (\nu - \nu_m)\langle \nabla \boldsymbol{z}_{n+1/2}^+, \nabla \boldsymbol{z}_{n+1/2}^-\rangle = {} & -\frac{|\nu - \nu_m|}{2}\big(\|\nabla \boldsymbol{z}_{n+1/2}^+\|^2 + \|\nabla \boldsymbol{z}_{n+1/2}^-\|^2\big) \\ & + \frac{|\nu - \nu_m|}{2}\|\nabla \boldsymbol{z}_{n+1/2}^+ + \operatorname{sign}(\nu - \nu_m)\nabla \boldsymbol{z}_{n+1/2}^-\|^2 \end{aligned}$$

gives

$$
\begin{aligned}
0 = {}& \frac{1}{\tau_n}\Big(\|\boldsymbol{z}^+_{n+1/2}\|^2 + \|\boldsymbol{z}^-_{n+1/2}\|^2 - \|\boldsymbol{z}^+_n\|^2 - \|\boldsymbol{z}^-_n\|^2\Big) \\
& + \frac{1}{\tau_n}\Big(\|\boldsymbol{z}^+_{n+1/2} - \boldsymbol{z}^+_n\|^2 + \|\boldsymbol{z}^-_{n+1/2} - \boldsymbol{z}^-_n\|^2\Big) \\
& + \frac{\nu+\nu_m}{2}\Big(\|\nabla \boldsymbol{z}^+_{n+1/2}\|^2 + \|\nabla \boldsymbol{z}^-_{n+1/2}\|^2\Big) - \frac{|\nu-\nu_m|}{2}\big(\|\nabla \boldsymbol{z}^+_{n+1/2}\|^2 \\
& + \|\nabla \boldsymbol{z}^-_{n+1/2}\|^2\big) \\
& + \frac{|\nu-\nu_m|}{2}\|\nabla \boldsymbol{z}^+_{n+1/2} + \operatorname{sign}(\nu-\nu_m)\nabla \boldsymbol{z}^-_{n+1/2}\|^2 \\
= {}& \frac{1}{\tau_n}\Big(\|\boldsymbol{z}^+_{n+1/2}\|^2 + \|\boldsymbol{z}^-_{n+1/2}\|^2 - \|\boldsymbol{z}^+_n\|^2 - \|\boldsymbol{z}^-_n\|^2\Big) \\
& + \frac{1}{\tau_n}\Big(\|\boldsymbol{z}^+_{n+1/2} - \boldsymbol{z}^+_n\|^2 + \|\boldsymbol{z}^-_{n+1/2} - \boldsymbol{z}^-_n\|^2\Big) \\
& + \frac{\nu+\nu_m-|\nu-\nu_m|}{2}\Big(\|\nabla \boldsymbol{z}^+_{n+1/2}\|^2 + \|\nabla \boldsymbol{z}^-_{n+1/2}\|^2\Big) \\
& \quad + \frac{|\nu-\nu_m|}{2}\|\nabla \boldsymbol{z}^+_{n+1/2} + \operatorname{sign}(\nu-\nu_m)\nabla \boldsymbol{z}^-_{n+1/2}\|^2,
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
0 = {}& \frac{1}{\tau_n}\Big(\|\boldsymbol{z}^+_{n+1/2}\|^2 + \|\boldsymbol{z}^-_{n+1/2}\|^2 - \|\boldsymbol{z}^+_n\|^2 - \|\boldsymbol{z}^-_n\|^2\Big) \\
& + \frac{1}{\tau_n}\Big(\|\boldsymbol{z}^+_{n+1/2} - \boldsymbol{z}^+_n\|^2 + \|\boldsymbol{z}^-_{n+1/2} - \boldsymbol{z}^-_n\|^2\Big) \\
& + \frac{\nu+\nu_m-|\nu-\nu_m|}{2}\Big(\|\nabla \boldsymbol{z}^+_{n+1/2}\|^2 + \|\nabla \boldsymbol{z}^-_{n+1/2}\|^2\Big) \\
& + \frac{|\nu-\nu_m|}{2}\|\nabla \boldsymbol{z}^+_{n+1/2} + \operatorname{sign}(\nu-\nu_m)\nabla \boldsymbol{z}^-_{n+1/2}\|^2.
\end{aligned}
$$

Similarly, from (19), multiplying by $\boldsymbol{z}^\pm_{n+1/2}$ we obtain

$$
\begin{aligned}
0 = {}& \frac{1}{\tau_n}\Big(\|\boldsymbol{z}^+_{n+1}\|^2 + \|\boldsymbol{z}^-_{n+1}\|^2 - \|\boldsymbol{z}^+_{n+1/2}\|^2 - \|\boldsymbol{z}^-_{n+1/2}\|^2\Big) \\
& - \frac{1}{\tau_n}\Big(\|\boldsymbol{z}^+_{n+1} - \boldsymbol{z}^+_{n+1/2}\|^2 + \|\boldsymbol{z}^-_{n+1} - \boldsymbol{z}^-_{n+1/2}\|^2\Big) \\
& + \frac{\nu+\nu_m-|\nu-\nu_m|}{2}\Big(\|\nabla \boldsymbol{z}^+_{n+1/2}\|^2 + \|\nabla \boldsymbol{z}^-_{n+1/2}\|^2\Big) \\
& + \frac{|\nu-\nu_m|}{2}\|\nabla \boldsymbol{z}^+_{n+1/2} + \operatorname{sign}(\nu-\nu_m)\nabla \boldsymbol{z}^-_{n+1/2}\|^2.
\end{aligned}
$$

Adding the last two relations above,

$$
\begin{aligned}
0 = &\frac{1}{\tau_n}\Big(\|\boldsymbol{z}^+_{n+1/2}\|^2 + \|\boldsymbol{z}^-_{n+1/2}\|^2 - \|\boldsymbol{z}^+_n\|^2 - \|\boldsymbol{z}^-_n\|^2\Big)\\
&+ \frac{1}{\tau_n}\Big(\|\boldsymbol{z}^+_{n+1}\|^2 + \|\boldsymbol{z}^-_{n+1}\|^2 - \|\boldsymbol{z}^+_{n+1/2}\|^2 - \|\boldsymbol{z}^-_{n+1/2}\|^2\Big)\\
&+ \frac{1}{\tau_n}\Big(\|\boldsymbol{z}^+_{n+1/2} - \boldsymbol{z}^+_n\|^2 + \|\boldsymbol{z}^-_{n+1/2} - \boldsymbol{z}^-_n\|^2\Big)\\
&- \frac{1}{\tau_n}\Big(\|\boldsymbol{z}^+_{n+1} - \boldsymbol{z}^+_{n+1/2}\|^2 + \|\boldsymbol{z}^-_{n+1} - \boldsymbol{z}^-_{n+1/2}\|^2\Big).\\
&+ \big(\nu + \nu_m - |\nu - \nu_m|\big)\Big(\|\nabla \boldsymbol{z}^+_{n+1/2}\|^2 + \|\nabla \boldsymbol{z}^-_{n+1/2}\|^2\Big)\\
&+ |\nu - \nu_m|\big\|\nabla \mathbf{z}^+_{n+1/2} + \operatorname{sign}(\nu - \nu_m)\nabla \mathbf{z}^-_{n+1/2}\big\|^2.
\end{aligned}
$$

and use the fact that by (15) and (19)

$$
\boldsymbol{z}^\pm_{n+1} - \boldsymbol{z}^\pm_{n+1/2} = \boldsymbol{z}^\pm_{n+1/2} - \boldsymbol{z}^\pm_n,
$$

we get

$$
\begin{aligned}
0 = &\frac{1}{\tau_n}\Big(\|\boldsymbol{z}^+_{n+1}\|^2 + \|\boldsymbol{z}^-_{n+1}\|^2 - \|\boldsymbol{z}^+_n\|^2 - \|\boldsymbol{z}^-_n\|^2\Big)\\
&+ \big(\nu + \nu_m - |\nu - \nu_m|\big)\Big(\|\nabla \boldsymbol{z}^+_{n+1/2}\|^2 + \|\nabla \boldsymbol{z}^-_{n+1/2}\|^2\Big)\\
&+ |\nu - \nu_m|\big\|\nabla \boldsymbol{z}^+_{n+1/2} + \operatorname{sign}(\nu - \nu_m)\nabla \boldsymbol{z}^-_{n+1/2}\big\|^2.
\end{aligned}
$$

Finally, we multiply by $\frac{\tau_n}{2}$
since

$$
\nu + \nu_m - |\nu - \nu_m| = 2\min\{\nu, \nu_m\}
$$

we have

$$
\begin{aligned}
0 = &\frac{1}{2}\Big(\|\boldsymbol{z}^+_{n+1}\|^2 + \|\boldsymbol{z}^-_{n+1}\|^2 - \|\boldsymbol{z}^+_n\|^2 - \|\boldsymbol{z}^-_n\|^2\Big)\\
&+ \tau_n \min\{\nu, \nu_m\}\Big(\|\nabla \boldsymbol{z}^+_{n+1/2}\|^2 + \|\nabla \boldsymbol{z}^-_{n+1/2}\|^2\Big)\\
&+ \tau_n \frac{|\nu - \nu_m|}{2}\big\|\nabla \boldsymbol{z}^+_{n+1/2} + \operatorname{sign}(\nu - \nu_m)\nabla \boldsymbol{z}^-_{n+1/2}\big\|^2
\end{aligned}
$$

and sum over n from 1 to $N-1$ to obtain the energy equality

$$\frac{1}{2}\Big(\|\boldsymbol{z}_N^+\|^2 + \|\boldsymbol{z}_N^-\|^2\Big) + \min\{\nu, \nu_m\} \sum_{n=1}^{N-1} \tau_n \Big(\|\nabla \boldsymbol{z}_{n+1/2}^+\|^2 + \|\nabla \boldsymbol{z}_{n+1/2}^-\|^2\Big)$$
$$+ \frac{|\nu - \nu_m|}{2} \sum_{n=1}^{N-1} \tau_n \big\|\nabla \boldsymbol{z}_{n+1/2}^+ + \operatorname{sign}(\nu - \nu_m) \nabla \boldsymbol{z}_{n+1/2}^-\big\|^2$$
$$= \frac{1}{2}\Big(\|\boldsymbol{z}_1^+\|^2 - \|\boldsymbol{z}_1^-\|^2\Big),$$

which concludes the proof of stability. ∎

2.3. CONSERVATION OF QUADRATIC INVARIANTS OF THE MHD EQUATIONS: ENERGY, CROSS-HELICITY AND MAGNETIC HELICITY

As mentioned above, the sequential implementation (13)-(18) is a refactorization of the linearized midpoint rule [13] (for the MHD equations). The midpoint rule is a symplectic method for general Hamiltonian systems, conserving all quadratic Hamiltonians [4, 10], unconditionally stable, A-stable and B-stable [14, 1].

In order to verify the conservation properties, let us note that the system (15)-(18)

$$\frac{\boldsymbol{z}_{n+1}^{\pm} - \boldsymbol{z}_n^{\pm}}{\tau_n} \mp (\boldsymbol{B}_\circ \cdot \nabla) \boldsymbol{z}_{n+1/2}^{\pm} + (\widehat{\boldsymbol{z}}_{n+1/2}^{\mp} \cdot \nabla) \boldsymbol{z}_{n+1/2}^{\pm}$$
$$- \frac{\nu + \nu_m}{2} \Delta \boldsymbol{z}_{n+1/2}^{\pm} - \frac{\nu - \nu_m}{2} \Delta \boldsymbol{z}_{n+1/2}^{\mp} + \nabla p_{n+1/2}^{\pm} = 0,$$
$$\nabla \cdot \boldsymbol{z}_{n+1/2}^{\pm} = 0,$$

writes as follows, in the fluid velocity and magnetic fields original variables First, let's write it as

$$\frac{\boldsymbol{z}_{n+1}^{\pm} - \boldsymbol{z}_n^{\pm}}{\tau_n} \mp (\boldsymbol{B}_\circ \cdot \nabla) \frac{\boldsymbol{z}_{n+1}^{\pm} + \boldsymbol{z}_n^{\pm}}{2} + (\widehat{\boldsymbol{z}}_{n+1/2}^{\mp} \cdot \nabla) \frac{\boldsymbol{z}_{n+1}^{\pm} + \boldsymbol{z}_n^{\pm}}{2}$$
$$- \frac{\nu + \nu_m}{2} \Delta \frac{\boldsymbol{z}_{n+1}^{\pm} + \boldsymbol{z}_n^{\pm}}{2} - \frac{\nu - \nu_m}{2} \Delta \frac{\boldsymbol{z}_{n+1}^{\mp} + \boldsymbol{z}_n^{\mp}}{2} + \nabla p_{n+1/2}^{\pm} = 0,$$
$$\nabla \cdot \boldsymbol{u}_n = 0, \quad \nabla \cdot \boldsymbol{B}_n = 0,$$

then add / subtract, and divide by two:

$$\frac{\boldsymbol{u}_{n+1}-\boldsymbol{u}_n}{\tau_n}-(\boldsymbol{B}_\circ\cdot\nabla)\frac{\boldsymbol{b}_{n+1}+\boldsymbol{b}_n}{2}+\mathcal{A}_n \\ -\frac{\nu+\nu_m}{2}\Delta\frac{\boldsymbol{u}_{n+1}+\boldsymbol{u}_n}{2}-\frac{\nu-\nu_m}{2}\Delta\frac{\boldsymbol{u}_{n+1}+\boldsymbol{u}_n}{2} \\ +\nabla\frac{p^+_{n+1/2}+p^-_{n+1/2}}{2}=0,$$

respectively,

$$\frac{\boldsymbol{b}_{n+1}-\boldsymbol{b}_n}{\tau_n}-(\boldsymbol{B}_\circ\cdot\nabla)\frac{\boldsymbol{u}_{n+1}+\boldsymbol{u}_n}{2}+\mathcal{B}_n \\ -\frac{\nu+\nu_m}{2}\Delta\frac{\boldsymbol{b}_{n+1}+\boldsymbol{b}_n}{2}+\frac{\nu-\nu_m}{2}\Delta\frac{\boldsymbol{b}_{n+1}+\boldsymbol{b}_n}{2} \\ +\nabla\frac{p^+_{n+1/2}-p^-_{n+1/2}}{2}=0,$$

where

$$\begin{aligned}\mathcal{A}_n &= \frac{1}{2}(\widehat{\boldsymbol{z}}^-_{n+1/2}\cdot\nabla)\boldsymbol{z}^+_{n+1/2}+\frac{1}{2}(\widehat{\boldsymbol{z}}^+_{n+1/2}\cdot\nabla)\boldsymbol{z}^-_{n+1/2}\\ &= \frac{1}{2}(\widehat{\boldsymbol{z}}^-_{n+1/2}\cdot\nabla)(\boldsymbol{u}_{n+1/2}+\boldsymbol{b}_{n+1/2})+\frac{1}{2}(\widehat{\boldsymbol{z}}^+_{n+1/2}\cdot\nabla)(\boldsymbol{u}_{n+1/2}-\boldsymbol{b}_{n+1/2})\\ &= \frac{1}{2}\big((\widehat{\boldsymbol{z}}^+_{n+1/2}+\widehat{\boldsymbol{z}}^-_{n+1/2})\cdot\nabla\big)\boldsymbol{u}_{n+1/2}-\frac{1}{2}\big((\widehat{\boldsymbol{z}}^+_{n+1/2}-\widehat{\boldsymbol{z}}^-_{n+1/2})\cdot\nabla\big)\boldsymbol{b}_{n+1/2}\\ &= (\widehat{\boldsymbol{u}}_{n+1/2}\cdot\nabla)\boldsymbol{u}_{n+1/2}-\big(\widehat{\boldsymbol{b}}_{n+1/2}\cdot\nabla\big)\boldsymbol{b}_{n+1/2},\end{aligned}$$

and

$$\begin{aligned}\mathcal{B}_n &= \frac{1}{2}(\widehat{\boldsymbol{z}}^-_{n+1/2}\cdot\nabla)\boldsymbol{z}^+_{n+1/2}-\frac{1}{2}(\widehat{\boldsymbol{z}}^+_{n+1/2}\cdot\nabla)\boldsymbol{z}^-_{n+1/2}\\ &= \frac{1}{2}(\widehat{\boldsymbol{z}}^-_{n+1/2}\cdot\nabla)(\boldsymbol{u}_{n+1/2}+\boldsymbol{b}_{n+1/2})-\frac{1}{2}(\widehat{\boldsymbol{z}}^+_{n+1/2}\cdot\nabla)(\boldsymbol{u}_{n+1/2}-\boldsymbol{b}_{n+1/2})\\ &= \frac{1}{2}\big((\widehat{\boldsymbol{z}}^-_{n+1/2}-\widehat{\boldsymbol{z}}^+_{n+1/2})\cdot\nabla\big)\boldsymbol{u}_{n+1/2}+\frac{1}{2}\big((\widehat{\boldsymbol{z}}^-_{n+1/2}+\widehat{\boldsymbol{z}}^+_{n+1/2})\cdot\nabla\big)\boldsymbol{b}_{n+1/2}\\ &= -(\widehat{\boldsymbol{b}}_{n+1/2}\cdot\nabla)\boldsymbol{u}_{n+1/2}+(\widehat{\boldsymbol{u}}_{n+1/2}\cdot\nabla)\boldsymbol{b}_{n+1/2}.\end{aligned}$$

Therefore

$$\frac{\boldsymbol{u}_{n+1}-\boldsymbol{u}_n}{\tau_n}-(\boldsymbol{B}_\circ\cdot\nabla)\boldsymbol{b}_{n+1/2}+(\widehat{\boldsymbol{u}}_{n+1/2}\cdot\nabla)\boldsymbol{u}_{n+1/2}-\big(\widehat{\boldsymbol{b}}_{n+1/2}\cdot\nabla\big)\boldsymbol{b}_{n+1/2} \\ -\nu\Delta\boldsymbol{u}_{n+1/2}+\nabla\frac{p^+_{n+1/2}+p^-_{n+1/2}}{2}=0,$$

respectively,

$$\frac{\boldsymbol{b}_{n+1}-\boldsymbol{b}_n}{\tau_n}-(\boldsymbol{B}_\circ\cdot\nabla)\frac{\boldsymbol{u}_{n+1}+\boldsymbol{u}_n}{2}-(\widehat{\boldsymbol{b}}_{n+1/2}\cdot\nabla)\boldsymbol{u}_{n+1/2}+(\widehat{\boldsymbol{u}}_{n+1/2}\cdot\nabla)\boldsymbol{b}_{n+1/2}$$
$$-\nu_m\Delta\boldsymbol{b}_{n+1/2}+\nabla\frac{p^+_{n+1/2}-p^-_{n+1/2}}{2}=0,$$

or

$$\frac{\boldsymbol{u}_{n+1}-\boldsymbol{u}_n}{\tau_n}-\big((\boldsymbol{B}_\circ+\widehat{\boldsymbol{b}}_{n+1/2})\cdot\nabla\big)\boldsymbol{b}_{n+1/2}+(\widehat{\boldsymbol{u}}_{n+1/2}\cdot\nabla)\boldsymbol{u}_{n+1/2}$$
$$-\nu\Delta\boldsymbol{u}_{n+1/2}+\nabla\frac{p^+_{n+1/2}+p^-_{n+1/2}}{2}=0,$$
$$\frac{\boldsymbol{b}_{n+1}-\boldsymbol{b}_n}{\tau_n}-\big((\boldsymbol{B}_\circ+\widehat{\boldsymbol{b}}_{n+1/2})\cdot\nabla\big)\boldsymbol{u}_{n+1/2}+(\widehat{\boldsymbol{u}}_{n+1/2}\cdot\nabla)\boldsymbol{b}_{n+1/2}$$
$$-\nu_m\Delta\boldsymbol{b}_{n+1/2}+\nabla\frac{p^+_{n+1/2}-p^-_{n+1/2}}{2}=0.$$

Since $\boldsymbol{B}_\circ\in\mathbb{R}^d$, this yields

$$\frac{\boldsymbol{u}_{n+1}-\boldsymbol{u}_n}{\tau_n}-(\widehat{\boldsymbol{B}}_{n+1/2}\cdot\nabla)\boldsymbol{B}_{n+1/2}+(\widehat{\boldsymbol{u}}_{n+1/2}\cdot\nabla)\boldsymbol{u}_{n+1/2}-\nu\Delta\boldsymbol{u}_{n+1/2}$$
$$+\nabla\frac{p^+_{n+1/2}+p^-_{n+1/2}}{2}=0,\qquad\nabla\cdot\boldsymbol{u}_{n+1/2}=0,$$
$$\frac{\boldsymbol{B}_{n+1}-\boldsymbol{B}_n}{\tau_n}+(\widehat{\boldsymbol{u}}_{n+1/2}\cdot\nabla)\boldsymbol{B}_{n+1/2}-(\widehat{\boldsymbol{B}}_{n+1/2}\cdot\nabla)\boldsymbol{u}_{n+1/2}$$
$$-\nu_m\Delta\boldsymbol{B}_{n+1/2}+\nabla\frac{p^+_{n+1/2}-p^-_{n+1/2}}{2}=0,\qquad\nabla\cdot\boldsymbol{B}_{n+1/2}=0,$$

where, similar to (11), the extrapolations are

$$\widehat{\boldsymbol{u}}_{n+1/2}=\boldsymbol{u}_{n-1/2}+\frac{\tau_n+\tau_{n-1}}{\tau_{n-1}}(\boldsymbol{u}_n-\boldsymbol{u}_{n-1/2}),$$
$$\widehat{\boldsymbol{B}}_{n+1/2}=\boldsymbol{B}_{n-1/2}+\frac{\tau_n+\tau_{n-1}}{\tau_{n-1}}(\boldsymbol{B}_n-\boldsymbol{B}_{n-1/2}).$$

Therefore, in the absence of external forcing terms, with zero kinematic viscosity ($\nu=0$) and zero magnetic diffusivity ($\nu_m=0$), the energy $\mathcal{E}_n=\frac{1}{2}\int_\Omega(\boldsymbol{u}_n^2+\boldsymbol{B}_n^2)\,d\boldsymbol{x}$, and the cross-helicity $\mathcal{H}_{Cn}=\frac{1}{2}\int_\Omega\boldsymbol{u}_n\cdot\boldsymbol{B}_n\,d\boldsymbol{x}$, two out of the three invariants of the MHD equations (1)-(2), are conserved (see e.g.,

[26]): First test

$$\frac{\boldsymbol{u}_{n+1}-\boldsymbol{u}_n}{\tau_n} - (\widehat{\boldsymbol{B}}_{n+1/2}\cdot\nabla)\boldsymbol{B}_{n+1/2} + (\widehat{\boldsymbol{u}}_{n+1/2}\cdot\nabla)\boldsymbol{u}_{n+1/2} - \nu\Delta\boldsymbol{u}_{n+1/2}$$
$$+\nabla\frac{p^+_{n+1/2}+p^-_{n+1/2}}{2} = 0, \qquad \nabla\cdot\boldsymbol{u}_{n+1/2}=0,$$
$$\frac{\boldsymbol{B}_{n+1}-\boldsymbol{B}_n}{\tau_n} - (\widehat{\boldsymbol{B}}_{n+1/2}\cdot\nabla)\boldsymbol{u}_{n+1/2} + (\widehat{\boldsymbol{u}}_{n+1/2}\cdot\nabla)\boldsymbol{B}_{n+1/2}$$
$$-\nu_m\Delta\boldsymbol{B}_{n+1/2} + \nabla\frac{p^+_{n+1/2}-p^-_{n+1/2}}{2} = 0, \qquad \nabla\cdot\boldsymbol{B}_{n+1/2}=0,$$

by $\boldsymbol{u}_{n+1/2}$ and $\boldsymbol{B}_{n+1/2}$, respectively, use the conservation of mass,

$$\frac{1}{2\tau_n}\big(\|\boldsymbol{u}_{n+1}\|^2-\|\boldsymbol{u}_n\|^2\big) - \int_\Omega (\widehat{\boldsymbol{B}}_{n+1/2}\cdot\nabla)\boldsymbol{B}_{n+1/2}\boldsymbol{u}_{n+1/2}\,d\boldsymbol{x} + \nu\|\nabla\boldsymbol{u}_{n+1/2}\|^2 = 0,$$
$$\frac{1}{2\tau_n}\big(\|\boldsymbol{B}_{n+1}\|^2-\|\boldsymbol{B}_n\|^2\big) - \int_\Omega \Big((\widehat{\boldsymbol{B}}_{n+1/2}\cdot\nabla)\boldsymbol{u}_{n+1/2} + (\widehat{\boldsymbol{u}}_{n+1/2}\cdot\nabla)\boldsymbol{B}_{n+1/2}\Big)\boldsymbol{B}_{n+1/2}\,d\boldsymbol{x}$$
$$+\nu_m\|\nabla\boldsymbol{B}_{n+1/2}\|^2 = 0,$$

and add to obtain, using the skew-symmetry of the trilinear form,

$$\frac{1}{2\tau_n}\big(\|\boldsymbol{u}_{n+1}\|^2+\|\boldsymbol{B}_{n+1}\|^2-\|\boldsymbol{u}_n\|^2-\|\boldsymbol{B}_n\|^2\big) - \int_\Omega (\widehat{\boldsymbol{B}}_{n+1/2}\cdot\nabla)\boldsymbol{B}_{n+1/2}\boldsymbol{u}_{n+1/2}\,d\boldsymbol{x}$$
$$-\int_\Omega \{(\widehat{\boldsymbol{B}}_{n+1/2}\cdot\nabla)\boldsymbol{u}_{n+1/2}\boldsymbol{B}_{n+1/2}\,d\boldsymbol{x}\} + \nu\|\nabla\boldsymbol{u}_{n+1/2}\|^2 + \nu_m\|\nabla\boldsymbol{B}_{n+1/2}\|^2 = 0,$$

or

$$\frac{1}{2\tau_n}\big(\|\boldsymbol{u}_{n+1}\|^2+\|\boldsymbol{B}_{n+1}\|^2-\|\boldsymbol{u}_n\|^2-\|\boldsymbol{B}_n\|^2\big) + \nu\|\nabla\boldsymbol{u}_{n+1/2}\|^2 + \nu_m\|\nabla\boldsymbol{B}_{n+1/2}\|^2 = 0,$$

For the cross helicity conservation, test

$$\frac{\boldsymbol{u}_{n+1}-\boldsymbol{u}_n}{\tau_n} - (\widehat{\boldsymbol{B}}_{n+1/2}\cdot\nabla)\boldsymbol{B}_{n+1/2} + (\widehat{\boldsymbol{u}}_{n+1/2}\cdot\nabla)\boldsymbol{u}_{n+1/2} - \nu\Delta\boldsymbol{u}_{n+1/2}$$
$$+\nabla\frac{p^+_{n+1/2}+p^-_{n+1/2}}{2} = 0, \qquad \nabla\cdot\boldsymbol{u}_{n+1/2}=0,$$
$$\frac{\boldsymbol{B}_{n+1}-\boldsymbol{B}_n}{\tau_n} - (\widehat{\boldsymbol{B}}_{n+1/2}\cdot\nabla)\boldsymbol{u}_{n+1/2} + (\widehat{\boldsymbol{u}}_{n+1/2}\cdot\nabla)\boldsymbol{B}_{n+1/2}$$
$$-\nu_m\Delta\boldsymbol{B}_{n+1/2} + \nabla\frac{p^+_{n+1/2}-p^-_{n+1/2}}{2} = 0, \qquad \nabla\cdot\boldsymbol{B}_{n+1/2}=0,$$

by $\boldsymbol{B}_{n+1/2}$ and $\boldsymbol{u}_{n+1/2}$, respectively, use the conservation of mass,

$$\begin{aligned}
&\frac{1}{\tau_n}\int_\Omega \big(\boldsymbol{u}_{n+1}\boldsymbol{B}_{n+1} - \boldsymbol{u}_n\boldsymbol{B}_n\big)\,d\boldsymbol{x} - \int_\Omega (\widehat{\boldsymbol{B}}_{n+1/2}\cdot\nabla)\boldsymbol{B}_{n+1/2}\boldsymbol{B}_{n+1/2}\,d\boldsymbol{x}\\
&+\int_\Omega (\widehat{\boldsymbol{u}}_{n+1/2}\cdot\nabla)\boldsymbol{u}_{n+1/2}\boldsymbol{B}_{n+1/2}\,d\boldsymbol{x} + \nu\int_\Omega \nabla\boldsymbol{u}_{n+1/2}\nabla\boldsymbol{B}_{n+1/2}\\
&-\int_\Omega (\widehat{\boldsymbol{B}}_{n+1/2}\cdot\nabla)\boldsymbol{u}_{n+1/2}\boldsymbol{u}_{n+1/2}\,d\boldsymbol{x} + \int_\Omega (\widehat{\boldsymbol{u}}_{n+1/2}\cdot\nabla)\boldsymbol{B}_{n+1/2}\boldsymbol{u}_{n+1/2}\,d\boldsymbol{x}\\
&\qquad\qquad + \nu_m\int_\Omega \nabla\boldsymbol{B}_{n+1/2}\nabla\boldsymbol{u}_{n+1/2}\,d\boldsymbol{x} = 0,
\end{aligned}$$

and again the skew-symmetry of the trilinear form

$$\begin{aligned}
&\frac{1}{\tau_n}\int_\Omega \big(\boldsymbol{u}_{n+1}\boldsymbol{B}_{n+1} - \boldsymbol{u}_n\boldsymbol{B}_n\big)\,d\boldsymbol{x}\\
&+\int_\Omega (\widehat{\boldsymbol{u}}_{n+1/2}\cdot\nabla)\boldsymbol{u}_{n+1/2}\boldsymbol{B}_{n+1/2}\,d\boldsymbol{x} + \nu\int_\Omega \nabla\boldsymbol{u}_{n+1/2}\nabla\boldsymbol{B}_{n+1/2}\\
&+\int_\Omega (\widehat{\boldsymbol{u}}_{n+1/2}\cdot\nabla)\boldsymbol{B}_{n+1/2}\boldsymbol{u}_{n+1/2}\,d\boldsymbol{x}\\
&\qquad\qquad + \nu_m\int_\Omega \nabla\boldsymbol{B}_{n+1/2}\nabla\boldsymbol{u}_{n+1/2}\,d\boldsymbol{x} = 0,
\end{aligned}$$

gives

$$\frac{1}{\tau_n}\int_\Omega \big(\boldsymbol{u}_{n+1}\boldsymbol{B}_{n+1} - \boldsymbol{u}_n\boldsymbol{B}_n\big)\,d\boldsymbol{x} + (\nu+\nu_m)\int_\Omega \nabla\boldsymbol{u}_{n+1/2}\nabla\boldsymbol{B}_{n+1/2} = 0.$$

Finally, the magnetic helicity $\mathcal{H}_{Mn} = \frac{1}{2}\int_\Omega \mathbb{A}_n\cdot\boldsymbol{B}_n d\boldsymbol{x}$ (where $\mathbb{A}_n$ is the vector potential, $\boldsymbol{B}_n = \nabla\times\mathbb{A}_n$) is also conserved provided that in the equations (13) the extrapolated terms $\widehat{\boldsymbol{z}}^{\mp}_{n+1/2}$ are replaced by the previous iterations $\boldsymbol{z}^{\mp}_{(\kappa-1)}$. In this case, the convergence of the iterates in Theorem 2.1 still holds, as expected, under a timestep restriction depending on the data.

3. CONCLUSIONS

Using a refactorization of the midpoint rule [13], we proposed and analyzed a partitioned method for the MHD equations in the Elsässer variables, which increases the accuracy of a first-order algorithm to second-order, with only few lines of code, at the same computational cost. We proved the convergence of the sub-iterates and the unconditional stability of the method. We also proved that the linearized midpoint rule conserves the energy and cross-helicity, while the magnetic helicity is conserved by the fully nonlinear method. Currently we are studying the partitioned, time-adaptive (variable time-step) implementation of the midpoint rule for the MHD equations in primal variables. The flexibility and ease of implementation of the 'refactorization' of the midpoint rule facilitates the partitioning of the MHD into the Navier-Stokes and Maxwell equations, for use of blackbox or legacy codes. Further modifications of each dedicated code, introducing turbulence models, ensemble calculations or uncertainty quantification techniques, are interesting research paths to be pursued.

References

[1] G. Akrivis, *Numerical Methods for initial value problems*, Lecture Notes from a course taught at BCAM, Basque Center for Applied Mathematics, Bilbao, Basque Country, Spain, 2012.

[2] H. Alfvén, *Existence of electromagnetic-hydrodynamic waves*, Nature, 150 (1942), p. 405.

[3] F. Armero, J. C. Simo, *Long-term dissipativity of time-stepping algorithms for an abstract evolution equation with applications to the incompressible MHD and Navier-Stokes equations*, Comput. Methods Appl. Mech. Engrg., 131 (1996), pp. 41–90.

[4] U. M. Ascher, S. Reich, *The midpoint scheme and variants for Hamiltonian systems: advantages and pitfalls*, SIAM J. Sci. Comput., 21 (1999), pp. 1045–1065.

[5] L. Barleon, V. Casal, L. Lenhart, *MHD flow in liquid-metal-cooled blankets*, Fusion Eng. Des., 14 (1991), pp. 401 – 412.

[6] J. D. Barrow, R. Maartens, C. G. Tsagas, *Cosmology with inhomogeneous magnetic fields*, Phys. Rep., 449 (2007), pp. 131–171.

[7] M. Berroukeche, E. K. Maschke, B. Saramito, *Convergence of an unconditionally stable semi-implicit scheme for linear incompressible MHD equations*, C. R. Acad. Sci. Paris Sér. I Math., 328 (1999), pp. 81–86.

[8] D. Biskamp, *Nonlinear magnetohydrodynamics*, vol. 1 of Cambridge Monographs on Plasma Physics, Cambridge University Press, Cambridge, 1993.

[9] D. Biskamp, *Magnetohydrodynamic turbulence*, Cambridge University Press, 2003.

[10] P. B. Bochev, C. Scovel, *On quadratic invariants and symplectic structure*, BIT, 34 (1994), pp. 337–345.

[11] P. Bodenheimer, G. P. Laughlin, M. Różyczka, H. W. Yorke, *Numerical methods in astrophysics*, Series in Astronomy and Astrophysics, Taylor & Francis, New York, 2007.

[12] A. Bourlioux, A. T. Layton, M. L. Minion, *High-order multi-implicit spectral deferred correction methods for problems of reactive flow*, J. Comput. Phys., 189 (2003), pp. 651–675.

[13] J. Burkardt, C. Trenchea, *Refactorization of the midpoint rule*, Appl. Math. Lett., 107 (2020), p. 106438.

[14] J. C. Butcher, *A stability property of implicit Runge-Kutta methods*, BIT Numerical Mathematics, 15 (1975), pp. 358–361.

[15] A. Cibik, F. G. Eroglu, S. Kaya, *Analysis of second order time filtered backward Euler method for MHD equations*, J. Sci. Comput., 82 (2020), pp. Art. 38, 25.

[16] G. G. Dahlquist, W. Liniger, O. Nevanlinna, *Stability of two-step methods for variable integration steps*, SIAM J. Numer. Anal., 20 (1983), pp. 1071–1085.

[17] P. A. Davidson, *An introduction to magnetohydrodynamics*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2001.

[18] E. Deriaz, B. Despres, G. Faccanoni, K. P. Gostaf, L.-M. Imbert-Gérard, G. Sadaka, R. Sart, *Magnetic equations with* **FreeFem++***: the Grad-Shafranov equation & the current hole*, in CEMRACS'10 research achievements: numerical modeling of fusion, vol. 32 of ESAIM Proc., EDP Sci., Les Ulis, 2011, pp. 76–94.

[19] M. Dobrowolny, A. Mangeney, P. Veltri, *Fully developed anisotropic hydromagnetic turbulence in interplanetary space*, Phys. Rev. Lett., 45 (1980), pp. 144–147.

[20] E. Dormy, M. Núñez, *Introduction [Special issue: Magnetohydrodynamics in astrophysics and geophysics]*, Geophys. Astrophys. Fluid Dyn., 101 (2007), p. 169.

[21] E. Dormy, A. M. Soward, eds., *Mathematical aspects of natural dynamos*, vol. 13 of Fluid Mechanics of Astrophysics and Geophysics, Grenoble Sciences. Universite Joseph Fourier, Grenoble, 2007.

[22] A. Dutt, L. Greengard, V. Rokhlin, *Spectral deferred correction methods for ordinary differential equations*, BIT, 40 (2000), pp. 241–266.

[23] W. M. Elsässer, *The hydromagnetic equations*, Phys. Rev., 79 (1950), pp. 183–183.

[24] A. Fierros Palacios, *The Hamilton-type principle in fluid dynamics*, Springer, Vienna, 2006. Fundamentals and applications to magnetohydrodynamics, thermodynamics, and astrophysics.

[25] J. A. Font, *General relativistic hydrodynamics and magnetohydrodynamics: hyperbolic systems in relativistic astrophysics*, in Hyperbolic problems: theory, numerics, applications, Springer, Berlin, 2008, pp. 3–17.

[26] U. Frisch, A. Pouquet, J. Leorat, A. Mazure, *Possibility of an inverse cascade of magnetic helicity in magnetohydrodynamic turbulence*, J. Fluid Mech., 68 (1975), pp. 769–778.

[27] S. Galtier, S. V. Nazarenko, A. C. Newell, A. Pouquet, *A weak turbulence theory for incompressible magnetohydrodynamics*, J. Plasma Phys., 63 (2000), pp. 447–488.

[28] J.-F. Gerbeau, C. Le Bris, T. Lelièvre, *Mathematical methods for the magnetohydrodynamics of liquid metals*, Numerical Mathematics and Scientific Computation, Oxford University Press, Oxford, 2006.

[29] P. Goldreich and S. Sridhar, *Toward a theory of interstellar turbulence. II: Strong Alfvénic turbulence*, ApJ, 438 (1995), pp. 763–775.

[30] E. Hairer, C. Lubich, G. Wanner, *Geometric numerical integration*, vol. 31 of Springer Series in Computational Mathematics, Springer, Heidelberg, 2010. Structure-preserving algorithms for ordinary differential equations, Reprint of the second (2006) edition.

[31] E. Hairer, S. P. Nørsett, G. Wanner, *Solving ordinary differential equations. I*, vol. 8 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, second ed., 1993. Nonstiff problems.

[32] H. Hashizume, *Numerical and experimental research to solve MHD problem in liquid blanket system*, Fusion Eng. Des., 81 (2006), pp. 1431 – 1438.

[33] T. Heister, M. Mohebujjaman, L. G. Rebholz, *Decoupled, unconditionally stable, higher order discretizations for MHD flow simulation*, Journal of Scientific Computing, 71 (2017), pp. 21–43.

[34] W. Hillebrandt, F. Kupka, eds., *Interdisciplinary aspects of turbulence*, vol. 756 of Lecture Notes in Physics, Springer-Verlag, Berlin, 2009.

[35] P. S. Iroshnikov, *Turbulence of a conducting fluid in a strong magnetic field*, Soviet Astronom. AJ, 7 (1964), pp. 566–571.

[36] R. H. Kraichnan, *Inertial-range spectrum of hydromagnetic turbulence*, Phys. Fluids, 8 (1965), pp. 1385–1387.

[37] L. D. Landau, E. M. Lifshitz, *Electrodynamics of continuous media*, Course of Theoretical Physics, Vol. 8. Translated from the Russian by J. B. Sykes and J. S. Bell, Pergamon Press, Oxford, 1960.

[38] W. Layton, H. Tran, C. Trenchea, *Stability of partitioned methods for magnetohydrodynamics flows at small magnetic Reynolds number*, in Recent advances in scientific computing and applications, vol. 586 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2013, pp. 231–238.

[39] ———, *Numerical analysis of two partitioned methods for uncoupling evolutionary MHD flows*, Numer. Methods Partial Differential Equations, 30 (2014), pp. 1083–1102.

[40] J. Leray, *Essai sur les mouvements plans d'une liquide visqueux que limitent des parois*, J. Math. Pur. Appl., Paris Ser. IX, 13 (1934), pp. 331–418.

[41] Y. Li, C. Trenchea, *Partitioned second order method for magnetohydrodynamics in Elsässer variables*, Discrete Contin. Dyn. Syst. Ser. B, 23 (2018), pp. 2803–2823.

[42] T. Lin, J. Gilbert, R. Kossowsky, *Sea-Water Magnetohydrodynamic Propulsion for Next-Generation Undersea Vehicles*, Defense Technical Information Center, 1990.

[43] E. Marsch, *Turbulence in the solar wind*, in Reviews in Modern Astronomy, G. Klare, ed., vol. 4 of Reviews in Modern Astronomy, Springer Berlin Heidelberg, 1991, pp. 145–156.

[44] M. L. Minion, *Semi-implicit spectral deferred correction methods for ordinary differential equations*, Commun. Math. Sci., 1 (2003), pp. 471–500.

[45] ———, *Semi-implicit projection methods for incompressible flow based on spectral deferred corrections*, Appl. Numer. Math., 48 (2004), pp. 369–387. Workshop on Innovative Time Integrators for PDEs.

[46] R. Moreau, *Magnetohydrodynamics*, vol. 3 of Fluid Mechanics and its Applications, Kluwer Academic Publishers Group, Dordrecht, 1990. Translated from the French by A. F. Wright.

[47] B. Punsly, *Black hole gravitohydromagnetics*, vol. 355 of Astrophysics and Space Science Library, Springer-Verlag, Berlin, second ed., 2008.

[48] J. V. Shebalin, W. H. Matthaeus, D. Montgomery, *Anisotropy in MHD turbulence due to a mean magnetic field*, J. Plasma Phys., 29 (1983), pp. 525–547.

[49] S. Smolentsev, R. Moreau, L. Bühler, C. Mistrangelo, *MHD thermofluid issues of liquid-metal blankets: Phenomena and advances*, Fusion Eng. Des., 85 (2010), pp. 1196–1205.

[50] H. J. Stetter, *Analysis of discretization methods for ordinary differential equations*, Springer-Verlag, New York-Heidelberg, 1973. Springer Tracts in Natural Philosophy, Vol. 23.

[51] R. Temam, *Navier-Stokes equations*, AMS Chelsea Publishing, Providence, RI, 2001. Theory and numerical analysis, Reprint of the 1984 edition.

[52] C. Trenchea, *Unconditional stability of a partitioned IMEX method for magnetohydrodynamic flows*, Appl. Math. Lett., 27 (2014), pp. 97–100.

[53] M. K. Verma, *Statistical theory of magnetohydrodynamic turbulence: recent results*, Physics Reports, 401 (2004), pp. 229 – 380.

[54] J. Verwer, *Component splitting for semi-discrete Maxwell equations*, BIT Numerical Mathematics, (2010), pp. 1–19. 10.1007/s10543-010-0296-y.

[55] I. Veselovsky, *Turbulence and waves in the solar wind formation region and the heliosphere*, Astrophys. Space Sci., 277 (2001), pp. 219–224.

[56] N. Wilson, A. Labovsky, C. Trenchea, *High Accuracy Method for Magnetohydrodynamics System in Elsässer Variables*, Comput. Methods Appl. Math., 15 (2015), pp. 97–110.