ROMAI J., v.15, no.2(2019), 139–153

HERMITE-HADAMARD-FEJER TYPE INEQUALITIES FOR P-CONVEX FUNCTIONS VIA α-GENERATOR

Erdal Ünlüyol[1], Yeter Erdaş[2]

[1] *Ordu University, Faculty of Arts and Sciences, Department of Mathematics, Ordu, Turkey*

[2] *Ordu University, Sciences Institute, Department of Mathematics, Ordu, Turkey*

eunluyol@yahoo.com, yeterrerdass@gmail.com

Abstract Differentiation and integration are basic operations of calculus and analysis. Indeed, they are infinitesimal versions of substraction and addition operations on numbers. From 1967 till 1970 Michael Grossman and Robert Katz gave definitions of a new kind of derivative and integral, converting the roles of substraction and addition into division and multiplication, respectively. And they established Non-Newtonian Calculus. This calculi is generated by generators. So in this study, firstly we give definition of $(\alpha; p)$ or p_α-convex function, and some new theorems for this function class via α-generator. Secondly, those theorems are generalized using Hermite-Hadamard-Fejer inequality for p_α-convex function by this generator. Finally, it is obtained some new corollaries depend on these theorems.

Keywords: Non-Newtonian calculus, $*$-Hermite-Hadamard-Fejer inequality.

2010 MSC: 26A51, 26D10, 26D15.

1. INTRODUCTION

The Non-Newtonian calculi were developed by Michael Grossman and Robert Katz, and it were written to nine books related to the Non-Newtonian calculi. Grossman and Katz published first book concerning with Non-Newtonian calculus at 1972 [1].

This calculi, which are mentioned above, are geometric calculus, bigeometric calculus, harmonic calculus, biharmonic calculus, quadratic calculus and biquadratic calculus. In the geometric calculus and the bigeometric calculus from within of these calculi, the derivative and integral are both multiplicative. The geometric derivative and the bigeometric derivative are closely related to the wellknown logarithmic derivative and elasticity, respectively. Also, the linear functions of classical calculus are the functions which having a constant derivative and besides the exponential functions in the geometric calculus are the functions which having a constant derivative, the power functions in the bigeometric calculus are the functions which having a constant derivative.

Among the Non-Newtonian calculi, geometric and bigeometric calculi have been often used.

Since this calculi has emerged, it has become a seriously alternative to the classical analysis developed by Newton and Leibnitz. Just like the classical analysis, Non-Newtonian calculi have many varieties as a derivative, an integral, a natural average, a special class of functions having a constant derivative and two fundamental theorems which reveal that the derivative and integral are inversely related. However, the results of obtained by Non-Newtonian calculus has also significantly different from the classical analysis. For example, infinitely many Non-Newtonian calculi have a nonlinear derivative or integral.

The Non-Newtonian calculi are useful mathematical tools in science, engineering and mathematics and provide a wide variety of possibilities, as a different perspective. Specific fields of application include: fractal theory, image analysis (e.g., in bio-medicine), growth/decay processes (e. g., in economic growth, bacterial growth and radioactive decay), finance (e.g., rates of return), the theory of elasticity in economics, marketing, the economics of climate change, atmospheric temperature, wave theory in physics, quantum physics and gauge theory, signal processing, information technology, pathogen counts in treated water, actuarial science, tumor therapy in medicine, materials science/engineering, demographics, differential equations etc.

Recently, studies related with Non-Newtonian have increased. Especially, these studies are emerging in the field of applied mathematics.[10]–[22]. Arithmetic is any system that satisfies the whole of the ordered field axiom whose domain is a subset of $\mathbb{R}$. There are infinite many types arithmetic, all of which are isomorphic, that is, structurally equivalent.

2. BASIC DEFINITIONS

Theorem 2.1. *Let I be an interval in $\mathbb{R}$, $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function, f is said to be Hermite-Hadamard Type Inequality, if*

$$f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}dx \tag{1}$$

for all $x \in I$ and $t \in [0,1]$.

Theorem 2.2. *[2]: Let $f : [a,b] \to \mathbb{R}$ be a convex function, $g : [a,b] \to \mathbb{R}$ function is integrable on $[a,b]$, nonnegative and symmetric to $\left(\frac{a+b}{2}\right)$. Then, for all $x \in [a,b]$*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx &\leq \int_a^b f(x)g(x)dx \\ &\leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx \end{aligned}$$

Hermite-Hadamard-Fejer Inequality holds.

Theorem 2.3. *[7] Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a p-convex function, $p \in \mathbb{R} \setminus 0$, $a, b \in I$ with $a < b$. If $f \in L[a,b]$ and $w : [a,b] \to \mathbb{R}$ is nonnegative,integrable and p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}$, then the following inequalities hold:*

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx \leq \frac{f(a)+f(b)}{2} \tag{2}$$

Now we give a short brief of Non-Newtonian Calculus [1].

2.1. SYSTEMS OF ARITHMETIC

Arithmetic is any system that satisfies the whole of the ordered field axiom whose domain is a subset of $\mathbb{R}$. There are many types arithmetic, all of which are isomorphic, that is, structurally equivalent.

A generator α is a one-to-one function whose domain is $\mathbb{R}$ and whose range is a subset $\mathbb{R}_\alpha$ of $\mathbb{R}$ where $\mathbb{R}_\alpha = \{\alpha(x) : x \in \mathbb{R}\}$. Each generator generates exactly one arithmetic, and conversely each arithmetic is generated by exactly one generator. If $I(x) = x$, for all $x \in \mathbb{R}$, the identity function's inverse is itself. In the special cases $\alpha = I$ and $\alpha = exp$, α generates the classical and geometric arithmetic, respectively.

2.1.1 α-Arithmetics. By α-arithmetic, we mean the arithmetic whose domain is $\mathbb{R}$ and whose operations are defined as follows: for $x, y \in \mathbb{R}_\alpha$ and generator α,

$$\begin{aligned}
\alpha - \texttt{addition},\ x\dot{+}y &= \alpha\{\alpha^{-1}(x)+\alpha^{-1}(y)\},\\
\alpha - \texttt{subtraction},\ x\dot{-}y &= \alpha\{\alpha^{-1}(x)-\alpha^{-1}(y)\},\\
\alpha - \texttt{multiplication},\ x\dot{\times}y &= \alpha\{\alpha^{-1}(x)\times\alpha^{-1}(y)\},\\
\alpha - \texttt{division},\ x\dot{/}y &= \alpha\{\alpha^{-1}(x)/\alpha^{-1}(y)\},\\
\alpha - \texttt{order},\ x\dot{<}y &\iff \alpha^{-1}(x)<\alpha^{-1}(y).
\end{aligned}$$

As a generator, we choose exp function acting from $\mathbb{R}$ into the set $\mathbb{R}_{exp} = (0,\infty)$ as follows:

$$\begin{aligned}
\alpha :\ \mathbb{R} &\longrightarrow \mathbb{R}_{exp}\\
x &\longrightarrow y = \alpha(x) = e^x.
\end{aligned}$$

It is obvious that α arithmetic reduces to the geometric arithmetic as follows:

$$\begin{aligned}
\texttt{geometric addition},\ x\dot{+}y &= e^{\{\ln x+\ln y\}} = x.y,\\
\texttt{geometric subtraction},\ x\dot{-}y &= e^{\{\ln x-\ln y\}} = x/y,\\
\texttt{geometric multiplication},\ x\dot{\times}y &= e^{\{\ln x\times\ln y\}} = x^{\ln y} = y^{\ln x},\\
\texttt{geometric division},\ x\dot{/}y &= e^{\{\ln x/\ln y\}} = x^{\frac{1}{\ln y}},\\
\texttt{geometric order},\ x\dot{<}y &\iff ln(x) < ln(y).
\end{aligned}$$

Definition 2.1. *[1] Let $\alpha(p) = \dot{p}$ for all $p \in \mathbb{Z}$. If for $y \in \mathbb{R}_\alpha$, $y\dot{+}\dot{0} = y$ and $y\dot{\times}\dot{1} = y$, then according to α-addition $\dot{0}(\alpha - zero)$ and $\dot{1}(\alpha - one)$ numbers are called identity and unit elements, respectively.*

Definition 2.2. *[1] Let $\dot{-}\dot{n} = \dot{0}\dot{-}\dot{n} = \alpha(-n)$ for all $n \in \mathbb{Z}$. Set of α-integers is defined and denoted by $\mathbb{Z}_\alpha$ as can be seen in the figure below:*

$$\begin{aligned}
\mathbb{Z}_\alpha &= \{\ldots, \dot{-}\dot{2}, \dot{-}\dot{1}, \dot{0}, \dot{1}, \dot{2}\ldots\}\\
&= \{\ldots, \alpha(-2), \alpha(-1), \alpha(0), \alpha(1), \alpha(2), \ldots\}.
\end{aligned}$$

Namely, $\mathbb{Z}_\alpha = \{\dot{n} : \dot{n} = \alpha(n), n \in \mathbb{Z}\}$.

Definition 2.3. *[23] Let α is a generator. Then $I_\alpha \subseteq \mathbb{R}_\alpha$ is said to an α-interval on $\mathbb{R}_\alpha$ if for all $x, y \in I_\alpha$;*

1 $\dot{(}x, y\dot{)} := \{z \in I_\alpha : x\dot{<}z\dot{<}y\} \subseteq I_\alpha$,

2 $\dot{(}x, y\dot{]} := \{z \in I_\alpha : x\dot{<}z\dot{\leq}y\} \subseteq I_\alpha$,

3 $\dot{[}x, y\dot{)} := \{z \in I_\alpha : x\dot{\leq}z\dot{<}y\} \subseteq I_\alpha$,

4 $\dot{[}x, y\dot{]} := \{z \in I_\alpha : x\dot{\leq}z\dot{\leq}y\} \subseteq I_\alpha$,

5 $\dot{(}x, \dot{+}\dot{\infty}\dot{)} := \{z \in I_\alpha : x\dot{<}z\dot{<}\dot{+}\dot{\infty}\} \subseteq I_\alpha$,

6 $\dot{(}\dot{-}\dot{\infty}, y\dot{)} := \{z \in I_\alpha : \dot{-}\dot{\infty}\dot{<}z\dot{<}y\} \subseteq I_\alpha$,

7 $\dot{[}x, \dot{+}\dot{\infty}\dot{)} := \{z \in I_\alpha : x\dot{\leq}z\dot{<}\dot{+}\dot{\infty}\} \subseteq I_\alpha$,

8 $\dot{(}\dot{-}\dot{\infty}, y\dot{]} := \{z \in I_\alpha : \dot{-}\dot{\infty}\dot{<}z\dot{\leq}y\} \subseteq I_\alpha$.

Remark 2.1. *[23] Alternatively, we can express these* (1)-(8) *α-intervals as follows respectively;*

$$(x, y)_\alpha, (x, y]_\alpha, [x, y)_\alpha, [x, y]_\alpha, (x, +\infty)_\alpha, (-\infty, y)_\alpha, [x, +\infty)_\alpha, (-\infty, y]_\alpha.$$

Remark 2.2. *[23] Afterwards $[x, y]_\alpha, (x, y)_\alpha$ are said to be respectively α-closed interval, α-open interval.*

Definition 2.4. *[23] Let L_α is an α-linear space and $A \subseteq L_\alpha$. A set is said to be an α-convex set, if for all $x, y \in A$*

$$B_\alpha = \{z \in L_\alpha : z = \theta_1 \dot{\times} x \dot{+} \theta_2 \dot{\times} y,\ \theta_1 \dot{+} \theta_2 = \dot{1},\ \dot{0} \dot{\leq} \theta_1, \theta_2 \dot{\leq} \dot{1}\} \subseteq A.$$

It is immediate that $z = \theta_1 \dot{\times} x \dot{+} \theta_2 \dot{\times} y,\ \ \theta_1 \dot{+} \theta_2 = \dot{1},\ \theta_1, \theta_2 \in [0,1]_\alpha$ for $x, y \in A, z \in B_\alpha$.

Corollary 2.1. *[23] The interval $I_\alpha := [x, y]_\alpha$ is an α-convex set on $\mathbb{R}_\alpha$.*

Corollary 2.2. *[23] $A \subset \mathbb{R}_\alpha$ is $\alpha-$convex set if and only if $\dot{[}x, y\dot{]} \subset A$ for all $x, y \in A$, such that $x \dot{\leq} y$.*

Definition 2.5. *Let $I_\alpha \subseteq \mathbb{R}_\alpha$ be an α-interval. A function $f : I_\alpha \subseteq \mathbb{R}_\alpha \to \mathbb{R}$ is said to be α-convex if the following inequality holds*

$$f(\lambda_1 \dot{\times} a + \lambda_2 \dot{\times} b) \quad \leq \quad \theta_1 f(a) + \theta_2 f(b).$$

Where $\lambda_1 \dot{+} \lambda_2 = \dot{1}$ and $\theta_1 + \theta_2 = 1$. If we choose $\lambda_1 = \alpha(t)$, $\lambda_2 = \alpha(1-t)$, $\theta_1 = t$ and $\theta_2 = 1-t$ we get,

$$f(\alpha(t) \dot{\times} a + \alpha(1-t) \dot{\times} b) \leq t f(a) + (1-t) f(b). \tag{3}$$

Definition 2.6. *[8] Let $I_\alpha \subseteq \mathbb{R}_\alpha$ be an α-interval. A function $f : I_\alpha \to \mathbb{R}$ is said to be α-harmonically convex if,*

$$f\left(\frac{a \dot{\times} b}{\alpha(t) \dot{\times} a + \alpha(1-t) \dot{\times} b}\dot{}\right) \leq t f(b) + (1-t) f(a) \tag{4}$$

inequality holds.

Definition 2.7. *[8] Let $g : \dot{[}a, b\dot{]} \to \mathbb{R}$ is a function. If the function meets the following requirement*

$$g\left(\frac{a \dot{\times} b}{x}\dot{}\right) = g\left(\frac{a \dot{\times} b}{a \dot{+} b \dot{-} x}\dot{}\right), \tag{5}$$

then the function g is an α-symmetric according to $\left(\frac{a \dot{+} b}{\dot{2}}\dot{}\right)$

Definition 2.8. *A function $g : \dot{[}a, b\dot{]} \subseteq \mathbb{R}_\alpha \dot{\setminus} \{\dot{0}\} \to \mathbb{R}$ is said to be α-harmonically symmetric with respect to $\frac{\alpha(2) \dot{\times} a \dot{\times} b}{a \dot{+} b}\dot{}$, if*

$$g(x) = g\left(\frac{\alpha(1)}{\frac{\alpha(1)}{a}\dot{} \dot{+} \frac{\alpha(1)}{b}\dot{} \dot{-} \frac{\alpha(1)}{x}\dot{}}\dot{}\right) \tag{6}$$

holds for all $x \in \dot{[}a, b\dot{]}$

Theorem 2.4. *[23] Let I_α be a closed interval in $\mathbb{R}_\alpha$, and $f : I_\alpha \to \mathbb{R}$ also be any α-convex function. Then the following double inequality holds for all $a, b \in I_\alpha$,*

$$f\left(\alpha\left(\frac{1}{2}\right)\dot{\times}(a\dot{+}b)\right) \leq \int_0^1 f(\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b)dt \leq \frac{f(a)+f(b)}{2}. \quad (7)$$

3. MAIN RESULTS

Definition 3.1. *Let $I_\alpha \subset (0,\infty)_\alpha$ be an α-interval and $p \in \mathbb{R}_\alpha \dot{\backslash} \{\dot{0}\}$. A function $f : I_\alpha \to \mathbb{R}$ is said to be $(\alpha; p)$ or p_α-convex, if*

$$f([\lambda_1\dot{\times}x^p\dot{+}\lambda_2\dot{\times}y^p]^{\frac{\dot{1}}{p}\cdot}) \leq \theta_1 f(x) + \theta_2 f(y) \quad (8)$$

inequality holds. Where $\lambda_1\dot{+}\lambda_2 = \dot{1}$ and $\theta_1 + \theta_2 = 1$. if we take $\lambda_1 = \alpha(t)$, $\lambda_2 = \alpha(1-t)$ and $\theta_1 = t$, $\theta_2 = 1-t$, then we get below inequality

$$f([\alpha(t)\dot{\times}x^p\dot{+}\alpha(1-t)\dot{\times}y^p]^{\frac{\dot{1}}{p}\cdot}) \leq tf(x) + (1-t)f(y) \quad (9)$$

for all $x, y \in \mathbb{R}_\alpha$ and $t \in [0,1]$.

Remark 3.1. *In* (9), *one can see the followings,*

1 *if one takes $p = \dot{1}$, one has* (3)

2 *if one takes $p = \dot{-}\dot{1}$, one has* (4)

Definition 3.2. *A function $w : [a,b]_\alpha \subset (0,\infty)_\alpha \to \mathbb{R}$ is said to be p_α-symmetric with respect to $\left[\frac{a^p\dot{+}b^p}{\dot{2}}\cdot\right]^{\frac{\dot{1}}{p}}$ if*

$$w(x) = w([a^p\dot{+}b^p\dot{-}x^p]^{\frac{\dot{1}}{p}\cdot})$$

holds for all $x \in [a,b]_\alpha$, $p \in \mathbb{R}_\alpha \dot{\backslash} \{\dot{0}\}$.

Remark 3.2. *In definition 3.2, one can see the following:*

1 *If we take $p = \dot{1}$, then it is α-symmetric function definition,*

2 *If we take $p = \dot{-}\dot{1}$, then it is α-harmonically symmetric function definition.*

Theorem 3.1. *Let $f : \dot{[}a,b\dot{]} \subset \mathbb{R}_\alpha \to \mathbb{R}$ be an α-convex function and $g : \dot{[}a,b\dot{]} \subset \mathbb{R}_\alpha \to \mathbb{R}$ is a nonnegative, integrable and α-symmetric according to $\left(\frac{a\dot{+}b}{\dot{2}}\cdot\right)$. Then, the following inequality holds*

$$\begin{aligned} f\left(\frac{a\dot{+}b}{\dot{2}}\cdot\right)\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}(g\circ\alpha)(x)dx &\leq \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}(f\circ\alpha)(x)(g\circ\alpha)(x)dx \\ &\leq \frac{f(a)+f(b)}{2}\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}(g\circ\alpha)(x)dx. \end{aligned}$$

Proof. For all $t \in [0,1]$, we can write below the inequality,

$$\begin{aligned} f\left(\frac{a\dot{+}b}{\dot{2}}.\right) &= f\left(\frac{\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b\dot{+}\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a}{\dot{2}}.\right) \\ &\leq \frac{f(\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b)}{2} + \frac{f(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a)}{2} \end{aligned} \tag{10}$$

Multiplying both sides of (10) by $g(f(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a))$ and integrating to t over $[0,1]$, we have the inequality holds;

$$\begin{aligned} &\int_0^1 f\left(\frac{a\dot{+}b}{\dot{2}}.\right) g(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a)dt \\ &\leq \int_0^1 \left[\frac{f(\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b)}{2} g(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a)dt\right] \\ &+ \int_0^1 \left[\frac{f(\alpha(t)\times b\dot{+}\alpha(1-t)\dot{\times}a)}{2} g(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a)dt\right] \\ &\Rightarrow f\left(\frac{a\dot{+}b}{\dot{2}}.\right) \int_0^1 g(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a)dt \\ &\leq \int_0^1 \frac{(f\circ\alpha)(t\alpha^{-1}(a)+(1-t)\alpha^{-1}(b))(g\circ\alpha)(t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a))}{2}dt \\ &+ \int_0^1 \frac{(f\circ\alpha)(t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a))(g\circ\alpha)(t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a))}{2}dt \end{aligned} \tag{11}$$

If we take $x = t\alpha^{-1}(b) + (1-t)\alpha^{-1}(a)$ in (11) and consider that g function is an α-symmetric according to $\left(\frac{a\dot{+}b}{\dot{2}}.\right)$ inequality becomes as follows

$$\begin{aligned} f\left(\frac{a\dot{+}b}{\dot{2}}.\right) \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} (g\circ\alpha)(x)dx &\leq \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \frac{(f\circ\alpha)(x)(g\circ\alpha)(x)}{2}dx \\ &+ \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \frac{(f\circ\alpha)(x)(g\circ\alpha)(x)}{2}dx \\ \Rightarrow f\left(\frac{a\dot{+}b}{\dot{2}}.\right) \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} (g\circ\alpha)(x)dx &\leq \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} (f\circ\alpha)(x)(g\circ\alpha)(x)dx \end{aligned} \tag{12}$$

So, proof of left hand of (10) completes. Now we prove the second part. Since, f is an α-convex function, then we can write the below the inequality,

$$f(\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b) + f(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a) \leq f(a) + f(b). \tag{13}$$

Multiplying both sides of (13) by $g(f(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a))$ and integrating to t over $[0,1]$, we get the followings,

$$\begin{aligned} &\int_0^1 f(\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b)g(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a)dt \\ + &\int_0^1 f(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a)g(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a)dt \\ \leq &\int_0^1 [f(a)+f(b)]g(\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a)dt. \end{aligned} \tag{14}$$

$$\begin{array}{rl} & \int_0^1 (f\circ\alpha)(t\alpha^{-1}(a)+(1-t)\alpha^{-1}(b))(g\circ\alpha)(t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a))dt \\ + & \int_0^1 (f\circ\alpha)(t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a))(g\circ\alpha)(t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a))dt \\ \leq & \int_0^1 [f(a)+f(b)](g\circ\alpha)(t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a))dt. \end{array}$$

If we take $x = t\alpha^{-1}(b) + (1-t)\alpha^{-1}(a)$ in (15) inequality and consider that g function is α-symmetric according to $\left(\frac{a\dot{+}b}{\dot{2}}.\right)$ inequality becomes as follows

$$\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} (f\circ\alpha)(x)(g\circ\alpha)(x)dx \leq \frac{f(a)+f(b)}{2}\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} (g\circ\alpha)(x)dx. \tag{15}$$

So, we have

$$\begin{array}{rcl} f\left(\frac{a\dot{+}b}{\dot{2}}.\right)\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}(g\circ\alpha)(x)dx & \leq & \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}(f\circ\alpha)(x)(g\circ\alpha)(x)dx \\ & \leq & \frac{f(a)+f(b)}{2}\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}(g\circ\alpha)(x)dx. \end{array}$$

The proof is completed. ∎

Corollary 3.1. *[2] we obtain the Hermite-Hadamard-Fejer inequality for $\alpha = I$ in Theorem 3.1. Namely,*

$$\begin{array}{rcl} f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx & \leq & \int_a^b f(x)g(x)dx \\ & \leq & \frac{f(a)+f(b)}{2}\int_a^b g(x)dx. \end{array}$$

Corollary 3.2. *we obtain the following inequality for $\alpha = exp$ in Theorem 3.1*

$$f\left(\frac{e^{\ln ab}}{2}\right)\int_{\ln a}^{\ln b} g(e^x)dx \leq \int_{\ln a}^{\ln b} f(e^x)g(e^x)dx \leq \frac{f(a)+f(b)}{2}\int_{\ln a}^{\ln b} g(e^x)dx.$$

Theorem 3.2. *Let $I_\alpha := [a,b]_\alpha$, $a\dot{\leq}b$ and $f : [a,b]_\alpha \subset \mathbb{R}_\alpha\dot{\setminus}\{\dot{0}\} \to \mathbb{R}$ be an α-harmonically convex function. Then we have the following inequality,*

$$\begin{array}{rcl} f\left(\frac{\alpha(2)\dot{\times}a\dot{\times}b}{a\dot{+}b}.\right) & \leq & \frac{\alpha^{-1}(a)\alpha^{-1}(b)}{\alpha^{-1}(b)-\alpha^{-1}(a)}\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}\frac{(f\circ\alpha)(x)}{x^2}dx \\ & \leq & \frac{f(a)+f(b)}{2}. \end{array}$$

Proof. Since f α- harmonically convex function for all $t \in [0,1], u, v \in \mathbb{R}_\alpha$, we obtain the following

$$f\left(\frac{u\dot{\times}v}{\alpha(t)\dot{\times}u\dot{+}\alpha(1-t)\dot{\times}v}.\right) \leq tf(v) + (1-t)f(u). \tag{16}$$

If we choose $t = \frac{1}{2}$ in (16), then we get the below inequality,

$$f\left(\frac{\dot{2}\dot{\times}u\dot{\times}v}{u\dot{+}v}.\right) \leq \frac{f(u)+f(v)}{2}. \tag{17}$$

In other words, if we choose,

$$u = \frac{a\dot{\times}b}{\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b}.$$

and

$$v = \frac{a\dot{\times}b}{\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a}.$$

in (17), then we have the followings,

$$\begin{aligned} f\left(\frac{\dot{2}\dot{\times}u\dot{\times}v}{u\dot{+}v}.\right) = f\left(\frac{\dot{2}\dot{\times}a\dot{\times}b}{a\dot{+}b}.\right) &\leq \frac{1}{2}\Bigg[f\left(\frac{a\dot{\times}b}{\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b}.\right) \\ &+ f\left(\frac{a\dot{\times}b}{\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a}.\right)\Bigg]. \end{aligned} \tag{18}$$

If we take the integrate to t over $[0,1]$ of (18), then we have the below inequality,

$$\begin{aligned} \int_0^1 f\left(\frac{\dot{2}\dot{\times}u\dot{\times}v}{u\dot{+}v}.\right) &\leq \frac{1}{2}\Bigg[\int_0^1 (f\circ\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(a)+(1-t)\alpha^{-1}(b)}\right)dt \\ + \int_0^1 (f\circ\alpha)&\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)}\right)dt\Bigg]. \end{aligned} \tag{19}$$

If we choose $x = \frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(a)+(1-t)\alpha^{-1}(b)}$ in (19), then we obtain the following inequality,

$$f\left(\frac{\dot{2}\dot{\times}u\dot{\times}v}{u\dot{+}v}.\right) \leq \frac{\alpha^{-1}(a)\alpha^{-1}(b)}{\alpha^{-1}(b)-\alpha^{-1}(a)}\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \frac{(f\circ\alpha)(x)}{x^2}dx$$

If we choose $u = a$ and $v = b$ in (17), then we can write the below inequality,

$$\begin{aligned} f\left(\frac{\alpha(2)\dot{\times}a\dot{\times}b}{a\dot{+}b}.\right) &\leq \frac{\alpha^{-1}(a)\alpha^{-1}(b)}{\alpha^{-1}(b)-\alpha^{-1}(a)}\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \frac{(f\circ\alpha)(x)}{x^2}dx \\ &\leq \frac{f(a)+f(b)}{2}. \end{aligned} \tag{20}$$

So, the proof is completed.

Corollary 3.3. *If we take $\alpha = I$ in Theorem 3.2, then we obtain the following inequality in [3],*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2} \tag{21}$$

Corollary 3.4. *If we take $\alpha = exp$ in Theorem 3.1, then we have the below inequality,*

$$f\left(2^{\frac{\ln a \ln b}{\ln ab}}\right) \leq \frac{\ln a \ln b}{\ln b - \ln a}\int_{\ln a}^{\ln b}\frac{f(e^x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}.$$

Theorem 3.3. *Let $I_\alpha \subset \mathbb{R}_\alpha$, $a\dot{\leq}b$ and $f : I_\alpha \subset \mathbb{R}_\alpha\dot{\setminus}\{\dot{0}\} \to \mathbb{R}$ be an α-harmonically convex function. $f \in L[a,b]_\alpha$ and $w : [a,b]_\alpha \subset \mathbb{R}_\alpha\dot{\setminus}\{\dot{0}\} \to \mathbb{R}$ nonnegative, integrable and α-symmetric to $\left[\frac{\alpha(2)\dot{\times}a\dot{\times}b}{a\dot{+}b}\right]$. Then we get inequality,*

$$\begin{aligned} & f\left(\frac{\alpha(2)\dot{\times}a\dot{\times}b}{a\dot{+}b}\dot{}\right)\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}\frac{(w\circ\alpha)(x)}{x^2}dx \leq \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}\frac{(f\circ\alpha)(x)(w\circ\alpha)(x)}{x^2}dx \\ \leq\ & \frac{f(a)+f(b)}{2}\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}\frac{(w\circ\alpha)(x)}{x^2}dx. \end{aligned}$$

Proof. If f α- harmonically convex function, we can write the following inequality,

$$f\left(\frac{\dot{2}\dot{\times}u\dot{\times}v}{u\dot{+}v}\dot{}\right) \leq \frac{f(u)+f(v)}{2}. \tag{22}$$

In other words, if we choose,

$$u = \frac{a\dot{\times}b}{\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b}\dot{}.$$

and

$$v = \frac{a\dot{\times}b}{\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a}\dot{}.$$

in (22), then we have the following inequality,

$$\begin{aligned} f\left(\frac{\dot{2}\dot{\times}u\dot{\times}v}{u\dot{+}v}\dot{}\right) = f\left(\frac{\dot{2}\dot{\times}a\dot{\times}b}{a\dot{+}b}\dot{}\right) &\leq \frac{1}{2}\Bigg[f\left(\frac{a\dot{\times}b}{\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b}\dot{}\right) \\ &+ f\left(\frac{a\dot{\times}b}{\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a}\dot{}\right)\Bigg]. \end{aligned} \tag{23}$$

Multiplying both sides of (23) by

$$w\left(\frac{a\dot{\times}b}{\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a}\dot{}\right)$$

and integrating to t over $[0,1]$, we get the below,

$$\begin{aligned} & \int_0^1 f\left(\frac{\dot{2}\dot{\times}a\dot{\times}b}{a\dot{+}b}\dot{}\right)w\left(\frac{a\dot{\times}b}{\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a}\dot{}\right)dt \\ \leq\ & \frac{1}{2}\Bigg[\int_0^1 f\left(\frac{a\dot{\times}b}{\alpha(t)\dot{\times}a\dot{+}\alpha(1-t)\dot{\times}b}\dot{}\right)w\left(\frac{a\dot{\times}b}{\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a}\dot{}\right)dt \\ +\ & \int_0^1 f\left(\frac{a\dot{\times}b}{\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a}\dot{}\right)w\left(\frac{a\dot{\times}b}{\alpha(t)\dot{\times}b\dot{+}\alpha(1-t)\dot{\times}a}\dot{}\right)dt\Bigg]. \end{aligned}$$

$$
\begin{aligned}
& f\left(\frac{\dot{2}\dot{\times}a\dot{\times}b}{a\dot{+}b}\dot{.}\right)\int_0^1(w\circ\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)}\right)dt \\
\leq\ & \frac{1}{2}\Bigg[\int_0^1(f\circ\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(a)+(1-t)\alpha^{-1}(b)}\right)(w\circ\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)}\right)dt \\
+\ & \int_0^1(f\circ\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)}\right)(w\circ\alpha)\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)}dt\right]\dot{.}
\end{aligned}
$$

If we choose

$$x=\left(\frac{\alpha^{-1}(a)\alpha^{-1}(b)}{t\alpha^{-1}(b)+(1-t)\alpha^{-1}(a)}\right)$$

in (24), then we obtain the below inequality holds,

$$f\left(\frac{\alpha(2)\dot{\times}a\dot{\times}b}{a\dot{+}b}\dot{.}\right)\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}\frac{(w\circ\alpha)(x)}{x^2}dx\leq\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}\frac{(f\circ\alpha)(x)(w\circ\alpha)(x)}{x^2}dx.$$

If we select $u=a$ and $v=b$ in (22), then we have the following inequality,

$$
\begin{aligned}
& f\left(\frac{\alpha(2)\dot{\times}a\dot{\times}b}{a\dot{+}b}\dot{.}\right)\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}\frac{(w\circ\alpha)(x)}{x^2}dx\leq\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}\frac{(f\circ\alpha)(x)(w\circ\alpha)(x)}{x^2}dx \\
\leq\ & \frac{f(a)+f(b)}{2}\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}\frac{(w\circ\alpha)(x)}{x^2}dx.
\end{aligned}
$$

∎

Corollary 3.5. *If we take $\alpha=I$ in Theorem 3.3, then we obtain the following inequality in [4],*

$$
\begin{aligned}
f\left(\frac{2ab}{a+b}\right)\int_a^b\frac{w(x)}{x^2}dx &\leq \int_a^b\frac{f(x)w(x)}{x^2}dx \\
&\leq \frac{f(a)+f(b)}{2}\int_a^b\frac{w(x)}{x^2}dx.
\end{aligned}
\tag{24}
$$

Corollary 3.6. *We obtain the following inequality for $\alpha=exp$ in Theorem 3.3,*

$$f\left(2^{\frac{\ln a\ln b}{\ln ab}}\right)\int_{\ln a}^{\ln b}\frac{w(e^x)}{x^2}dx\leq\int_{\ln a}^{\ln b}\frac{f(e^x)w(e^x)}{x^2}dx\leq\frac{f(a)+f(b)}{2}\int_{\ln a}^{\ln b}\frac{w(e^x)}{x^2}dx.$$

Theorem 3.4. *Let $f:I_\alpha\subset(0,\infty)_\alpha\to\mathbb{R}$ be a p_α-convex function, $p\in\mathbb{R}_\alpha\dot{\setminus}\{\dot{0}\}$, $a,b\in I_\alpha$ with $a\dot{<}b$. If $f\in L[a,b]_\alpha$ and $w:[a,b]_\alpha\to\mathbb{R}$ is nonnegative, integrable and p_α-symmetric with respect to $\left[\frac{a^p\dot{+}b^p}{\dot{2}}\dot{.}\right]$, then the following inequality holds,*

$$
\begin{aligned}
& f\left(\left[\frac{a^p\dot{+}b^p}{\dot{2}}\dot{.}\right]^{\frac{\dot{1}}{p}\dot{.}}\right)\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}\frac{(w\circ\alpha)(x)}{x^{1-\alpha^{-1}(p)}}dx\leq\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}\frac{(f\circ\alpha)(w\circ\alpha)(x)}{x^{1-\alpha^{-1}(p)}}dx \\
\leq\ & \frac{f(a)+f(b)}{2}\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)}\frac{(w\circ\alpha)(x)}{x^{1-\alpha^{-1}(p)}}dx.
\end{aligned}
$$

Proof. Let $p\dot{>}\dot{0}$. Since $f : I_\alpha \subset (0,\infty)_\alpha \to \mathbb{R}$ is a p_α-convex function, for all $x, y \in I_\alpha$, when $t = \frac{1}{2}$ is taken in (9) inequality we have the below,

$$f\left(\left[\frac{x^p \dot{+} y^p}{\dot{2}}\right]^{\frac{\dot{1}}{p}\dot{}}\right) \leq \frac{f(x)+f(y)}{2}. \tag{25}$$

If we choose $x = [\alpha(t)\dot{\times}a^p \dot{+} \alpha(1-t)\dot{\times}b^p]^{\frac{\dot{1}}{p}\dot{}}$ and $y = [\alpha(t)\dot{\times}b^p \dot{+} \alpha(1-t)\dot{\times}a^p]^{\frac{\dot{1}}{p}\dot{}}$ in (25), then we get the following inequality,

$$\begin{aligned} f\left(\left[\frac{a^p \dot{+} b^p}{\dot{2}}\right]^{\frac{\dot{1}}{p}\dot{}}\right) &\leq \frac{f([\alpha(t)\dot{\times}a^p \dot{+} \alpha(1-t)\dot{\times}b^p]^{\frac{\dot{1}}{p}\dot{}})}{2} \\ &+ \frac{f([\alpha(t)\dot{\times}b^p \dot{+} \alpha(1-t)\dot{\times}a^p]^{\frac{\dot{1}}{p}\dot{}})}{2}. \end{aligned} \tag{26}$$

Multiplying both sides of (26) by

$$w\big([\alpha(t)\dot{\times}a^p \dot{+} \alpha(1-t)\dot{\times}b^p]^{\frac{\dot{1}}{p}}\big).$$

and integrating respect to t over $[0,1]$, then we get the below inequality,

$$\begin{aligned} &\int_0^1 f\left(\left[\frac{a^p \dot{+} b^p}{\dot{2}}\right]^{\frac{\dot{1}}{p}\dot{}}\right) w([\alpha(t)\dot{\times}a^p \dot{+} \alpha(1-t)\dot{\times}b^p]^{\frac{\dot{1}}{p}}).dt \\ \leq\ &\int_0^1 \frac{f([\alpha(t)\dot{\times}a^p \dot{+} \alpha(1-t)\dot{\times}b^p]^{\frac{\dot{1}}{p}\dot{}}) w([\alpha(t)\dot{\times}a^p \dot{+} \alpha(1-t)\dot{\times}b^p]^{\frac{\dot{1}}{p}}).}{2} dt \\ +\ &\int_0^1 \frac{f([\alpha(t)\dot{\times}b^p \dot{+} \alpha(1-t)\dot{\times}a^p]^{\frac{\dot{1}}{p}\dot{}}) w([\alpha(t)\dot{\times}a^p \dot{+} \alpha(1-t)\dot{\times}b^p]^{\frac{\dot{1}}{p}}).}{2} dt \end{aligned}$$

$$\begin{aligned} &f\left(\left[\frac{a^p \dot{+} b^p}{\dot{2}}\right]^{\frac{\dot{1}}{p}\dot{}}\right) \int_0^1 (w\circ\alpha)([t\alpha^{-1}(a)^{\alpha^{-1}(p)} + (1-t)\alpha^{-1}(b)^{\alpha^{-1}(p)}]^{\frac{1}{\alpha^{-1}(p)}})dt \\ \leq\ &\int_0^1 \frac{(f\circ\alpha)([t\alpha^{-1}(a)^{\alpha^{-1}(p)} + (1-t)\alpha^{-1}(b)^{\alpha^{-1}(p)}]^{\frac{1}{\alpha^{-1}(p)}})}{2} dt \\ &\times \frac{(w\circ\alpha)([t\alpha^{-1}(a)^{\alpha^{-1}(p)} + (1-t)\alpha^{-1}(b)^{\alpha^{-1}(p)}]^{\frac{1}{\alpha^{-1}(p)}})}{2} dt \\ +\ &\int_0^1 \frac{(f\circ\alpha)([t\alpha^{-1}(b)^{\alpha^{-1}(p)} + (1-t)\alpha^{-1}(a)^{\alpha^{-1}(p)}]^{\frac{1}{\alpha^{-1}(p)}})}{2} dt \\ &\times \frac{(w\circ\alpha)([t\alpha^{-1}(a)^{\alpha^{-1}(p)} + (1-t)\alpha^{-1}(b)^{\alpha^{-1}(p)}]^{\frac{1}{\alpha^{-1}(p)}})}{2} dt. \end{aligned}$$

If we take $x = \big[\alpha(t)\dot{\times}a^p \dot{+} \alpha(1-t)\dot{\times}b^p\big]^{\frac{\dot{1}}{p}\dot{}}$ in (27), then we can write the below inequality,

$$f\left(\left[\frac{a^p \dot{+} b^p.}{\dot{2}}.\right]^{\frac{\dot{1}}{p}\dot{}}\right) \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \frac{(w\circ\alpha)(x)}{x^{1-\alpha^{-1}(p)}} dx \leq \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \frac{(f\circ\alpha)(w\circ\alpha)(x)}{x^{1-\alpha^{-1}(p)}} dx.$$

So, the proof of the left hand of (25) completes. Now we prove the second part. We know that

$$\frac{f([\alpha(t)\dot{\times}a^p\dot{+}\alpha(1-t)\dot{\times}b^p]^{\frac{\dot{1}}{p}\dot{}})}{2} + \frac{f([\alpha(t)\dot{\times}b^p\dot{+}\alpha(1-t)\dot{\times}a^p]^{\frac{\dot{1}}{p}\dot{}})}{2} \leq \frac{f(a)+f(b)}{2}.$$

the inequality holds. Multiplying both sides of (27) by

$$w([\alpha(t)\dot{\times}a^p\dot{+}\alpha(1-t)\dot{\times}b^p]^{\frac{\dot{1}}{p}\dot{}})$$

integrating respect to t over $[0,1]$ and

$$x = [\alpha(t)\dot{\times}a^p\dot{+}\alpha(1-t)\dot{\times}b^p]^{\frac{\dot{1}}{p}\dot{}}$$

changing variables we have the following inequality.

$$\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \frac{(f\circ\alpha)(w\circ\alpha)(x)}{x^{1-\alpha^{-1}(p)}}dx \leq \frac{f(a)+f(b)}{2}\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \frac{(w\circ\alpha)(x)}{x^{1-\alpha^{-1}(p)}}dx.$$

So, proof is completed. ∎

Corollary 3.7. *If we take $\alpha = I$ in Theorem 3.4, then we obtain the following inequality in [6],*

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx \leq \frac{f(a)+f(b)}{2}.$$

Corollary 3.8. *the following inequality holds for $\alpha = exp$ in Theorem 3.4,*

$$f\left(\left[e^{\frac{\ln a^{e^p}\ln b^{e^p}}{2}}\right]^{e^{\frac{1}{p}}}\right)\int_{\ln a}^{\ln b}\frac{w(e^x)}{x^{1-\ln p}}dx \leq \int_{\ln a}^{\ln b}\frac{f(e^x)w(e^x)}{x^{1-\ln p}}dx \leq \frac{f(a)+f(b)}{2}\int_{\ln a}^{\ln b}\frac{w(e^x)}{x^{1-\ln p}}dx.$$

Remark 3.3. *You can easily seen that the following inequality for (25),*

1. *If one takes $p = \dot{1}$ and $(w\circ\alpha)(x) = 1$, one has (7)*
2. *If one takes $p = \dot{1}$, one has (10)*
3. *If one takes $p = \dot{-}\dot{1}$ and $(w\circ\alpha)(x) = 1$, one has (16)*
4. *If one takes $p = \dot{-}\dot{1}$, one has (22)*
5. *If one takes $\alpha = I$ and $p = \dot{1}$, one has (1)*
6. *If one takes $\alpha = I$, $p = \dot{-}\dot{1}$ and $w(x) = 1$, one has (2)*
7. *If one takes $\alpha = I$ and $p = \dot{-}\dot{1}$, one has (24)*
8. *If one takes $\alpha = I$ and $w(x) = 1$, one has (5)*
9. *If one takes $\alpha = I$, $p = \dot{1}$ and $(w\circ\alpha)(x) = 1$, one has (1)*

References

[1] M. Grosmann, R. Katz, *Non- Newtonian Calculus*, Lee Press Pigeon Cove, Massachusetts, 1972.

[2] L. Fejer, *ber die Fourierreihen*, II. Math. Naturwiss. Anz Ungar. Akad., Wiss 369-390, 24, 1906.

[3] İ. İdscan, 2014, *Hermite-Hadamard type inequalities for harmonically convex functions.* Hacet. J. Math. Stat., 43(6), 935-942.

[4] F. Chen, S. Wu, *Fejér and Hermite-Hadamard type inequalities for harmonically convex functions.* J. Appl. Math., 386806, 2014.

[5] U. Kadak, *Non-Newtonian Calculus and its Applications*, Ph. D. Thesis, Gazi University, 2015.

[6] İ. İdscan, *Hermite-Hadamard type inequalities for p-convex functions*, Int. J. Anal. Appl. 11(2), 137-145, 2016.

[7] M. Kunt, İ. İdscan, *Hermite-Hadamard-Fejér type inequalities for p-convex functions*, Arab J. Math. Sci, 23, 215-230, 2017.

[8] E. Ünlüyol, Y. Erdaş, *Some New Inequalities and Hermite-Hadamard-Fejer Inequality via Non-Newtonian Calculus*, 4th International Conference on Analysis and Its Applications, p.33, 2018.

[9] E. Ünlüyol, Y. Erdaş, S. Salaş, *∗-Hermite-Hadamard-Fejer Inequality and Some New Inequality via ∗-Calculus*, Sigma Journal of Engineering and Natural Sciences accepted, 2019.

[10] D. Aniszewska, *Multiplicative Runge-Kutta Metods*, Nonlinear Dyn., 50 , 265-272, 2007.

[11] A. E. Bashirov, E. M. Kurpınar, A. Özyapıcı, *Multiplicative calculus and its applications*, J. Math. Anal. Appl. 337, 36-48 (2008).

[12] M. Rıza, A.Özyapıcı, E. Mısırlı, *Multiplicative Finite Difference Methods*, Q. Appl. Math., 67, 745-754, 2009.

[13] A. Uzer, *Multiplicative type complex calculus as an alternative to the classical calculus*, Comp. Math. Appl., 60, 2725-2737, 2010.

[14] E. Mısırlı, Y. Gürefe, *Multiplicative Adams Bashforth Moulton methods*, Numre. Algor., 57, 425-439, 2011.

[15] A.E. Bashirov, M. Rıza, *On complex multiplicative differentiation*, Twns. J. Appl. Eng. Math., 1(1), 75-85, 2011.

[16] A. F. Çakmak, F. Başar, *Some new results on sequence spaces with respect to non-Newtonian calculus*, J. Inequal. Appl., 2012:228, 2012.

[17] U. Kadak, H. Efe, *Matrix Transformations between Certain Sequence Spaces over the Non- Newtonian Complex Field*, The Scientific World Journal, vol. 2014, Article ID 705818, 12 pages, 2014. doi:10.1155/2014/705818.

[18] U. Kadak, H. Efe, *The Construction of Hilbert Spaces over the Non-Newtonian Field*, Int. J. Anal., vol. 2014, Article ID 746059, 2014. doi:10.1155/2014/746059.

[19] U. Kadak, M. Kirişçi, A. F. Çakmak, *On the classical paranormed sequence spaces and related duals over the non-Newtonian complex field*, J. Funct. Spaces, vol. 2015, Article ID 416906, 2015. doi:10.1155/2015/416906.

[20] S. Tekin, F. Başar, *Some basic results on the sets of sequences with geometric calculus*, AIP Conf. Proc., 1470, 95-98, 2012.

[21] S. Tekin, F. Başar, *Certain sequence spaces over the non- Newtonian complex field*, Abstr. Appl. Anal., 2013, Article ID 739319, 2013.

[22] C. Türkmen, F. Başar, *Some basic results on the geometric calculus*, Commun. Fac. Sci Univ. Ank. Series. A1, 61(2), 17-34, 2012.

[23] E. Ünlüyol, S. Salaş, *Convexity and Hermite-Hadamard Type Inequality via Non-Newtonian Calculus*, Konuralp Math. Journal, 7(2), 352-358, 2019.