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THE EVOLUTION OF A *SEIRS-SI* MALARIA MODEL: DETERMINISTIC AND STOCHASTIC PERSPECTIVES

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Abstract In order to study malaria transmission, we first propose a 6-dimensional deterministic ODE model which keeps track of both host (human) and vector (mosquito) populations and uses standard incidence terms to model the transmission of the disease. First, the stability of the equilibria is characterized in terms of an explicitly determined basic reproduction number, obtained via the next generation method. Then, to examine the effects of a randomly fluctuating environment, we introduce multiple perturbations of white noise type and discuss the asymptotic behavior of the solutions of the corresponding stochastic model around the steady states of the initial deterministic model.

Keywords: malaria transmission, deterministic model, stochastic model, basic reproduction number, Itô lemma.
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1. INTRODUCTION

Malaria is a disease caused by parasites of the genus *Plasmodium*, which spread to humans through the bites of infected female *Anopheles* mosquitoes, the vector of this disease. Although there are over 100 species of this genus, only four of them (*P. falciparum*, *P. malariae*, *P. ovale* and *P. vivax*) are known to infect humans.

The symptoms of malaria usually appear in 10-15 days after the infected mosquito bite. The most severe form of this disease is caused by *P. falciparum*, found worldwide in tropical and subtropical areas. Its symptoms, which are initially mild and may include fever, headaches and muscular weakness can worsen to severe blood loss, acute renal failure, generalized convulsions and circulatory collapse, ending in coma and death. It is of paramount importance that this form of malaria be treated within 24 h from the onset of clinical symptoms.

The other forms of malaria, although potentially causing significant health damage, are not normally life-threatening. However, both *P. vivax* and *P. ovale* have dormant stages in the human liver as parts of their life cycles, which makes them difficult to eradicate. Relapses caused by these parasites can occur months after exposure. *P. vivax*, found mostly in Asia and Latin America, is probably the most prevalent human malaria parasite [1]. Latent blood infections with *P. malariae*, found worldwide, can persist for a longer timespan, often measured in years, but are very rarely lethal.

Although this disease is nowadays preventable and curable, its burden is tremendous. In 2017, there were approximately 219 million cases of malaria worldwide (down from an estimated total of 239 million in 2010), resulting in approximately 435,000 deaths, most of those occurring on the African continent [2]. In the United States, about 1,700 new cases of malaria are diagnosed each year, the vast majority being travelers and immigrants returning from countries in sub-Saharan Africa and South Asia, where this disease is endemic.

As far as China is concerned, the endemicity of malaria is caused primarily by *P. falciparium* and *P. vivax*, the main areas of endemicity being the jungles and mountainous areas of Central and South China. Although the burden of this disease is much lighter than in Africa, the incidence of malaria being reduced to less than 6 cases per one million residents in 2010 due to intensive eradication measures taken by the Chinese government [3], an increase in the number of cases of imported malaria due to the growth of the Chinese overseas travel has been reported [4].

In 2000, Ngwa and Shu [5] formulated and analyzed a deterministic ODE model for endemic malaria involving human and mosquito populations of variable sizes, sufficient conditions for the existence of the endemic and disease-free equilibria, expressed in terms of the basic reproduction number, being derived. Further, numerical simulations suggested that the endemic equilibrium is unique and globally stable whenever it exists. Also, a framework for studying control strategies for malaria control has been proposed on the basis of the above-mentioned analysis.

In 2006, Chitnis et al. [6] introduced an ODE model for the spread of malaria in human and mosquito populations. In the absence of disease-induced death, they proved that a supercritical (forward) bifurcation occurs at $R_0 = 1$. However, it was observed from numerical simulations that for large values of the disease-induced death rate, a subcritical (backward) bifurcation may occur at $R_0 = 1$ instead. Okosun et al. [7] proposed a deterministic model for the transmission of malaria which employs a mass action incidence term. It has been determined that a subcritical (backward) bifurcation may occur at $R_0 = 1$, necessary conditions for the optimal control of the disease being also determined. Further, the impact of a combined vaccination and treatment strategy on the the transmission of disease has been investigated from a numerical viewpoint.

To account for the influence of randomly fluctuating environments, Liu et al. [14] proposed and investigated from a stability viewpoint a two-group stochastic $SEIR$ model with infinite delays which also involves random changes described by Brownian motions, related approaches being developed by Chang et al. [12] and Liu et al. [13]. Also, Zhou et al. [15] et al. discussed the global stability of a stochastic $SIRS$ model with general nonlinear incidence rate by means of constructing suitable Lyapunov functionals.

In this paper, we formulate a deterministic model of malaria transmission which accounts for human and mosquito populations of variable sizes. The model consists of a 6-dimensional system of nonlinear ODEs which keeps track of 4 compartments for humans (leading to a $SEIRS$ submodel) and of 2 compartments for mosquitoes (leading to a SI submodel), these submodels being coupled via disease transmission terms involving the so-called standard incidence rate. Our model is inspired by those considered in Ngwa and Shu [5], Ngwa [8], Oduro et al. [9], Okosun et al. [10] and Rafikov [11]. The perturbing influence of the environment is then accounted for by considering multiple stochastic perturbations of white noise type and discussing the asymptotic behavior of the solutions of the stochastic system.

This remaining part of this paper is organized as follows. In Section 2, we introduce a deterministic model for the transmission of malaria. In Section 3, we propose the corresponding stochastic model and discuss the asymptotic behavior of its solutions around the equilibria of the deterministic system. Finally, in Section 4 we illustrate our theoretical results via numerical simulations and give several concluding remarks.

2. A DETERMINISTIC MODEL

To introduce our model, the total host (human) population, denoted by N_h, is divided into the following subpopulations:

S_h individuals who are susceptible to infection with malaria,

E_h individuals exposed to the malaria parasite who are not yet infective,

I_h individuals with malaria symptoms,

R_h recovered individuals.

The total vector (mosquito) population, denoted by N_v, is also divided into the following subpopulations:

S_v susceptible mosquitoes;

I_v infective mosquitoes.

Table 1 Parameters of the malaria model

Parameter	Description
μ_h	Natural death rate of humans
μ_v	Natural death rate of mosquitoes
Λ_h	Human birth rate
Λ_v	Mosquito birth rate
ϵ	Mosquito biting rate
ϕ	Contact rate between mosquitoes and humans
β	Probability of humans getting infected
λ	Probability of mosquitoes getting infected
α_1	Humans progression rate from exposed to infected
τ	Proportion of effectively treated individuals
ψ	Disease induced death reate
κ	Relapse rate
b	Spontaneous recovery rate

In view of the above compartment splitting, we can now state our deterministic malaria model as seen below:

$$\begin{cases} \dfrac{dS_h}{dt} = \Lambda_h - \mu_h S_h + \kappa R_h - \dfrac{\beta\epsilon\phi I_v S_h}{N_h}, \\ \dfrac{dE_h}{dt} = \dfrac{\beta\epsilon\phi I_v S_h}{N_h} - \mu_h E_h - \alpha_1 E_h, \\ \dfrac{dI_h}{dt} = \alpha_1 E_h - (\psi + \mu_h) I_h - b I_h, \\ \dfrac{dR_h}{dt} = b I_h - \mu_h R_h - \kappa R_h, \\ \dfrac{dS_v}{dt} = \Lambda_v - \mu_v S_v - \dfrac{\lambda\epsilon\phi I_h S_v}{N_h}, \\ \dfrac{dI_v}{dt} = \dfrac{\lambda\epsilon\phi I_h S_v}{N_h} - \mu_v I_v, \end{cases} \tag{1}$$

the biological significance of each parameter being given in Table 1. By a straightforward argument, one can obtain the feasible set for (1) as seen below.

Theorem 2.1. *The positively invariant set of model* (1) *is given by*

$$\mathcal{D} = \left\{ (S_h, E_h, I_h, R_h, S_v, I_v) \in \mathbb{R}_+^6 | N_h \leq \frac{\Gamma_h}{\mu_h},\ N_v \leq \frac{\Gamma_v}{\mu_v} \right\}.$$

By employing the next generation method exposed in [18] with the notations therein, the basic reproduction number (reproduction ratio) R_0 is given by

$$R_0 = \rho(FV^{-1})$$

Here, $\rho(A)$ denotes the spectral radius of a matrix A,

$$F = \begin{pmatrix} 0 & 0 & \beta\epsilon\phi \\ 0 & 0 & 0 \\ 0 & \frac{\lambda\epsilon\phi\Lambda_v\mu_h}{\Lambda_h\mu_v} & 0 \end{pmatrix}$$

and

$$V = \begin{pmatrix} \mu_h + \alpha_1 & 0 & 0 \\ -\alpha_1 & \psi + \mu_h + b & 0 \\ 0 & 0 & \mu_v \end{pmatrix}.$$

As a result, the basic reproduction number R_0 can be expressed in the form

$$R_0 = \sqrt{R_{0h} \cdot R_{0v}},$$

where

- $R_{0h} = \frac{\beta\epsilon\phi\alpha_1\mu_h}{\Lambda_h(\mu_h+\alpha_1)(\psi+\mu_h+b)}$ is the number of humans infected by a single mosquito during its infectious period (the reproduction number of the mosquito to human transmission);
- $R_{0v} = \frac{\lambda\epsilon\phi\Lambda_v}{\mu_v^2}$ is number of mosquitoes infected by a single human during the duration of his/her infectious period (the reproduction number of the human to mosquito transmission),

both subreproduction numbers being computed assuming that all humans and mosquitoes are susceptible. It can then be seen that the model has a disease-free equilibrium regardless of the value of R_0 and an endemic equilibrium if and only if $R_0 > 1$, whose stability can be determined via the results of [18].

Theorem 2.2. *The disease-free equilibrium* $E_0 = (S_h^0, 0, 0, 0, S_v^0, 0)$ *is locally asymptotically stable if* $R_0 < 1$ *and unstable if* $R_0 > 1$.

Theorem 2.3. *The unique endemic equilibrium* $E^* = (S_h^*, E_h^*, I_h^*, R_h^*, S_v^*, I_v^*)$ *is locally asymptotically stable if* $R_0 > 1$.

3. A STOCHASTIC MODEL

Since unforeseen environmental disturbances which have the potential to perturb the steady states of the system may occur, it is meaningful to include the effects of uncertainty into our deterministic model. We thereby consider the following stochastic model

$$\begin{cases} dS_h = \left(\Lambda_h - \mu_h S_h + \kappa R_h - \dfrac{\beta\epsilon\phi I_v S_h}{N_h}\right) dt + \sigma_1 S_h dB_1(t), \\ dE_h = \left(\dfrac{\beta\epsilon\phi I_v S_h}{N_h} - \mu_h E_h - \alpha_1 E_h\right) dt + \sigma_2 E_h dB_2(t), \\ dI_h = (\alpha_1 E_h - (\psi + \mu_h) I_h - b I_h) dt + \sigma_3 I_h dB_3(t), \\ dR_h = (b I_h - \mu_h R_h - \kappa R_h) dt + \sigma_4 R_h dB_4(t), \\ dS_v = \left(\Lambda_v - \mu_v S_v - \dfrac{\lambda\epsilon\phi I_h S_v}{N_h}\right) dt + \sigma_5 S_v dB_5(t), \\ dI_v = \left(\dfrac{\lambda\epsilon\phi I_h S_v}{N_h} - \mu_v I_v\right) dt + \sigma_6 I_v dB_6(t), \end{cases} \tag{2}$$

where $B_i(t)(i = 1, \ldots, 6)$ are mutually independent standard Brownian motions defined over a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_{t\geq 0}\}$ satisfying the usual conditions (i.e., it is right continuous and $\{\mathcal{F}_0\}$ contains all P-null sets). Here, $\sigma_i(i = 1, \ldots, 6)$ denote the intensities of the perturbations.

Definition 3.1. *Consider the $n-$dimensional stochastic differential equation*

$$du(t) = \mathbf{A}(t,\ u)dt + \mathbf{B}(t,\ u)dW(t) \ \text{for}\ t \geq t_0. \tag{3}$$

Let $V(t,\ u) \in C^{1,2}$ be a nonnegative continuously differentiable function, once with respect to t and twice with respect to u. Then the differential operator $\boldsymbol{L}$ applied to the function $V(t,u)$ corresponding to the stochastic differential equations (3) *with drift and diffusion coefficients $A(t,\ u)$ and $B(t,\ u)$, respectively, is given by*

$$\boldsymbol{L}V(t,\ u) = \frac{\partial V(t,\ u)}{\partial t} + A^T\frac{\partial V(t,\ u)}{\partial u} + \frac{1}{2}\text{trace}\left[B^T\frac{\partial^2 V(t,\ u)}{\partial^2 u}B\right].$$

3.1. EXISTENCE OF THE GLOBAL AND POSITIVE SOLUTION

Denote

$$\mathbf{R}_+^6 = \{(x_1, \cdots, x_6) | x_i > 0,\ i = 1, 2, \cdots, 6\}.$$

First, we are concerned with the well-posedness of (2) in a biological sense, which amounts to preservation of positivity and exclusion of blow-up.

Theorem 3.1. *For any given initial data in $\mathbf{R}^6_+$, the model (2) has a unique global solution that will also remain in $\mathbf{R}^6_+$ with probability 1.*

Proof One easily sees that the model (2) has a unique solution on $[0, \tau_e)$, where τ_e is the explosion time. To show that this unique solution is global, we only need to prove that $\tau_e = \infty$ a.s.. In fact, choose $\epsilon_0 > 0$ small enough such that the components of the initial data $S_h(0)$, $E_h(0)$, $I_h(0)$, $R_h(0)$, $S_v(0)$ and $I_v(0)$ all belong to $(\epsilon_0, \frac{1}{\epsilon_0},)$. For each $0 < \varepsilon \leq \epsilon_0$, we define the stopping time by

$$\tau_\varepsilon = \inf \{t \in [0, \tau_e) : \min\{S_h, E_h, I_h, R_h, S_v, I_v\} < \epsilon_0 \text{ or } \max\{S_h, E_h, I_h, R_h, S_v, I_v\} > \frac{1}{\epsilon_0}\right\}.$$

Since $\tau_\varepsilon \leq \tau_e$ a.s. and τ_ε increases as $\varepsilon \to 0$, it suffices to show that $\tau_0 = \lim_{\varepsilon\to\infty} \tau_\varepsilon = +\infty$ a.s., which implies that $\tau_e = +\infty$ a.s.

Suppose that this is not the case. Then there is a pair of constants $T > 0$ and $\delta \in (0, 1)$ such that $P(\tau_0 \leq T) > \delta$. As a result, there is a pair of positive constants $\epsilon_1 \leq \epsilon_0$ and ρ such that $P(\tau_\varepsilon \leq T) \geq \delta$ and $P(N_h(t) \geq \rho) = 1$ for all $t \in [0, \tau_\varepsilon)$ and any positive $\varepsilon \leq \epsilon_1$.

We define a function $V : \mathbf{R}^6_+ \to \mathbf{R}_+$ as follows

$$\begin{aligned} V(S_h, E_h, I_h, R_h, S_v, I_v) = {} & (S_h + 1 - \ln S_h) + (E_h + 1 - \ln E_h) \\ & + (I_h + 1 - \ln I_h) + (R_h + 1 - \ln R_h) \\ & + (S_v + 1 - \ln S_v) + (I_v + 1 - \ln I_v), \end{aligned} \tag{4}$$

In what follows, we shall denote, whenever necessary,

$$V(S_h(t), E_h(t), I_h(t), R_h(t), S_v(t), I_v(t)) \doteq V(t).$$

Also, for $a, b \in \mathbf{R}$, we shall denote $\max(a, b)$ by $a \vee b$ and $\min(a, b)$ by $a \wedge b$. Applying Itô lemma, we see that

$$\begin{aligned} dV = {} & \mathbf{L}V dt + (1 - \frac{1}{S_h})\sigma_1 S_h dB_1(t) + (1 - \frac{1}{E_h})\sigma_2 E_h dB_2(t) \\ & + (1 - \frac{1}{I_h})\sigma_3 I_h dB_3(t) + (1 - \frac{1}{R_h})\sigma_4 R_h dB_4(t) + (1 - \frac{1}{S_v})\sigma_5 S_v dB_5(t) \\ & + (1 - \frac{1}{I_v})\sigma_6 I_v dB_6(t), \end{aligned}$$

where

$$\begin{aligned}
\mathbf{L}V =&(1-\frac{1}{S_h})\left[\Lambda_h-\mu_h S_h+\kappa R_h-\frac{\beta\epsilon\phi I_v}{N_h}S_h\right]\\
&+(1-\frac{1}{E_h})\left[\frac{\beta\epsilon\phi I_v}{N_h}S_h-\mu_h E_h-\alpha_1 E_h\right]\\
&+(1-\frac{1}{I_h})\left[\alpha_1 E_h-(\psi+\mu_h)I_h-bI_h\right]+(1-\frac{1}{R_h})\left[bI_h-\mu_h R_h-\kappa R_h\right]\\
&+(1-\frac{1}{S_v})\left[\Lambda_v-\mu_v S_v-\frac{\lambda\epsilon\phi I_h}{N_h}S_v\right]+(1-\frac{1}{I_v})\left[\frac{\lambda\epsilon\phi I_h}{N_h}S_v-\mu_v I_v\right]\\
&+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_2^2+\frac{1}{2}\sigma_3^2+\frac{1}{2}\sigma_4^2+\frac{1}{2}\sigma_5^2+\frac{1}{2}\sigma_6^2\\
\le&(\Lambda_h+\Lambda_v+4\mu_h+\alpha_1+\kappa+\psi+b+2\mu_v\\
&+\frac{1}{2}(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2+\sigma_5^2+\sigma_6^2))+2(\lambda\epsilon\phi+\frac{\beta\epsilon\phi}{\rho}+1)V\\
\doteq&k_1+k_2V,
\end{aligned}$$

in which

$$k_1=\Lambda_h+\Lambda_v+4\mu_h+\alpha_1+\kappa+\psi+b+2\mu_v+\frac{1}{2}(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2+\sigma_5^2+\sigma_6^2)$$

and

$$k_2=2(\lambda\epsilon\phi+\frac{\beta\epsilon\phi}{\rho}+1).$$

Thus

$$\begin{aligned}
dV\le&(k_1+k_2V)+\sigma_1(S_h-1)dB_1(t)+\sigma_2(E_h-1)dB_2(t)\\
&+\sigma_3(I_h-1)dB_3(t)+\sigma_4(R_h-1)dB_4(t)+\sigma_5(S_v-1)dB_5(t)\\
&+\sigma_6(I_v-1)dB_6(t).
\end{aligned}\tag{5}$$

Integrating both sides of (5) from 0 to $\tau_\epsilon\wedge T$ and then taking expectation, it is seen that

$$EV(\tau_\epsilon\wedge T)\le V(0)+k_1(\tau_\epsilon\wedge T)+k_2E\int_0^{\tau_\epsilon\wedge T}V(t)dt$$

Using Gronwall inequality, we obtain

$$EV(\tau_\epsilon\wedge T)\le(V(0)+k_1(\tau_\epsilon\wedge T))e^{k_2(\tau_\epsilon\wedge T)}$$

Let $\Omega_\epsilon=\{\tau_\epsilon\le T\}$ for any positive $\epsilon\le\epsilon_1$. Then, as seen above, $P(\Omega_\epsilon)\ge\delta$. Note that for every $\omega\in\Omega_\epsilon$, at least one of the components of $V(\tau_\epsilon)$ equals either ϵ or $\frac{1}{\epsilon}$. Then

$$\begin{aligned}&V(S_h(\tau_\epsilon,\omega),E_h(\tau_\epsilon,\omega),I_h(\tau_\epsilon,\omega),R_h(\tau_\epsilon,\omega),S_v(\tau_\epsilon,\omega),I_v(\tau_\epsilon,\omega))\\&\quad\geq \min\{\epsilon+1-\ln\epsilon,\frac{1}{\epsilon}+1+\ln\epsilon\}.\end{aligned}$$

Consequently,

$$\begin{aligned}+\infty &> (V(S_h(0),E_h(0),I_h(0),R_h(0),S_v(0),I_v(0))+k_1T)e^{k_2T}\\&\geq E(I_{\Omega_\epsilon}(\omega)V(S_h(\tau_\epsilon,\omega),E_h(\tau_\epsilon,\omega),I_h(\tau_\epsilon,\omega),R_h(\tau_\epsilon,\omega),S_v(\tau_\epsilon,\omega),\\&\qquad I_v(\tau_\epsilon,\omega)))\\&> \delta\min\{\epsilon+1-\ln\epsilon,\frac{1}{\epsilon}+1+\ln\epsilon\}\to+\infty(\epsilon\to 0).\end{aligned}$$

where $I_{\Omega_\epsilon}(\omega)$ is the indicator function of Ω_ϵ. Letting $\epsilon\to 0$ then leads to a contradiction, as seen above. One consequently sees that $\tau_0=+\infty$, a.s., which implies that $\tau_e=+\infty$, a.s.. This completes the proof.□

3.2. ASYMPTOTIC BEHAVIOR AROUND THE DISEASE-FREE EQUILIBRIUM OF THE DETERMINISTIC MODEL

Due to the stochastic perturbations, the disease-free equilibrium is no longer a steady state for the system (2). It is then meaningful to investigate the asymptotic behavior of the solutions of (2) around this equilibrium, and this can be done provided that the strength of the perturbations does not exceed a certain level.

Theorem 3.2. *Let $\big(S_h(t),E_h(t),I_h(t),R_h(t),S_v(t),I_v(t)\big)$ be a solution of the system* (2) *starting in $\mathbf{R}^6_+$. Let us also denote*

$$\xi_1=\max\{\sigma_1^2,\frac{1}{2}\sigma_2^2,\frac{1}{2}\sigma_3^2,\frac{1}{2}\sigma_4^2\},\quad \xi_2=\max\{\sigma_5^2,\frac{1}{2}\sigma_6^2\}.$$

If $\xi_1<\mu_h$ and $\xi_2<\mu_v$, then

$$\limsup_{t\to+\infty}\frac{1}{t}E\int_0^t((S_h-S_h^0)^2+E_h^2+I_h^2+R_h^2)d\tau\leq\frac{\sigma_1^2{S_h^0}^2+\frac{\Lambda_h}{\mu_h}C_1}{\mu_h-\xi_1},$$

$$\limsup_{t\to+\infty}\frac{1}{t}E\int_0^t((S_v-S_v^0)^2+I_v^2)d\tau\leq\frac{\sigma_5^2{S_v^0}^2+\frac{\Lambda_v}{\mu_v}C_2}{\mu_v-\xi_2},$$

in which

$$C_1=(\psi+2\mu_h)S_h^0,\quad C_2=2\mu_vS_v^0.$$

Proof. We start by proving the first inequality, which quantifies the dynamics of the human compartments. First of all, let us define a functional $V_{11} : \mathbf{R}_+^4 \to [0,\infty)$ by

$$V_{11}(S_h, E_h, I_h, R_h) = \frac{1}{2}(S_h - S_h^0 + E_h + I_h + R_h)^2 \tag{6}$$

Using Itô formula, we see that

$$\begin{aligned} dV_{11} =& (S_h - S_h^0 + E_h + I_h + R_h)(dS_h + dE_h + dI_h + dR_h) \\ &+ (\frac{1}{2}\sigma_1^2 S_h^2 + \frac{1}{2}\sigma_2^2 E_h^2 + \frac{1}{2}\sigma_3^2 I_h^2 + \frac{1}{2}\sigma_4^2 R_h^2)dt. \end{aligned} \tag{7}$$

Since $\frac{\Lambda_h}{\mu_h} = S_h^0$, it follows that

$$\begin{aligned} & dS_h + dE_h + dI_h + dR_h \\ &= \left[-\mu_h(S_h - S_h^0) - (\psi + \mu_h)I_h - \mu_h E_h - \mu_h R_h\right] dt \\ &\quad + \sigma_1 S_h dB_1(t) + \sigma_2 E_h dB_2(t) + \sigma_3 I_h dB_3(t) + \sigma_4 R_h dB_4(t). \end{aligned} \tag{8}$$

Substituting (8) into (7), we get

$$\begin{aligned} dV_{11} =& \Big[(S_h - S_h^0 + E_h + I_h + R_h)(-\mu_h(S_h - S_h^0) - (\psi + \mu_h)I_h \\ & -\mu_h E_h - \mu_h R_h) + \frac{1}{2}\sigma_1^2 S_h^2 + \frac{1}{2}\sigma_2^2 E_h^2 + \frac{1}{2}\sigma_3^2 I_h^2 + \frac{1}{2}\sigma_4^2 R_h^2\Big] dt \\ & + (S_h - S_h^0 + E_h + I_h + R_h) \\ & \cdot [\sigma_1 S_h dB_1(t) + \sigma_2 E_h dB_2(t) + \sigma_3 I_h dB_3(t) + \sigma_4 R_h dB_4(t)] \\ \le & \Big[-(\mu_h - \xi_1)((S_h - S_h^0)^2 + E_h^2 + I_h^2 + R_h^2) + \sigma_1^2 S_h^{0^2} \\ & + C_1(S_h + E_h + I_h + R_h)\Big] dt + (S_h - S_h^0 + E_h + I_h + R_h) \\ & \cdot [\sigma_1 S_h dB_1(t) + \sigma_2 E_h dB_2(t) + \sigma_3 I_h dB_3(t) + \sigma_4 R_h dB_4(t)], \end{aligned} \tag{9}$$

where $C_1 = (\psi + 2\mu_h)S_h^0$ and $\xi_1 = \max\{\sigma_1^2, \frac{1}{2}\sigma_2^2, \frac{1}{2}\sigma_3^2, \frac{1}{2}\sigma_4^2\}$.

Subsequently, we define $V_{12} : \mathbf{R}_+^4 \to \mathbf{R}_+$ by

$$V_{12}(S_h, E_h, I_h, R_h) = S_h + E_h + I_h + R_h. \tag{10}$$

Using again Itô formula, we obtain

$$\begin{aligned} dV_{12} =& [\Lambda_h - \mu_h S_h - (\psi + \mu_h)I_h - \mu_h E_h - \mu R_h] dt + \sigma_1 S_h dB_1(t) \\ & + \sigma_2 E_h dB_2(t) + \sigma_3 I_h dB_3(t) + \sigma_4 R_h dB_4(t). \end{aligned} \tag{11}$$

Next, we define $V_{13} : \mathbf{R}_+^4 \to \mathbf{R}_+$ by

$$V_{13} = V_{11} + \frac{C_1}{\mu_h} V_{12}. \tag{12}$$

It follows from (9) and (11) that

$$\begin{aligned} dV_{13} \leq & \left[-(\mu_h - \xi_1)((S_h - S_h^0)^2 + E_h^2 + I_h^2 + R_h^2) + \sigma_1^2 {S_h^0}^2 + \frac{\Lambda_h}{\mu_h} C_1 \right] dt \\ & + (S_h - S_h^0 + E_h + I_h + R_h + \frac{C_1}{\mu_h}) \big[\sigma_1 S_h dB_1(t) + \sigma_2 E_h dB_2(t) \\ & + \sigma_3 I_h dB_3(t) + \sigma_4 R_h dB_4(t) \big]. \end{aligned}$$

Integrating both sides from 0 to t and then taking expectation, we see that

$$\begin{aligned} EV_{13}(t) - V_{13}(0) \leq & E \int_0^t -(\mu_h - \xi_1)((S_h - S_h^0)^2 + E_h^2 + I_h^2 + R_h^2) d\tau \\ & + (\sigma_1^2 {S_h^0}^2 + \frac{\Lambda_h}{\mu_h} C_1) t + E \int_0^t (S_h - S_h^0 + E_h + I_h + R_h + \frac{C_1}{\mu_h}) \\ & \cdot [\sigma_1 S_h dB_1(t) + \sigma_2 E_h dB_2(t) + \sigma_3 I_h dB_3(t) + \sigma_4 R_h dB_4(t)]. \end{aligned}$$

Then

$$\begin{aligned} 0 \leq EV_{13}(t) \leq & V_{13}(0) - (\mu_h - \xi_1) E \int_0^t \left((S_h - S_h^0)^2 + E_h^2 + I_h^2 + R_h^2 \right) d\tau \\ & + (\sigma_1^2 {S_h^0}^2 + \frac{\Lambda_h}{\mu_h} C_1) t. \end{aligned}$$

Since $\mu_h > \xi_1$, dividing both sides by $t(\mu_h - \xi_1)$ and then letting $t \to \infty$, we have

$$\limsup_{t \to +\infty} \frac{1}{t} E \int_0^t ((S_h - S_h^0)^2 + E_h^2 + I_h^2 + R_h^2) d\tau \leq \frac{\sigma_1^2 {S_h^0}^2 + \frac{\Lambda_h}{\mu_h} C_1}{\mu_h - \xi_1}.$$

Consequently, the first of the two desired inequalities is now proved. Let us turn now our attention to the second one, which characterizes the dynamics of the vector populations and will be established via a similar argument.

We define the functional $V_{21} : \mathbf{R}_+^2 \to [0, \infty)$ by

$$V_{21}(S_v, I_v) = \frac{1}{2}(S_v - S_v^0 + I_v)^2. \tag{13}$$

It follows from the Itô formula that

$$dV_{21} = (S_v - S_v^0 + I_v)(dS_v + dI_v) + (\frac{1}{2}\sigma_5^2 S_v^2 + \frac{1}{2}\sigma_6^2 I_v^2) dt. \tag{14}$$

Since $\frac{\Lambda_v}{\mu_v} = S_v^0$, we obtain that

$$dS_v + dI_v = \left(-\mu_v (S_v - S_v^0) - \mu_v I_v \right) dt + \sigma_5 S_v dB_5(t) + \sigma_6 I_v dB_6(t). \tag{15}$$

Substituting (15) into (14), we obtain

$$\begin{aligned}dV_{21} =&((S_v - S_v^0 + I_v)(-\mu_h(S_v - S_v^0) - \mu_v I_v) + \frac{1}{2}\sigma_5^2 S_v^2 + \frac{1}{2}\sigma_6^2 I_v^2)dt \\ &+ (S_v - S_v^0 + I_v)(\sigma_5 S_v dB_5(t) + \sigma_6 I_v dB_6(t)) \\ \le&(-(\mu_h - \xi_2)((S_v - S_v^0)^2 + I_v^2) + \sigma_5^2 {S_v^0}^2 + C_2(S_v + I_v))dt \\ &+ (S_v - S_v^0 + I_v)(\sigma_5 S_v dB_5(t) + \sigma_6 I_v dB_6(t)).\end{aligned} \tag{16}$$

Here $C_2 = 2\mu_v S_v^0$ and $\xi_2 = \max\{\sigma_5^2, \frac{1}{2}\sigma_6^2\}$. We then define $V_{22} : \mathbf{R}_+^2 \to \mathbf{R}_+$ by

$$V_{22}(S_v, I_v) = S_v + I_v. \tag{17}$$

Applying Itô formula, we obtain

$$dV_{22} = \left(\mu_v(\frac{\Lambda_v}{\mu_v} - S_v) - \mu_v I_v\right) dt + \sigma_5 S_v dB_5(t) + \sigma_6 I_v dB_6(t). \tag{18}$$

Next, we define $V_{23} : \mathbf{R}_+^2 \to \mathbf{R}_+$

$$V_{23} = V_{21} + \frac{C_2}{\mu_v} V_{22} \tag{19}$$

From (16) and (18), we obtain

$$\begin{aligned}dV_{23} \le& \left(-(\mu_v - \xi_2)((S_v - S_v^0)^2 + I_v^2) + \sigma_5^2 {S_v^0}^2 + \frac{\Lambda_v}{\mu_v} C_2\right) dt \\ &+ (S_v - S_v^0 + I_v + \frac{C_2}{\mu_v})(\sigma_5 S_v dB_5(t) + \sigma_6 I_v dB_6(t)).\end{aligned}$$

Integrating both sides from 0 to t and then taking expectation, we see that

$$\begin{aligned}EV_{23}(t) - V_{23}(0) \le& E\int_0^t \left(-(\mu_v - \xi_2)((S_v - S_v^0)^2 + I_v^2\right) d\tau \\ &+ (\sigma_5^2 {S_v^0}^2 + \frac{\Lambda_h}{\mu_h} C_2)t \\ &+ E\int_0^t (S_v - S_v^0 + I_v + \frac{C_2}{\mu_v})(\sigma_5 S_v dB_5(t) + \sigma_6 I_v dB_6(t)).\end{aligned}$$

Then

$$\begin{aligned}0 \le EV_{23}(t) \le& V_{23}(0) - (\mu_h - \xi_2)E\int_0^t \left((S_v - S_v^0)^2 + I_v^2\right) d\tau \\ &+ (\sigma_5^2 {S_v^0}^2 + \frac{\Lambda_v}{\mu_v} C_2)t.\end{aligned}$$

Noting that $\mu_v > \xi_2$, dividing both sides by $t(\mu_v - \xi_2)$ and then letting $t \to \infty$, we have

$$\limsup_{t\to+\infty} \frac{1}{t} E \int_0^t ((S_v - S_v^0)^2 + I_v^2) d\tau \leq \frac{\sigma_5^2 S_v^{0\,2} + \frac{\Lambda_v}{\mu_v} C_2}{\mu_v - \xi_2}.$$

This completes the proof. ∎

3.3. ASYMPTOTIC BEHAVIOR AROUND THE ENDEMIC EQUILIBRIUM OF THE DETERMINISTIC MODEL

Due to the same stochastic perturbations, the endemic equilibrium is not a steady state for the system (2) either. By using an argument which is essentially similar to the one displayed in the previous Subsection, we now investigate the asymptotic behavior of the solutions of (2) around the endemic equilibrium. Again, this happens provided that the perturbations do not exceed certain strength limitations.

Theorem 3.3. *Let us define*

$$\xi_3 = \max\{\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2\}, \quad \xi_4 = \max\{\sigma_5^2, \sigma_6^2\}$$

Assume that

$$\xi_3 < \mu_h \text{ and } \xi_4 < \mu_v.$$

Then for any given initial data in $\mathbf{R}_+^6$, *the corresponding solution of* (2) *satisfies*

$$\limsup_{t\to+\infty} \frac{1}{t} E \int_0^t ((S_h - S_h^*)^2 + (E_h - E_h^*)^2 + (I_h - I_h^*)^2 + (R_h - R_h^*)^2) d\tau$$
$$\leq \frac{\sigma_1^2 S_h^{*2} + \sigma_2^2 E_h^{*2} + \sigma_3^2 I_h^{*2} + \sigma_4^2 R_h^{*2} + \frac{\Lambda_h}{\mu_h} C_3}{\mu_h - \xi_3};$$
$$\limsup_{t\to+\infty} \frac{1}{t} E \int_0^t ((S_v - S_v^*)^2 + (I_v - I_v^*)^2) d\tau \leq \frac{\sigma_5^2 S_v^{*2} + \sigma_6^2 I_v^{*2} + \frac{\Lambda_v}{\mu_v} C_4}{\mu_v - \xi_4},$$

in which

$$C_3 = (\psi + 2\mu_h)(S_h^* + E_h^* + I_h^* + R_h^*), \quad C_4 = 2\mu_v(S_v^* + I_v^*).$$

Proof. Define the functional $V_{31} : \mathbf{R}_+^4 \to [0, \infty)$,

$$V_{31}(S_h, E_h, I_h, R_h) = \frac{1}{2}(S_h - S_h^* + E_h - E_h^* + I_h - I_h^* + R_h - R_h^*)^2. \quad (20)$$

Using Itô formula, we obtain

$$\begin{aligned} dV_{31} =&(S_h - S_h^* + E_h - E_h^* + I_h - I_h^* + R_h - R_h^*) \\ &\cdot (dS_h + dE_h + dI_h + dR_h) \\ &+ (\frac{1}{2}\sigma_1^2 S_h^2 + \frac{1}{2}\sigma_2^2 E_h^2 + \frac{1}{2}\sigma_3^2 I_h^2 + \frac{1}{2}\sigma_4^2 R_h^2) dt, \end{aligned} \tag{21}$$

where

$$\begin{aligned} &dS_h + dE_h + dI_h + dR_h \\ &\quad = [\mu_h(\Lambda_h - \mu_h S_h - (\psi + \mu_h) I_h - \mu_h E_h + \mu_h R_h] \, dt \\ &\qquad + \sigma_1 S_h dB_1(t) + \sigma_2 E_h dB_2(t) + \sigma_3 I_h dB_3(t) + \sigma_4 R_h dB_4(t) \\ &\quad = \big[- \mu_h (S_h - S_h^*) - \mu_h (E_h - E_h^*) - (\psi + \mu_h)(I_h - I_h^*) \\ &\qquad - \mu (R_h - R_h^*) \big] dt + \sigma_1 S_h dB_1(t) + \sigma_2 E_h dB_2(t) \\ &\qquad + \sigma_3 I_h dB_3(t) + \sigma_4 R_h dB_4(t). \end{aligned}$$

Therefore

$$\begin{aligned} dV_{31} =&\Bigg[(S_h - S_h^* + E_h - E_h^* + I_h - I_h^* + R_h - R_h^*) \\ &\quad \cdot (-\mu_h (S_h - S_h^*) - \mu_h (E_h - E_h^*) - (\psi + \mu_h)(I_h - I_h^*) \\ &\quad - \mu (R_h - R_h^*)) + \frac{1}{2}\sigma_1^2 S_h^2 + \frac{1}{2}\sigma_2^2 E_h^2 + \frac{1}{2}\sigma_3^2 I_h^2 + \frac{1}{2}\sigma_4^2 R_h^2 \Bigg] dt \\ &+ (S_h - S_h^* + E_h - E_h^* + I_h - I_h^* + R_h - R_h^*) \\ &\cdot \big(\sigma_1 S_h dB_1(t) + \sigma_2 E_h dB_2(t) + \sigma_3 I_h dB_3(t) + \sigma_4 R_h dB_4(t)\big). \end{aligned}$$

Consequently,

$$\begin{aligned} dV_{31} \leq &\Bigg[- (\mu_h - \sigma_1^2)(S_h - S_h^*)^2 - (\mu_h - \sigma_2^2)(E_h - E_h^*)^2 \\ &- (\psi + \mu_h - \sigma_3^2)(I_h - I_h^*)^2 - (\mu_h - \sigma_4^2)(R_h - R_h^*)^2 + \sigma_1^2 {S_h^*}^2 + \sigma_2^2 {E_h^*}^2 \\ &+ \sigma_3^2 {I_h^*}^2 + \sigma_4^2 {R_h^*}^2 + (2\mu_h E_h + (\psi + 2\mu_h) I_h^* + 2\mu_h R_h^*) S_h \\ &+ (2\mu_h S_h^* + (\psi + 2\mu_h) I_h^* + 2\mu_h R_h^*) E_h + (\psi + 2\mu_h)(S_h^* + E_h^* + R_h^*) I_h \\ &+ (2\mu_h S_h^*) R_h \Bigg] dt + (S_h - S_h^* + E_h - E_h^* + I_h - I_h^* + R_h - R_h^*) \\ &\cdot \big(\sigma_1 S_h dB_1(t) + \sigma_2 E_h dB_2(t) + \sigma_3 I_h dB_3(t) + \sigma_4 R_h dB_4(t)\big) \end{aligned}$$

$$
\begin{aligned}
\leq & \Big[-(\mu_h - \xi_3)\big((S_h - S_h^*)^2 + (E_h - E_h^*)^2 + (I_h - I_h^*)^2 + (R_h - R_h^*)^2\big) \\
& + C_3(S_h + E_h + I_h + R_h) + \sigma_1^2 {S_h^*}^2 + \sigma_2^2 {E_h^*}^2 + \sigma_3^2 {I_h^*}^2 + \sigma_4^2 {R_h^*}^2 \Big] dt \\
& + (S_h - S_h^* + E_h - E_h^* + I_h - I_h^* + R_h - R_h^*) \\
& \quad \cdot (\sigma_1 S_h dB_1(t) + \sigma_2 E_h dB_2(t) + \sigma_3 I_h dB_3(t) + \sigma_4 R_h dB_4(t)),
\end{aligned}
$$

where $\xi_3 = \max\{\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2\}$ and $C_3 = (\psi + 2\mu_h)(S_h^* + E_h^* + I_h^* + R_h^*)$.
Define also $V_{32} : \mathbf{R}_+^4 \to \mathbf{R}_+$,

$$
V_{32}(S_h, E_h, I_h, R_h) = S_h + E_h + I_h + R_h. \tag{22}
$$

Applying Itô formula again, we see that

$$
\begin{aligned}
dV_{32} = & [\Lambda_h - \mu_h S_h - (\psi + \mu_h) I_h - \mu_h E_h + \mu_h R_h]\, dt + \sigma_1 S_h dB_1(t) \\
& + \sigma_2 E_h dB_2(t) + \sigma_3 I_h dB_3(t) + \sigma_4 R_h dB_4(t)
\end{aligned}
$$

Next, we define $V_{33} : \mathbf{R}_+^4 \to \mathbf{R}_+$,

$$
V_{33} = V_{31} + \frac{C_3}{\mu_h} V_{32}. \tag{23}
$$

Using Itô formula again, we obtain

$$
\begin{aligned}
dV_{33} = & -(\mu_h - \xi_3)\Big[(S_h - S_h^*)^2 + (E_h - E_h^*)^2 + (I_h - I_h^*)^2 + (R_h - R_h^*)^2\Big] dt \\
& + (\sigma_1^2 {S_h^*}^2 + \sigma_2^2 {E_h^*}^2 + \sigma_3^2 {I_h^*}^2 + \sigma_4^2 {R_h^*}^2 + \frac{\Lambda_h}{\mu_h} C_3) dt \\
& + (S_h - S_h^* + E_h - E_h^* + I_h - I_h^* + R_h - R_h^* + \frac{C_3}{\mu_h}) \\
& \quad \cdot \big(\sigma_1 S_h dB_1(t) + \sigma_2 E_h dB_2(t) + \sigma_3 I_h dB_3(t) + \sigma_4 R_h dB_4(t)\big).
\end{aligned}
$$

Integrating both sides from 0 to t and then taking the expectation, we obtain

$$
\begin{aligned}
& EV_{33}(t) - V_{33}(0) \\
& \leq E\int_0^t \Big[-(\mu_h - \xi_3)((S_h - S_h^*)^2 + (E_h - E_h^*)^2 + (I_h - I_h^*)^2 \\
& \quad + (R_h - R_h^*)^2\Big] d\tau + \left(\sigma_1^2 S_h^{*2} + \sigma_2^2 E_h^{*2} + \sigma_3^2 I_h^{*2} + \sigma_4^2 R_h^{*2} + \frac{\Lambda_h}{\mu_h} C_3\right) t \\
& \quad + E\int_0^t \big(S_h - S_h^* + E_h - E_h^* + I_h - I_h^* + R_h - R_h^* + \frac{C_3}{\mu_h}\big)\big(\sigma_1 S_h dB_1(t) \\
& \quad + \sigma_2 E_h dB_2(t) + \sigma_3 I_h dB_3(t) + \sigma_4 R_h dB_4(t)\big).
\end{aligned}
$$

Then

$$\begin{aligned}0 \leq EV_{33}(t) \leq & V_{33}(0) - (\mu_h - \xi_3)E\int_0^t \Big[(S_h - S_h^*)^2 + (E_h - E_h^*)^2 \\ & + (I_h - I_h^*)^2 + (R_h - R_h^*)^2\Big]d\tau \\ & + \big(\sigma_1^2 S_h^{*2} + \sigma_2^2 E_h^{*2} + \sigma_3^2 I_h^{*2} + \sigma_4^2 R_h^{*2} + \frac{\Lambda_h}{\mu_h}C_3\big)t\end{aligned}$$

Noting that $\mu_h > \xi_3$, dividing both sides by $t(\mu_h - \xi_3)$ and and letting $t \to \infty$, we have

$$\begin{aligned}&\limsup_{t\to+\infty} \frac{1}{t}E\int_0^t ((S_h - S_h^*)^2 + (E_h - E_h^*)^2 + (I_h - I_h^*)^2 + (R_h - R_h^*)^2)d\tau \\ &\leq \frac{\sigma_1^2 S_h^{*2} + \sigma_2^2 E_h^{*2} + \sigma_3^2 I_h^{*2} + \sigma_4^2 R_h^{*2} + \frac{\Lambda_h}{\mu_h}C_3}{\mu_h - \xi_3}.\end{aligned}$$

We now turn our atention to the second estimation. We further define $V_{41} : \mathbf{R}_+^2 \to [0, \infty)$,

$$V_{41} = \frac{1}{2}(S_v - S_v^* + I_v - I_v^*)^2. \tag{24}$$

It follows from the Itô formula that

$$dV_{41} = (S_v - S_v^* + I_v - I_v^*)(dS_v + dI_v) + (\frac{1}{2}\sigma_5^2 S_v^2 + \frac{1}{2}\sigma_6^2 I_v^2)dt, \tag{25}$$

where

$$\begin{aligned}dS_v + dI_v &= \left(\mu_v(\frac{\Lambda_v}{\mu_v} - S_v^*) - \mu_v I_v^*\right)dt + \sigma_5 S_v dB_5(t) + \sigma_6 I_v dB_6(t) \\ &= (-\mu_v(S_v - S_v^*) - \mu_v(I_v - I_v^*))\,dt + \sigma_5 S_v dB_5(t) \\ &\quad + \sigma_6 I_v dB_6(t).\end{aligned}$$

Therefore,

$$\begin{aligned}dV_{41} =& \Big(-\mu_v(S_v - S_v^*)^2 - \mu_v(I_v - I_v^*)^2 - 2\mu_v(S_v - S_v^*)(I_v - I_v^*) + \frac{1}{2}\sigma_5^2 S_v^2 \\ & + \frac{1}{2}\sigma_6^2 I_v^2\Big)dt + (S_v - S_v^* + I_v - I_v^*)(\sigma_5 S_v dB_5(t) + \sigma_6 I_v dB_6(t)) \\ \leq & \Big(-(\mu_v - \sigma_5^2)(S_v - S_v^*)^2 - (\mu_v - \sigma_6^2)(I_v - I_v^*)^2 + 2\mu_v(S_v I_v^* + S_v^* I_v) \\ & + \sigma_5^2 S_v^{*^2} + \sigma_6^2 I_v^{*^2}\Big)dt + (S_v - S_v^* + I_v - I_v^*) \\ & \cdot (\sigma_5 S_v dB_5(t) + \sigma_6 I_v dB_6(t))\end{aligned}$$

$$\leq\Big(-(\mu_v-\xi_4)((S_v-S_v^*)^2+(I_v-I_v^*)^2)+C_4(S_v+I_v)+\sigma_5^2{S_v^*}^2 + \sigma_6^2{I_v^*}^2\Big)dt+(S_v-S_v^*+I_v-I_v^*)(\sigma_5S_vdB_5(t)+\sigma_6I_vdB_6(t)).$$

Here, $\xi_4=\max\{\sigma_5^2,\sigma_6^2\}$ and $C_4=2\mu_v(S_v^*+I_v^*)$.

We then define $V_{42}:\mathbf{R}_+^2\to\mathbf{R}_+$,

$$V_{42}=S_v+I_v. \tag{26}$$

Using Itô formula, we obtain

$$dV_{42}=(\Lambda_v-\mu_vS_v-\mu_vI_v)\,dt+\sigma_5S_vdB_5(t)+\sigma_6I_vdB_6(t).$$

Next, we define $V_{43}:\mathbf{R}_+^2\to\mathbf{R}_+$,

$$V_{43}=V_{41}+\frac{C_4}{\mu_v}V_{42} \tag{27}$$

and we obtain the following inequality

$$dV_{43}\leq\left(-(\mu_v-\xi_4)((S_v-S_v^*)^2+(I_v-I_v^*)^2)+\sigma_5^2{S_v^*}^2+\sigma_6^2{I_v^*}^2+\frac{\Lambda_v}{\mu_v}C_4\right)dt + (S_v-S_v^*+I_v-I_v^*+\frac{C_4}{\mu_h})(\sigma_5S_vdB_5(t)+\sigma_6I_vdB_6(t)). \tag{28}$$

Integrating both sides from 0 to t and then taking the expectation yields

$$EV_{43}(t)\leq V_{43}(0)-(\mu_v-\xi_4)E\int_0^t\big((S_v-S_v^*)^2+(I_v-I_v^*)^2\big)d\tau + (\sigma_5^2S_v^{*2}+\sigma_6^2I_v^{*2}+\frac{\Lambda_v}{\mu_v}C_4)t.$$

Since we assumed that $\mu_v>\xi_4$, dividing both sides by $t(\mu_v-\xi_4)$ and letting $t\to\infty$, we obtain

$$\limsup_{t\to+\infty}\frac{1}{t}E\int_0^t((S_v-S_v^*)^2+(I_v-I_v^*)^2)d\tau\leq\frac{\sigma_5^2S_v^{*2}+\sigma_6^2I_v^{*2}+\frac{\Lambda_v}{\mu_v}C_4}{\mu_v-\xi_4}.$$

This completes the proof. ∎

4. NUMERICAL SIMULATIONS AND CONCLUDING REMARKS

In this section, we shall give several numerical simulations meant to illustrate and complement our theoretical findings. The concrete values of the

Table 2 Concrete values for each parameter of the deterministic malaria model

Parameter	Estimated value	Reference
ϕ	0.502	[16]
ϵ	0.2	[16]
β	0.8333	[17]
λ	0.09	[16]
μ_h	0.04	estimated
μ_v	0.1429	[17]
κ	0.0014	estimated
α_1	0.0588	[16]
Λ_h	0.4	estimated
Λ_v	20, 200	estimated
ψ	0.05	[19]
b	0.005	[20]

parameters which appear in the deterministic model (1) are given in Table 2, together with the references used for these values. If we choose $\Lambda_v = 20$, then

$$R_0 = \sqrt{R_{0h} \cdot R_{0v}} \approx 0.6811,$$

which implies that malaria can be eradicated, as shown in Figure 1, even though the dynamics of both infective human and mosquito populations start with acute spikes.

If we choose $\Lambda_v = 200$, then

$$R_0 = \sqrt{R_{0h} \cdot R_{0v}} \approx 2.1537,$$

which implies that malaria can not be eradicated and there exist endemic steady states in human and mosquito populations, although at low population sizes, as shown in Figure 2.

In this regard, if we choose $\sigma_1 = 0.18$, $\sigma_2 = 0.18$, $\sigma_3 = 0.18$, $\sigma_4 = 0.10$, $\sigma_5 = 0.25$, $\sigma_6 = 0.15$, then $\xi_1 = 0.0324 < \mu_h$, $\xi_2 = 0.0625 < \mu_v$, $\xi_3 = 0.0324 < \mu_h$ and $\xi_4 = 0.0625 < \mu_v$, so that the hypotheses of Theorems 3.2 and 3.3 are

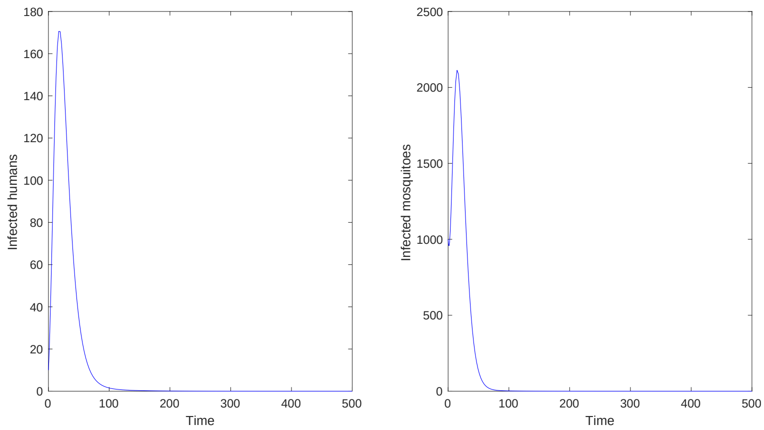

Fig. 1. Dynamics of the infected human and mosquitoes populations of the deterministic model with $R_0 < 1$

satisfied. Figures 3 and 4 then illustrate the asymptotic behavior around the disease-free and the endemic equilibrium of the deterministic model, respectively.

In this paper, in order to investigate malaria transmission, we first propose a 6-dimensional model which consists of a combination of a $SEIRS$ model for the human population and a SI model for the mosquito population, standard incidence terms being used for both human to mosquito and mosquito to human transmission. The basic reproduction number of this model is determined via the next generation approach, the stability of the equilibria being then expressed in terms of this number, understood as a threshold parameter.

To consider the influence of a randomly fluctuating environment, multiple stochastic perturbations of white noise type are considered. The well-posedness of the model in a biological sense is first investigated, then the asymptotic behavior of the solutions of the stochastic model around the steady states of the deterministic model is investigated in terms of estimations involving averaged expectation. Finally, numerical simulations are given in order to illustrate our theoretical findings.

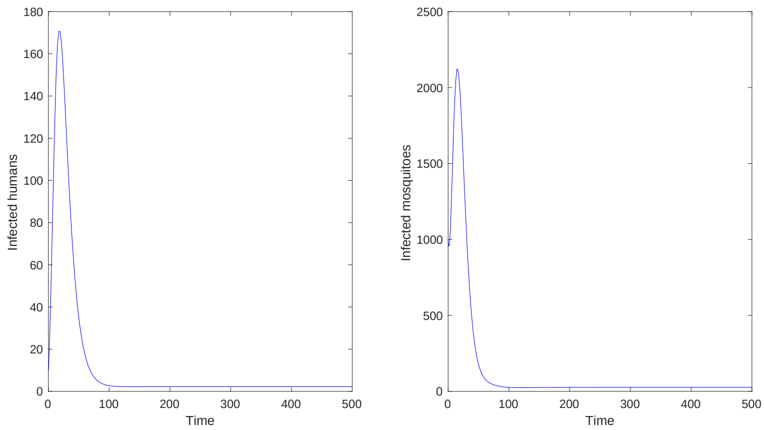

Fig. 2. Dynamics of the infected human and mosquito populations for the deterministic model with $R_0 > 1$

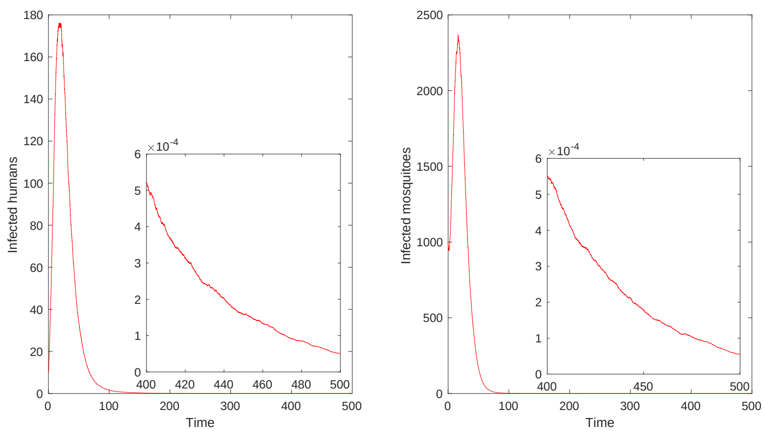

Fig. 3. Dynamics of the infected human and mosquito populations for the stochastic model

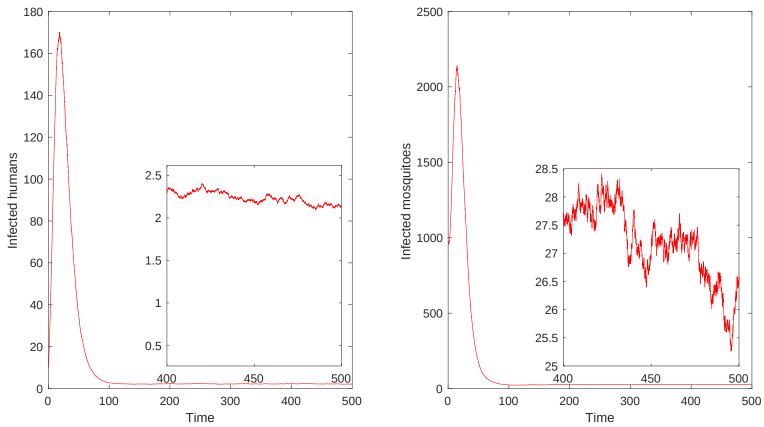

Fig. 4. Dynamics of the infected human and mosquito populations for the stochastic model

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