ROMAI J., v.17, no.1(2021), 1–16

130 YEARS OF EFFORT FOR SOLVING THE POINCARÉ'S CENTER-FOCUS PROBLEM

Mitrofan Choban[1], Tatiana Rotaru[2]

[1] *Sciences Academy of Moldova, Chişinău, Republic of Moldova,*

[1] *Tiraspol State University, Chişinău, Republic of Moldova,*

[2] *National Council for Accreditation and Attestation, Chişinău, Republic of Moldova*

Abstract It is well known that many mathematical models use differential equation systems and apply the qualitative theory of differential equations, introduced by Poincaré and Lyapunov. One of the problems that persists in order to control the behavior of systems of this type, is to distinguish between a focus or a center (the Center and Focus Problem).

The solving of this problem goes through the computation of the Poincaré-Lyapunov constants. In the case of polynomial right-hand sides it follows from Hilbert's theorem on the finiteness of bases of polynomial ideals that in this sequence only finitely many are essential and that the remaining ones are consequences of them. Hence, this problem is divided in two parts: in the first, to estimate the number of essential constants; in the second, to determine the minimal upper border of the indexes of a complete system of essential constants. The first part is called the Weak Center and Focus Problem.

The problem of estimation the maximal number of algebraically independent essential constants is called the Generalized Center and Focus Problem. Recently M. N. Popa and V. V. Pricop have solved the Generalized Center and Focus Problem. The present article contains: some moments related to the history of the Center and Focus Problem; the contribution of the Sibirsky's school in the solving of the Center and Focus Problem; methodological aspects of the Popa - Pricop solution of the Generalized Center and Focus Problem.

The problem of the estimation of the minimal upper border of the indexes of a complete system of algebraically independent essential constants is open. Another open problem consists on determining what differential systems are integrable.

Keywords: Poincaré-Lyapunov constants, center and focus problem, generalized center and focus problem.

2020 MSC: 34C60, 37G15.

Editor's Note

The article "130 years of the effort the solving of the Poincaré's center-focus problem", signed by academician Mitrofan Choban and journalist Tatiana Rotaru, was published in Romanian in Journal of Science and Innovation, Culture and Arts of the Academy of Sciences of Moldova "Akademos", 3(30), 2013, pp. 13-21. The work remains current today and had continuity. Based on the examined problem, have been published two monographs by M. N.

Popa V. V. Pricop - one in Russian (2018) and recent - in English: Popa M. N., Pricop V. V. The Center and Focus Problem: Algebraic Solutions and Hypoteheses. Ed. Taylor&Frances Group, 2021, 215 p. The cited above article is reproduced in English in this number of ROMAI Journal to follow the beginning and the stages of solving an old mathematical problem that a troubled the minds of mathematicians for more than a century. The publication of the English version is done with the accord of the Editorial Board of "Akademos".

The much regretted academician Mitrofan Choban (05.01.1942-02.02.2021) was an illustrious international mathematician, patriot and patriarch of science and education in the Republic of Moldova, a great friend and activist of ROMAI, who would have turned 80 in February 2022, if a relentless illness wouldn't have take him from our ranks.

We shall always miss him.

1. FROM THE HISTORY OF MATHEMATICS

Talking about mathematics or mathematicians is a challenge with the risk of being misunderstood or even rejected from the start. Gone are the days when research disciplines were not built strictly, and dialogue between their representatives was a normal way of existence and collaboration. But the nineteenth century brought many surprising discoveries to human civilization. Many of them are the result of logical analysis of phenomena or the mathematical one: Gauss discovered by calculation the asteroids Ceres, Palass, Vesta, Juno; Galle also based on calculations of the identity of the planet Neptune (1846); Mendeleev, starting from the atomic mass, he systematized the chemical elements and anticipated the existence of many new ones; Schliemann, based on Homer's descriptions, determined the location of Troy, etc. It is mathematical research that has helped to solve a number of problems that have plagued the minds of scientists for nearly 2500 years, beginning with Platon, Aristotle, Euclid, Archimedes, as well as the creation of new disciplines in the field.

At the beginning of the twentieth century, mathematics proliferated so much that it became, figuratively speaking, a Kingdom of the Universe of Science, although this word in Greek means "learning", "study", "science". We consider indisputably the fact that science is also an art, an art of human depth and strength of thought. A little earlier, in the 19th century, the brilliant French mathematician Henri Poincaré (1854-1912) created new fields of research, as topology, qualitative theory of dynamical systems, etc.

The development of mathematics in Romania was deeply connected with Poincaré's work and activity: he was a member of the Commission for the defense of doctoral theses for many high-performance mathematicians and physicists such as Nicolae Coculescu, Gheorghe Ţiţeica, Anton Davidoglu, Dragomir Hurmuzescu, Dimitrie Pompeiu, Constantin Nicolau and others. By quantitative methods, Spiru Haret had demonstrated the instability of the Solar System. The qualitative approach, as well as in a much broader framework, led Poincaré to confirm this fact. The results of the KAM theory (Kolmogorov-Arnold-Moser) showed that the Solar System is in a state of relative stability. As a result of this fruitful cooperation with Romanian researchers, Henri Poincaré was awarded the honorary titles of Doctor Honoris Causa of Kolosvar University (Cluj) and Honorary Member of the Romanian Academy (1909).

Henri Poincaré [16] formulated a series of problems, the solution of which determines the further development of science. A sensational news, in this sense, was in the first years of this millennium the solution of the Poincaré Conjecture by the enigmatic Russian mathematician Grigori Perelman. His 2002 demonstration ranked first in the top of the most important scientific discoveries in decades. Poincaré's conjecture or Poincaré's hypothesis, first stated by Poincaré in 1904, states that if S is a compact 3-dimensional variety (a closed, bordered or borderless 3-dimensional surface, immersed in a 4-dimensional space), in which any circle can be continuously deformed until it becomes a point, then this space S is equivalent, from a topological point of view (homeomorphic), to a 3-dimensional sphere. Solving a big problem generates the formulation of other new problems, which determine the further development of science. It is hoped that this famous result of Perelman's will help solve the problem of classifying three-dimensional varieties - another important problem stated by Poincaré in 1904, and especially the study of the Universe.

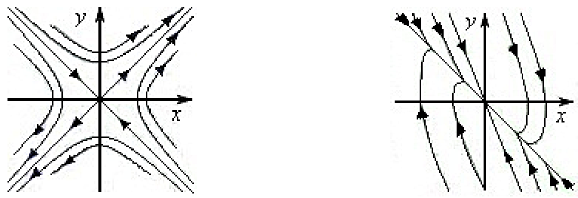
One of the famous problems of the qualitative theory of differential equations is the Center and Focus Problem, formulated by Poincaré 130 years ago ([15]. In 1881-1899 he studied the periodic and asymptotic solutions of differential equations, developed the method of the small parameter, the method of fixed points, the method of integral invariants, which became classical methods of research not only in mechanics and astronomy, but also in static physics, quantum mechanics. Working on the problems of celestial mechanics, he simultaneously laid the foundation of a new science - topology, which he called "Analysis situs".

2. THE CENTER AND FOCUS PROBLEM

Let

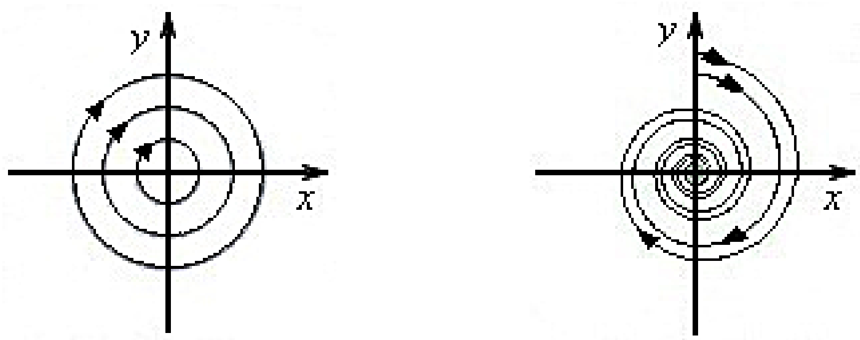
$$\dot{x} = X(x, y), \ \dot{y} = Y(x, y) \tag{1}$$

be an autonomous system of differential equations. We admit that the functions $X(x, y)$ and $Y(x, y)$ are analytical. The solutions of this system of equations are called integral curves. The qualitative theory has as its starting point the stability theory and the problem of the movement of three and more bodies in the celestial mechanics. Henri Poincaré would say that even if the differential equation is not solved explicitly, it is possible to determine the character of the behavior of the solutions (integral curves) and proposed a classification of the singular points of the solutions: saddle, focus, center, node.



a) saddle b) node

Fig. 1. Singular points of the first type



c) center d) focus

Fig. 2. Singular points of the second type

It is known that if the roots of the characteristic equation of the singular point $O(0,0)$ are imaginary, then it can be center or focus (singular point of the second type). In the case of a center the singular point is surrounded by the closed trajectories and in the case of a focus it is surrounded by spirals. **The Center and Focus Problem** is to determine *the condition* under which a singular point is a center. In general case the Center Problem is algebraically unsolvable [9, 1, 19].

The Center and Focus Problem has deep ties to **David Hilbert's 16th problem**. In 1900, at the Second International Congress of Mathematicians, Hilbert posed 23 important problems for the further development of science. The 16th problem, which remains unsolved at present, concerned algebraic curves and surfaces. Today, this problem is divided into two parts related to different areas. The maximal number of closed branches of an n-order algebraic curves was set by Harnack. The first part of 16th problem is to determine the position of these branches relative to each other. For $n = 6$ are obtained 11 branches and Hilbert assumed that there is one branch that contains another branch, and outside it there are the other nine branches or inverse. However, in 1970 D. A. Gudkov determined that there were cases when five branches fell outside and inside the curve. This has shown that the first part of the problem is much more complicated. Various properties and extraordinary examples have been described by I. G. Petrovski, O. A. Oleinic, V. I. Arnold, V. A. Rohlin, O. Ya. Viro and others. This part of the problem now refers to algebraic geometry (see [15]).

In the second part of 16th problem, which also remains unsolved and completes the Center and Focus Problem, for a polynomial vector field of order n it is required to determine the upper bound $H(n)$ of the number of cycles and their relative position. It is well known that the number of limit cycles is always finite. The number $H(n)$ is called the Hilbert's number.

Researches on the 16th problem has been quite dramatic. In 1923 Henri Dulac [5] proposed a demonstration that the number $H(n, v)$ is finite for any polynomial vector field v of order n. In 1955 Ivan G. Petrovski and Evgeny M. Landis announced the complete solution of the second part of 16th problem, but in 1960 it was determined that their demonstration had serious shortcomings. A great surprise was the work of Yulii Ilyashenko in 1981, in which it was established that the work of Dulac in 1923 also contains gaps, which with great efforts were removed over 10 years by Yulii Ilyashenko and Jean Ecalle (see [6, 8, 15]).

These results have boosted the researches of the polynomial vector fields v of order n. In this case, the second part of 16th problem is a particular case of the **Problem of Global Finiteness**: *In any analytically finite-parametrized family of analytic vector fields on the sphere with the compact parameter space B (from the K-dimensional Euclidean space) the number $H(n, p)$ of limit cycle is uniformly bounded for all values p of the parameter in B.* This problem was formulated by Yu. Ilyashenko in 1994 and is called **the Hilbert-Arnold Problem** (see [8]). In 1986 V. I. Arnold (see [1, 8]), for a smooth vector field denoted on a sphere, introduced the notions of polycycle, the bifurcation number $B(k)$ of maximal cycling of non - trivial polycycle of field, of elementary singular point, of elementary polycycle and of elementary bifurcation number $E(k)$ of maximal cycling of non-trivial elementary polycycle. Thus, **Hilbert-Arnold's Local Problem** was formulated: *to prove that the number $B(k)$ is finite and to estimate this number from above.* The positive solution of the global problem is a consequence of positive answer to the local problem. It is established that $B(1) = 1$ and $B(2) = 2$. For $B(3)$ there is currently only one calculation strategy. V. Yu. Kaloshin established that $E(k) < 25k$ (see [8,9,10,11]).

We examine the case when the functions $X(x, y)$ and $Y(x, y)$ are polynomials. For the Center and Focus Problem to be algebraically solvable, the linear parts of the polynomials $X(x, y)$ and $Y(x, y)$ must not be zero. Under these conditions the system (1) can be written in the form

$$\frac{dx}{dt} = \sum_{i=0}^{\ell} P_{m_i}(x,y), \ \frac{dy}{dt} = \sum_{i=0}^{\ell} Q_{m_i}(x,y) \tag{2}$$

where $P_{m_i}(x,y)$ and $Q_{m_i}(x,y)$ are homogeneous polynomials of degree $m_i \geq 1$ in x, y, and $m_0 = 1$. The set $\{1, m_1, m_2, ..., m_\ell\}$ consists of a finite number ($\ell < \infty$) of distinct natural numbers. The coefficients and variables in polynomials $P_{m_i}(x,y)$ and $Q_{m_i}(x,y)$ take values from the fields of real numbers $\mathbb{R}$. Hereafter we denote system (2) by $s(1, m_1, m_2, ..., m_\ell)$.

The fundamental results on the Center and Focus Pproblem were obtained by **A. M. Lyapunov** (1857-1918) [12]. Henri Poincaré and Aleksandr Lyapunov laid the foundations *of methods of the qualitative theory of differential equations.*

As established, the conditions of a center consists of an infinite sequence of polynomials is equal to zero (focus quantities, Lyapunov's constants, Poincaré-Lyapunov constants)

$$L_1, L_2, ..., L_k, ... \tag{3}$$

which depend on the coefficients of the polynomials on the right sides of system $s(1, m_1, ..., m_\ell)$.

If at least one of quantities (3) is not zero, then origin of coordinates $O(0,0)$ for system $s(1, m_1, ..., m_\ell)$ is a focus. These conditions are necessary and sufficient.

From Hilbert's Theorem on the finiteness of basis of polynomial ideals it follows that *the essential center conditions*, which imply vanishing of an infinite sequence of polynomials (3), consist of a finite number of polynomials, the rest ones are the consequences of them.

Taking into account this result, the Center and Focus Problem can be formulated in the following way: *what finite number of polynomials* (*essential center conditions*)

$$L_{n_1}, L_{n_2}, ..., L_{n_\omega} \quad (n_i \in \{1, 2, ..., k, ...\};\ i = \overline{1, \omega};\ \omega < \infty) \tag{4}$$

is necessary for their equality to zero annuls all polynomials from (3)?

Hence the Center and Focus Problem consists of two parts.

The first part relates to finding the number ω that determines the upper bound of the number of focus quantities which constitute the essential center conditions.

The second part consists in finding the set $\Omega = \{n_1, n_2, ..., n_\omega\}$ of indices of essential conditions.

We will consider the first part as *the Weak Center and Focus Problem.*

The Generalized Center and Focus Problem is to determine the upper bound of the number λ of algebraically independent elements from $\Pi = \{L_i :\ i \in \Omega\}$.

The problem of determining essential center conditions (4) with number ω is a rather complicated problem and it is completely solved only for systems $s(1,2)$ and $s(1,3)$, for which we have $\omega = 3$ and $\omega = 5$, respectively (see [3, 23]).

Until now it is not known the number ω for a system $s(1,2,3)$, which seems to be not a complicated system.

There exists a hypothesis formulated by Professor H. Żołądek (Poland), mostly based on intuition, that for system $s(1,2,3)$ the number $\omega \leq 13$. Till now this hypothesis has not been disproved, but there is a recent paper from 2010, which confirms that 12 focus quantities is not enough for solving the Center and Focus Problem in the complex plane for system $s(1,2,3)$ [7].

Lie algebra method and Sibirsky's graded algebras allow us to solve the Generalized Center and Focus Problem.

If the Center and Focus Problem is solved negatively for system (2)*, having at the origin a singular point of second type (center or focus), then solution of the Generalized Center and Focus Problem can be considered as the final solution of this problem.*

3. THE EFFORT OF RESEARCHES BETWEEN CENTER AND FOCUS

Until we move to the basic topic - The Center and Focus Problem - we will mention some theories that, in one way or another, contributed to its appearance and formulation 130 years ago by Poincaré and finding a solution (until its generalized form) only now, in Chişinău. From **Artur Cayley** (1821-1895), Cambridge, England, he started an *invariants theory.* **Marius Sophus Lie** (1842-1899), Christiania, Norway, developed the theory of Lie groups and algebras - a new kind of algebraic structure that bears his name - both being applied in various fields of real science, including geometry and study differential equations. **Constantin Sibirsky** (1928-1990), Chişinău, Republic of Moldova, founded *the theory of algebraic invariants*, which is applied in the qualitative theory of equations knowing that this has to do with Lie's theory.

But who and how established this connection? In 1976, acad. Constantin Sibirsky, head of laboratory at the Institute of Mathematics and Informatics of ASM, founder of the scientific school of differential equations in the Republic of Moldova, published the monograph *Algebraic invariants of differential equations and matrices* (see [21]), which had a great resonance in the world of mathematicians. Three years later, in 1979, the American professor **C. S. Coleman** published a review of this scientific paper, in which he specified that it was written *in the spirit of the research of the Norwegian mathematician Marius Sophus Lie.* What these investigations consisted of it was not clear to Moldavian mathematicians. They only knew that the Norwegian had created a new direction in mathematical research, but the tangent between it and Moldavian research and how Lie's methods could be applied in practice was unknown.

Four years ago, one of the undersigned of this paper (journalist T. Rotaru: n.r.) wrote and prepared for print an article of memoirs, entitled *A troubled life between center and focus*, signed by Ana Sibirsky, wife of the regretted mathematician, acad. Constantin Sibirsky (see [20]). Then, from the first source, she learned about the troubles and researches of a scientist in identifying the scientific truth. At that time, the founder of the Moldavian scientific school in the field of the qualitative theory of differential equations was preoccupied with the elaboration of the theory of algebraic invariants for their application to the solution of the problems related to *the qualitative theory of differential equations*. The respective theory, elaborated by Poincaré in the years 1880-1882, allows to determine the character of the behavior of the solutions (integral curves) in case of differential equation is not solved explicitly. As mentioned above, Poincaré proposed a classification of the singular points of the solutions. But the problem of distinguishing them without explicit knowledge of the solutions proved to be very complicated.

Remembering those time, Prof. Mihail Popa confessed: "I never thought that I would ever deal with the Center and Focus Problem. But, after establishing the connection between Lie algebras and the Sibirsky graded algebras of invariants, I understand that the way is open to solve this problem, formulated by Henri Poincaré 130 years ago".

It should be noted that a large number of works in scientific centers of France, Russia, Belarus, China, Great Britain, Spain, Poland, Slovenia, Canada, USA, etc. are dedicated to the Center and Focus Problem and published in the world literature. Only in the Republic of Moldova their number is more than 100. At different stages the disciples of the academician C. S. Sibirsky (c.m. Nicolae Vulpe, prof. Alexandru Suba, dr. Iurie Calin, dr. Valeriu Baltag, dr. Dumitru Cozma and others), examined various issues of this problem and obtained important results. Some aspects of the development of mathematics in the Republic of Moldova are described in the book [5].

The mathematician Mihail Popa went his own way, starting from establishing the connection between Lie algebras and the Sibirsky graded algebras of invariants - a working tool in further searches. In this context, we will make some clarifications: the way to solve the Center and Focus Problem was initially determined by the Russian mathematician Alexander Lyapunov. But applying this method even for the simplest differential systems, you were faced with some enormous calculations, which could not be overcome even with the help of the most modern computers. That is why the Moldavian researcher took as a basis the Generalized Center and Focus Problem for the mentioned differential systems, avoiding the calculation of the Poincaré-Lyapunov quantities for each system. The Poincaré-Lyapunov quantities sequence (3) was replaced by a series of Lie algebras and a series of linear subspaces of Sibirsky graded algebras of invariants. To estimating the maximal number of algebraically independent focus quantities he applied these algebras. As a result, a finite numerical estimation was obtained for algebraically independent focal quantities that participate in solving the Center and Focus Problem for any system of polynomial differential equations (2). An analysis of the activity of Professor Mihail Popa is contained in the recently published article [4].

Lie algebra of the group $GL(2,\mathbb{R})$ and graded algebra of unimodular comitants and invariants of system $s(1,m_1,...,m_\ell)$

It is known that the system $s(1,m_1,...,m_\ell)$ admits the group $GL(2,\mathbb{R})$, to which the reductive Lie algebra $L_4 = \langle X_1, X_2, X_3, X_4\rangle$ corresponds, that consists of operators of linear representation of this group in the space of phase variables and coefficients of polynomials of this system [17].

This algebra generates a graded Sibirsky algebra of invariant polynomials with respect to the unimodular group $SL(2,\mathbb{R}) \subset GL(2,\mathbb{R})$ [16], which we write in the form

$$S_{1,m_1,...,m_\ell} = \sum_{(d)} S^{(d)}_{1,m_1,...,m_\ell}, \tag{5}$$

where (d) is called a type of the space $S^{(d)}_{1,m_1,...,m_\ell}$ and has the form [22, 17]

$$(d) = (\delta, d_0, d_1, ..., d_\ell), \tag{6}$$

and

$$S^{(d)}_{1,m_1,...,m_\ell} \tag{7}$$

is a finite-dimensional linear space of invariant polynomials (homogeneous comitants, invariants) of degree δ with respect to the phase variables x, y and of degree d_i with respect to the coefficients of the polynomials $P_{m_i}(x,y)$ and $Q_{m_i}(x,y)$ $(i = \overline{0,\ell})$ of system (2).

It is known that this algebra is finitely determined.

Using Lie algebra L_4 it can be shown that the maximal number of algebraically independent elements (Krull dimension) of algebra [18] is

$$\varrho(S_{1,m_1,...,m_\ell}) = 2\left(\sum_{i=1}^{\ell} m_i + \ell\right) + 3. \tag{8}$$

It is obvious that if Krull's dimension of algebra $S_{1,m_1,...,m_\ell}$ is $\varrho(S_{1,m_1,...,m_\ell})$ then for any invariant variety $V = \{i_1 = 0, i_2 < 0;\ i_1, i_2 \in S_{1,m_1,...,m_\ell}\}$ where i_1 is the trace of the matrix of the linear part of the system (2), (i_2 does not influence variety [18]) in this algebra $S_{1,m_1,...,m_\ell}$ no more than $\varrho(S_{1,m_1,...,m_\ell})$ algebraically independent elements will be found.

4. SOLVING THE GENERALIZED CENTER AND FOCUS PROBLEM

Remember that the focal quantities of system $s(1, m_1, ..., m_\ell)$ which has at origin of coordinates a singular point of the second type (center or focus), forms an infinite series of polynomials from the coefficients of this system which was written in the form (3).

It can be shown that to each focus quantity L_k ($k = \overline{1, \infty}$) one can associate a finite-dimensional linear spaces of invariant polynomials (unimodular comitants) [18]

$$S_{1,m_1,...,m_\ell}^{(d^{(k)})} \ (k = 1, 2, ...), \tag{9}$$

where

$$(d^{(k)}) = (\delta^{(k)}, d_0^{(k)}, d_1^{(k)}, ..., d_\ell^{(k)}) \tag{10}$$

is a type of a space (9) which was defined above.

The spaces (9) are characterized by the following fact [18]: they contain at least one homogeneous polynomial with respect to x and y (comitant), in which the coefficients are some quantities, named generalized focus pseudo-quantities. They are characterized by the fact that on the invariant variety $V = \{i_1 = 0, i_2 < 0;\ i_1, i_2 \in S_{1,m_1,...,m_\ell}\}$ some of these focus pseudo-quantities, except for a numerical constant, go to the corresponding focus quantity L_k, and the others go to zero. Invariant polynomials i_1 and i_2 does not depend on variables x and y, and i_1 which we will call null focus pseudo-quantity, belongs to the space $S_{1,m_1,...,m_\ell}^{(0,1,...,0)}$. Thus we obtain the sequence of spaces $\mathbb{R} = S_{1,m_1,...,m_\ell}^{(0,0,...,0)}$, $S_{1,m_1,...,m_\ell}^{(0,1,...,0)}$,..., $S_{1,m_1,...,m_\ell}^{(\delta^{(k)}, d_0^{(k)}, d_1^{(k)}, ..., d_\ell^{(k)})}$,... from $S_{1,m_1,...,m_\ell}$. Using them, we form a graded subalgebra $S'_{1,m_1,...,m_\ell}$ of algebra $S_{1,m_1,...,m_\ell}$ i.e. $S'_{1,m_1,...,m_\ell} \subset S_{1,m_1,...,m_\ell}$ (see [18]). From this inclusion it follows that between their Krull dimensions the following inequality holds $\varrho(S'_{1,m_1,...,m_\ell}) \leq \varrho(S_{1,m_1,...,m_\ell})$. With the help of this inequality and the above formula for Krull's dimension $\varrho(S_{1,m_1,...,m_\ell})$ it can be shown that the following is true:

Lemma 1 [18]. *The maximal number of algebraically independent invariants and comitants, which contain as coefficients null and generalized focus pseudo-quantities of system $s(1, m_1, ..., m_\ell)$, does not exceed the numerical upper bound* $2\left(\sum_{i=1}^{\ell} m_i + \ell\right) + 3$.

Taking into account properties of these comitants from algebra $S'_{1,m_1,...,m_\ell}$ which are related to focal quantities on the variety $V = \{i_1 = 0, i_2 < 0;\ i_1, i_2 \in S_{1,m_1,...,m_\ell}\}$ and Lemma 1, we obtain that there takes place:

Theorem 1 [18]. *The maximal number of algebraically independent focus quantities λ of system $s(1, m_1, ..., m_\ell)$ on the variety $V = \{i_1 = 0, i_2 <$*

$0;\, i_1, i_2 \in S_{1,m_1,...,m_\ell}\}$, *that take part in solving the Center and Focus Problem, does not exceed the the numerical upper bound* $2\left(\sum\limits_{i=1}^{\ell} m_i + \ell\right) + 3$.

Recall that for the differential systems $s(1,2)$ and $s(1,3)$ the number of essential conditions of center is $\omega = 3$ and 5, respectively, and for the differential system $s(1,2,3)$ according to one hypothesis it is $\omega \leq 13$. From Theorem 1 we have that maximal number of algebraically independent focus quantities of the differential system $s(1,2)$ does not exceed 9, for the differential system $s(1,3)$ does not exceed 11, and for the differential system $s(1,2,3)$ does not exceed 17.

These arguments, as well as the fact that the system $s(1, m_1, ..., m_\ell)$ on variety V has at the origin of coordinates a singular point of second type (center or focus), allow us to conclude that the following can be true

General Hypothesis [18]. *If the system* $s(1, m_1, ..., m_\ell)$ *has at the origin of coordinates a singular point of the second type (center or focus), then the number of essential conditions of center* ω *for this system does not exceed the the numerical upper bound* $2\left(\sum\limits_{i=1}^{\ell} m_i + \ell\right) + 3$.

Remark. *The expression* $2\left(\sum\limits_{i=1}^{\ell} m_i + \ell\right) + 3$ *is equal to maximal number of all possible nonzero coefficients of the right-hand sides of the system* $s(1, m_1, ..., m_\ell)$ *minus one.*

5. METHODOLOGICAL CONCLUSIONS

The question arises: how can it be explained that till now we do not have a solution to the Center and Focus Problem for any system $s(1, m_1, ..., m_\ell)$.

First of all, it is obvious that the Center and Focus Problem is a difficult one.

Till now, no general methods have been found for studying the Poincaré-Lyapunov constants in the sequence (3). In particular, there is no general solution strategy. However, the specified way in solving the Center and Focus Problem for system $s(1,2,3)$ is connected with cumbersome computations with application of supercomputers. These difficulties are also insurmountable for other more complicated systems.

From a psychological point of view, there are also impediments in terms of human conservatism to research problems in the traditional, classical way. History confirms that new, unusual methods with great difficulty are approved and appreciated at their fair value. However, according to Kurt Gödel (1906-1978) *incompleteness theorem*, the resources created so far are not sufficient for further studies. Therefore, it is undeniable that subsequent success depends largely on the new means created.

Solving the Center and Focus Problem: traditional aspect is equivalent to determining the essential conditions of the center

$$L_{n_1}, L_{n_2}, ..., L_{n_\omega} \quad (n_i \in \{1, 2, ..., k, ...\};\ i = \overline{1, \omega};\ \omega < \infty)$$

which involves *knowing the number* ω and the set $\Omega = \{n_1, n_2, ..., n_\omega\}$ the finiteness of which results from Hilbert's Theorem on the finiteness of basis of polynomial ideals. This is similar to finiteness theorem of Hilbert's number $H(n)$ in the second part of 16th Problem.

The problem of determining a finite number ω or obtaining for it an argued numerical upper bound (even in the form of a hypothesis), which until now is not known for any system $s(1, m_1, ..., m_\ell)$ is important for the complete solution of the Center and Focus Problem.

Formally the Center and Focus Problem consists in *determining the conditions* that guarantee that a singular point of the second type is a center.

Through counterweights, for example, *matter and antimatter, the world and anti-world*, we penetrate the essence of the universe, constituting amazing symmetries in the world of known phenomenas. From a mathematical point of view, such symmetries are constructed using *the principle of duality.* To construct a duality - means to determine a correspondence between certain types of objects, to which each property of the initial object corresponds a certain property of the respective object to this correspondence. In any duality, *objects* and some of their *properties* have *dual objects and properties.* This method, which is its *reasoning by anti-analogy*, it determines that many objects, different in form and content, are constructed, from the point of view of formal logic, in one and the same way. Any duality between two theories establishes at a certain level an isomorphism between these theories.

Let A and B be two theories, and $\beta : A \to B$ an application to which each object $a \in A$ and properties of objects from A correspond to object $\beta(a)$ and property $\beta(w)$, respectively. It is assumed that this application is logically continuous in the sense: that if the object a possessed the property w, then the object $\beta(a)$ possessed the property $\beta(w)$. The application β can also have a symmetry for certain properties: the properties w and $\beta(w)$ are symmetric (dual), if the object a possessed the property w only and only if object $\beta(a)$ possessed property $\beta(w)$. Under these conditions, for each problem π in theory A is corresponded to a certain problem $\beta(\pi)$ of theory B. From the logical continuity of the application β we obtain:

- If the problem π is solved positively, then the problem $\beta(\pi)$ is solved positively;
- If the problem $\beta(\pi)$ is solved negatively, then the problem π is not solved positively;

- If the properties in the problem π are symmetric, then the problems π and $\beta(\pi)$ are equivalent.

We note that the problem $\beta(\pi)$ is a generalized form of the original problem π. Solving the generalized forms is important if for the initial problem, for a long time, no solutions are found. Moreover, *the solutions to the generalized problem propose strategies and hypotheses for solving the initial problem.* Some estimates from the solution of the generalized problem can serve as working hypotheses for the initial problem.

The study of a new problem or an unsolved problem, applying the methods of solving a known problem is done by various methods: the method of substituting variables; border crossing method etc. They are well known since ancient times. With the boundary crossing method, Hopf, for example, built the solutions of the quasi-linear equations.

Amazing constructions have been proposed by V. I. Arnold in the study of the critical points of the functions defined on varieties with the help of semi-simple Lie groups [2].

In the second half of the last century, the famous Russian mathematician Victor P. Maslov proposed a new theory - *Idempotent Analysis* - based on changing the usual operations $\{+, \times\}$ with two other operations (see [13,14]). By this exceptional method succeeded:

- reduction of Bellman and Hamilton-Jacobi equation theory to linear equation theory;
- studying Fourier transforms using the Legendre transformations;
- research with known methods of many problems in quantum physics, thermodynamics, superconductivity, etc.

Any constructed duality is a valuable event for those theories. The dualities of projective geometry, the Pontryagin duality in the theory of locally compact abelian groups, the Kolmogorov-Gelfand dualism of compact spaces and functional Banach algebras, the dualities of Serre and Alexander in topology, the Radu Miron duality of Cartan and Finsler spaces, the De Morgan duality are well known in sets theory, Stone duality between zero-dimensional compact spaces and Boolean rings, wave-corpuscle duality in theoretical physics, Kramers-Wannier dualism in statistical physics, etc.

All these examples have a simple explanation: many problems, different in form and content, can be studied with a single mathematical method and from a single point of view.

Now let's fix two polynomials $P = P(x, y)$ and $Q = Q(x, y)$ with non-zero linear parts. Then

$$P = \sum_{i=0}^{\ell} P_{m_i}(x, y),\ \ Q = \sum_{i=0}^{\ell} Q_{m_i}(x, y) \tag{11}$$

where $P_{m_i}(x, y)$ and $Q_{m_i}(x, y)$ are homogeneous polynomials of degree $m_1 \geq 1$ in x, y, and $m_0 = 1$. The set $\{1, m_1, ..., m_\ell\}$ determine a system $s(1, m_1, ..., m_\ell)$ of the form (2). An infinite series of polynomials (Poincaré-Lyapunov constants) $p = \{L_k : k = 1, 2, 3, ...\}$ of the form (3) were constructed for this system. For each focal quantities L_k ($k = \overline{1, \infty}$)) it can be corresponded to a finite-dimensional linear spaces of invariant polynomials, unimodular (comitants) $S^{(d^{(k)})}_{1,m_1,...,m_\ell}$ of the form (9), that forming a sequence of the type a. In this way, a correspondence is constructed between the sequence p of polynomials (3) and the sequence of linear spaces of the form (9). The first type of sequences form the P class, and the second type of sequences form class A. Therefore, we have built a correspondence $f : P \to A$. Probably this correspondence is a functor (a symmetry) in some sense. For each sequence $p \in P$ is determined the number $E(p)$ of the essential conditions of center and the set $\Omega = \{n_i : i = \overline{1, \omega}\}$, and for each sequence $a \in A$ is determined the number $gE(a)$ of algebraically independent focal quantities, i.e. the number of generalized essential conditions of the center and the set $g\Omega = \{m_i : i = \overline{1, \lambda}\}$, respectively. Consider $a = f(p)$, if $p = \varphi(s(1, m_1, ..., m_\ell), P, Q)$ and $a = \psi(s(1, m_1, ..., m_\ell), P, Q)$ are respectively determined by system $s(1, m_1, ..., m_\ell)$) and the polynomials P, Q with the respective decompositions.

Theorem 1 says that

$$gE(a) \leq 2\left(\sum_{i=1}^{\ell} m_i + \ell\right) + 3, \text{if} a = \psi(s(1, m_1, ..., m_\ell), P, Q).$$

This inequality is the solution of the Generalized Center and Focus Problem. The General Hypothesis assumes that $E(p) \leq 2\left(\sum\limits_{i=1}^{\ell} m_i + \ell\right) + 3$ for $p = \varphi(s(1, m_1, ..., m_\ell), P, Q)$.

There are simple examples for which inequality $E(p) > gE(f(p))$ holds, but for all known examples $E(p) \leq 2\left(\sum\limits_{i=1}^{\ell} m_i + \ell\right) + 3$. This was the basis for launching the General Hypothesis. Probably, the truth about this inequality is contained in the functor properties of the correspondence f. But to solve the Weak Center and Focus Problem is sufficient a numeric function $h : \mathbb{N} \to \mathbb{N}$ defined on the sequence of natural numbers $\mathbb{N}$ and set to $E(p) \leq h(gE(f(p)))$. In this context, there is another unresolved problem: to study, from a general point of view, the second part of the Center and Focus Problem, that is, to determine the subset of $g\Omega = \{m_i : i = \overline{1, \lambda}\}$ of the set $\Omega = \{n_1, n_2, ..., n_\omega\}$ of indices of algebraically independent focal quantities.

Let's admit that at some point the Center and Focus Problem is solved positively. In this situation, will the study of particular cases be of interest?

We think so. Whether for some system $\omega = 3$ and $1000^{1000} \in \Omega$. Who and by what means will determine the type of the singular point? It is well known that the simplex method, proposed by George Dantzig in 1947, is a general method of solving the linear programming problem formulated by L. V. Kantorovich in 1939. But research in this area continues.

Therefore, the general solution of the initial or generalized problem reflects the ways of examining the different particular cases. Moreover, because the sequence of form (3) is infinite, the solutions of the initial problem and the generalized problem allow to *discover effectively* particular cases for which the number ω and those of Ω are accessible for a deeper study of the properties of integral curves of certain types. Therefore, the role of generalized solutions is enormous in the deep study of different classes of differential systems (1).

References

[1] Arnold V. I., Afrajmovich, V. S., Ilyashenko Yu. S., Shilnikov, L. P. *Bifurcation theory.* Current problems in mathematics. Fundamental directions. Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 5, 1986, pp. 5-218.

[2] Arnold V. I. *Critical points of functions on a manifold with boundary, the simple Lie groups Bk, Ck and F4 and singularities of evolutes.* Russian Mathematical Surveys 33:5 (1978), pp. 99-116.

[3] Bautin N. N. *On the number of limit cycles which appear with the variation of coefficients from equilibrium position of focus or center type.* Math. Sb. 30(72) (1952), 181-196 (Amer. Math. Soc. Transl., 100, 1954, pp. 397-413).

[4] Ciobanu M., Cojocaru S., Şubă A., Rotaru T. *Matematicianul Mihail Popa la ceas de glorie: 130 de ani de zbucium pentru soluţionarea problemei lui Poincaré.* Literatura şi Arta, 1 august, 2013.

[5] Ciobanu M. M., Valuţă I. I. *Elemente de istorie a matematicii şi matematica în Republica Moldova.* Chişinău, Tipogr. AŞM, 2006.

[6] Dulac H. *Sur les cycles limits.* Bull. Soc. Math. France, 51, 1923, pp. 45-188.

[7] Graf V., Bothmer H. C., Kroker J. *Focal Values of Plane Cubic Centers.* Qual. Theory Dyn. Syst. 9, (2010), pp. 319-324.

[8] Ilyashenko Yu. *Centennial History of Hilbert's 16th problem.* Bulletin of the AMS 39(3), (2002), pp. 301-354.

[9] Ilyashenko Yu. *The origin of limit cycles under perturbation of the equation $dw/dz = -Rz/Rw$, where $R(z, w)$ is a polynomial.* Mathematics of the USSR Sbornik 7, (1969), pp. 353-364

[10] Ilyashenko Yu., Kaloshin V. *Bifurcations of planar and spatial polycycles: Arnold's program and its development.* Fields Inst. Commun. 24 (1999), pp. 241-271.

[11] Kaloshin V. *The Existential Hilbert 16-th Problem and an Estimate for Cyclicity of Elementary Polycycles.* Invent, math. 151 (2003), pp. 451-512.

[12] Liapunoff A. *Probléme général de la stabilité du mouvement.* Annales de la Faculté des Sciences de Touluose. Sér 2, 9, (1907), pp. 204-477. Reproduction in Annals of Mathematics Studies 17, Princeton: Princeton University Press, 1947, reprinted, Kraus Reprint Corporation, New York, 1965.

[13] Litvinov G. L. *The Maslov dequantizan, idempotent mathematics: a very brief introduction.* Idempotent Mathematics and Mathematical Phisics. Contemporary Mathematics, 377 (2005), pp 1-17.

[14] Maslov V. R. *On a new principle of superposition for optimization problems.* Russian Mathematical Surveys 42:3, (1987), pp. 43-54.

[15] Mathematical Encyclopedia *Center and focus problem.* http://dic.academic.ru/dic.nsf/encmathematics/6088/?????).

[16] Poincaré H. *Mémoire sur les courbes définies par une équation différentielle.* J. Math. Pures et Appl. (Sîr. 3) 7 (1881) pp. 375-422; (Sér. 3) 8 (1882) pp. 251-296; (Sér. 4) 1 (1885) 167-244; (Sér. 4) 2 (1886), pp. 151-217.

[17] Popa M. N. *Algebraic methods for differential systems.* Editura the Flower Power, Universitatea din Piteşti, Seria Matematică Aplicată şi Industrială, 2004, (in Romanian).

[18] Popa M. N., Pricop V. V. *Applications of algebraic methods in solving the center-focus problem.* Buletinul AŞM, Matematica, 1(71), (2013), pp. 45-71.

[19] Schlomiuk D. *New developments based on the mathematical legasy of C. S. Sibirschi.* Buletinul AŞM, Matematica, 1(71), (2013), pp. 3 - 10.

[20] Sibirsky Ana. *O viaţă zbuciumată între centru şi focar*, "Literatura şi Arta" din 11 decembrie 2008.

[21] Sibirsky K. S. *Algebraic invariants of differential equations and matrices.* Kishinev, Ştiinţa, 1976 (in Russian).

[22] Vulpe N. I. *Polynomial bases of comitants of differential systems and their application in qualitative theory.* Kishinev, Ştiinţa, 1986 (in Russian).

[23] Żołądek H. *On certain generalization of the Bautins theorem.* Nonlinearity, 7 (1994), pp. 273-279.