CONTENTS

IN MEMORIAM –

ACADEMICIAN MITROFAN M. CHOBAN

05.01.1942 – 02.02.2021

President of the Mathematical Society of Republic Moldova (1999-2021). Vice-President of the Romanian Society of Applied and Industrial Mathematics-ROMAI (1995-2021). Founder of the school of general topology in the Republic of Moldova. Mathematics professor and researcher at the Tiraspol State University for over 50 years. His original contributions to mathematics can hardly be fully estimated.

Mitrofan Cioban (or Choban – as he chosed to sign his papers) was born on January 5, 1942 in Copceac, Tighina county, Moldova, Romania to farmers Mihail and Tecla Cioban. He was the fourth of the couple's seven children. His parents, encouraged Mitrofan to get as good an education as was possible at the time. They sent him to a boarding school in the neighboring village of Volontiri. Then, in 1959, he finished his high school studies at the high-school from village Volontiri. After graduating from high school, he worked for a year in the local agricultural cooperative.

At the age of 17, M. Choban decided to become a mathematician.

Because he spoke no Russian at the time, he had to give up on his early dream of becoming a ship designer. At the age of 17 he decided to become a mathematician. In 1960 he enrolled at the Tiraspol State Pedagogical Institute (Moldova), at the Faculty of Physics and Mathematics within the Tiraspol State Pedagogical Institute (Tiraspol State University) the first higher education institution in Moldova. Within the Faculty of Physics and Mathematics, the young Mitrofan met great university professors such as: P. Osmatescu, C. Cozlovschi, M. Cozlovschi, Gh. Gleizer, I. Valuta, etc. Soon he joined a seminar in topology led by Professor P. Osmatescu. Thus, without initially realizing, the topology seminar cultivated for the young student a great passion for research in topology for a lifetime.

The Topology seminar of Pavel Alexandrov.

After a year of study in Tiraspol, at the initiative of Professor Petru Osmatescu, Mitrofan Choban and two other young people were sent to study at Moscow State University. Ion Valuță, being at that time the doctoral student of Professor A. Kurosh, was invited by the Academician P. S. Alecsandrov, to attend the conversation with M. Choban. Professor I. Valuță remembers: "To the question asked, by P.S. Alexandrov, M. Choban doesn't answer! Either he didn't understand the question well enough, or he couldn't figure out how to answer in Russian. P.S. Alexandrov asked the young people to wait and told them that the results would be announced very soon. I thought I should go out too, but he told me to stay. The academician told me: «You have sent three students to university, but we will admit only two for study.» I thanked him, but at the same time I dared to tell him that I had a misunderstanding. He asked me what it was. The young man who did not answer anything, in my opinion, had special abilities for scientific research. P. S. Alexandrov's answer struck me: «If you say so, then we will accept everyone to study. We will always be able to expel the weakest in mathematics». In a short time, the young mathematician demonstrated his creative potential in mathematics."

Whithin the Moscow State University, M. Choban started attending the Topology seminar of Academician Pavel Alexandrov. The scientific coordinator during his years of study at the University was Professor A. V. Arhangel 'skii.

Whithin the Topology seminar he did not hesitate to express his opinion when discussing scientific reports of venerable specialists. For example, it was believed that one result of the American mathematician Arthur Stone is final and not subject to further development. At the end of April 1964, while discussing A. Stone's results, unexpectedly, student M. Choban said that this was not the case. And, of course, many doubted that he was right. Nevertheless, M. Choban did not back down, although many looked with a grin at the insolent student. Pavel Sergeevich turned out to be at his best - he invited Mitrofan Choban to present his opinion on the development of the idea from the works of A. Stone in a week or, of course, to present his apology to everyone for his daring challenge. A week later, surprisingly to everyone, Choban presented a wonderful scientific report, which formed the basis of his famous scientific article on factorial mappings with separable preimages.

Profesor Stoyan Yordanov Nedev, Bulgaria, remembers: ”Apparently, this report put M. Choban on a special position - he was recognized as a highly qualified specialist in the field of topology. He finished the third year of study with two excellent works, which were soon published in the journal Reports of the Academy of Sciences of the USSR. Despite the fact that he did not know Russian, German and English, he perfectly knew almost all the works written in these languages over the past 70 years.”

This was a generalization of one of A.H. Stone’s results in the first paper Mitrofan had refereed. He has proved the following:

Theorem 1. If f : *X—>Y is a quotient mapping of a metrizable space X onto a Tychonoff first-countable separable space Y, and all fibers under f are separable, then Y is metrizable.*

After this case, P.S. Aleksandrov has repeatedly said: „*if Mitrofan said this, then this is beyond doubt*”. P. S. Alexandrov recommended the first article of Mitrofan Choban containing Theorem 1 for publication in Doklady AN SSSR, one of the most prestigious Soviet mathematical journals. It had appeared in print in 1966, and is his first publication. In fact, his first mathematical result had been obtained earlier, in November 1964, but was published later.

Professor A. V. Arhagel’skii remembers ”Mitrofan became a «star» in the main seminar on General topology of P.S. Alexandroff. This made this seminar even more attractive to students, more popular with them. Mitrofan was active not only in mathematics. He was also a sportsman, participated in Greco-Roman wrestling competitions and won the title of the Champion of Moscow University. At the MGU the name Mitrofan is quite rare and therefore everyone called Mitrofan Choban simply Mitrofan.”

It is necessary to mention that student M. Choban managed to obtain a series of beautiful results in topology. He published a valuable paper in the prestigious journal Sovietic Mathematiceskij Doklady (see Sov. Math. Dokl) . Thus, M. Choban's publications during the student period at Moskow University are:

1. *On the behavior of metrizability under quotient s-mappings. Dokl. Akad. Nauk SSSR,* ***166****:3 (1966), 562–565.*
2. *Behavior of metrizability under monotone quotient mappings. Dokl. Akad. Nauk SSSR,* ***168****:3 (1966), 535–538.*
3. *Behavior of metrizability under factorial s-mappings. Abstracts. Moscow. International Congress of Mathematicians. Section 8, M. (1966), 30.*
4. *Certain metrization theorems for p-spaces, Dokl. Akad. Nauk SSSR,* ***173****:6 (1967), 1270–1272.*
5. *Finite-to-one open maps, Dokl. Akad. Nauk SSSR,* ***174****:1 (1967), 41–44.*
6. *Perfect mappings and spaces of countable type, Vestnik Moskov. Univ. Ser. 1. Mat. Mekh., 1967, 6, 87–93.*

He brilliantly graduated this Faculty in 1967. Here he attracted the attention of the famous and exigent teaching staff. He was recommended to Professor A.V. Arhangel'skii, who marked his whole evolution as a mathematician, so that at the beginning of year 1967, he was privileged to study and work in the prestigious collective of researchers in topology, brilliantly dominated by the great personality of A.N. Kolmogorov. A year after his graduation of the faculty he has begun preparing his Doctor's Degree - the chosen specialty being Topology - at the State University "M. Lomonosov" of Moscow. He graduated in 1970 the Ph.D. in Mathematics with the thesis "Relations between classes of topological spaces", his adviser being Professor A.V. Arhangel'skii.

Professional career at Tiraspol State University.

Academician Mitrofan Choban was a mathematics professor and researcher at the Tiraspol State University for over 50 years**.** Professor Mitrofan Choban started his didactic career at the Tiraspol State University (Moldova) in 1970 and continuously worked in this university till 2021. He was in succession Senior Lecturer at the Department of Geometry and Didactics of Mathematics (1970-1974), Associate Professor at the Department of Geometry and Didactics of Mathematics (1975-1976), the Head of the Department of Geometry and Didactics of Mathematics (1976-1983), ViceRector for Science (1983-2002), President of the University (2002-2009) and the Head of the Department of Algebra, Geometry and Topology (2009-2021). Since 1981, Professor Mitrofan Choban was adviser for PhD Thesis as well as for Dr. Sc. Thesis. He advised 22 doctors of sciences and 4 Doctors Habilitat in Mathematics. His teaching activity concerned the courses of "Geometry" as well as of "Set Theory and Topology". He also taught several special courses: *Functional spaces, Topological groups, Algebraic theory of automata, Topological universal algebras, etc.*

Scientific performance.

In 1980, he became Dr. Habilitat in Mathematics with the thesis "Set-valued mappings and their applications" (scientific consultant again being A.V. Arhangel'skij). In 1995, he was elected corresponding member of the Academy of

Sciences of Moldova. Then, in 2000 he was elected Member of the Academy of Moldova, the highest forum of Moldavian spirituality and the highest recognition which a scholar may receive.

His scientific concerns group the following main directions: topology, topological algebra, descriptive set theory, functional analysis, topological optimization theory, measure theory, etc. He solved a number of well-known problems, formulated in the last 100 years by P.S. Alexandrov, A. V. Arhagel'skii, WW Comfort, F. Hausdorff, A. N. Kolmogorov, AI Maltsev, E. Michael, I. Namioka, B.A. Pasynkov, A. Pelczynski, R. Pheps, Z.Frolik, A. Stone, Z. Semadeni etc.

Mitrofan Cioban published in academic journals from 1966 to 2021, mostly under the name of Choban, but also under the name Čoban, and occasionally Chobanu, or Coban.

Thus, Professor Choban authored more than 300 papers and 20 books in many branches of Mathematics. He brought important contributions in: Hausdorff's problem on Borelian classes of sets; Alexandroff's problem about the structure of compact subsets of countable pseudocharacter in topological groups; Arhangel'skii's problem on the zero-dimensional representation of topological universal algebras; two Maltsev's problems on free topological universal algebras; two Michael's problems about G -sections of open mappings of compact spaces and of the k-coverings of open compact mappings of paracompact spaces; Phelps' problem about the structure of the set of points of Gateaux differentiability of convex functionals (with P.Kenderov and J.Revalski); Tichonoff's problem about well-possedness of optimization problems in the Banach spaces of continuous functions (with P.Kenderov and J.Revalski); Confort's problem about Baire isomorphism of compact groups; Pasynkov's problem about Raikov completion of topological groups; Arhangel'skii's problem on metrizability of o-metrizable topological groups (with S.Nedev); Pelczinski's and Semadeni's problems about structure of Banach spaces of continuous functions on special compact subsets of quotient spaces of topological groups, etc.

He attended more then 100 scientific forums: 1) International mathematical congresses (Moscow, Zurich, Berlin), conferences (Moscow, New York, Baku, Sofia, Pitești, Oradea, Sozopol, București, Timișoara, Brașov, Chișinău, Novosibirsk, Tbilisi, Lecce, Iași, Constanța, Sicilia, Livov, Varna, Borovets, Ohrid), 2) Symposiums (Prague, Eger, Burgas, Genova, Marseille), 3) All-Union mathematical conferences and symposiums (Minsk, Moscow, Tiraspol, Chișinău, Livov, Sankt-Petersburg, Novosibirsk, Tobolsk, Tartu), 4) several national conferences. Having a great prestige in the world of Mathematics, Professor Mitrofan Choban has been invited to lecture by the well-known institutions: Institute of Mathematics and Informatics of the Academy of Science of Bulgaria, the Universities of Oradea, Tartu, Tbilisi, Tashkent, Tsukuba, Bishkek, North Bay (Canada). Moreover, he was invited speaker of the forums: V-th Prague Topol. Symp. (1981), Topological Colloq., Eger, Ungary (1983), International Moscow Topological Conference (1979), Soviet-Japan Topological Symposium, Niigata

(1991), Workshop on General Topology and Geometric Topology, Tsukuba (1991), Workshop ”Well-Posedness in Optimization, Margarita di Liguri”, Italy (1991), International Conference on group theory, Timișoara, Romania (1991), Workshop ”Well-Posedness in Stability and Optimization”, Sozopol, Bulgaria (1993), Conferences on Applied and Industrial Mathematics, Romania (1994-2019), International Congress of Mathematical Society of South Europe, Borovets, Bulgaria (2003), International Conference ”Geometric Topology, Discrete Geometry and Set Theory” in celebration of the centennial of Ljumila V.Keldysh, Moscow (2004), International Conference ”Quality in Formal and non Formal Education”, Iași, Romania (2010), Centennial Conference ”Alexandru Myller” Mathematical Seminar”, Iași, Romania (2010), ICTA Islamabad, Pakistan (2011), 8-th International Conference on Applied Mathematics, Baia Mare, Romania (2011), etc.

Due to his prestige in the world of Mathematics he became: 1. Member of the Editorial Boards of: - Buletinul Academiei de Științe a Moldovei, Matematica, ROMAI Journal, Scientific Annals of Oradea University, Qusigroups and related systems; 2. President of the Mathematical Society of Republic Moldova (1999-2021); 3. Vice-President of the Romanian Society of Applied and Industrial Mathematics (ROMAI) (1995-2021); 4. Member of the Moscow Mathematical Society; 5. Member of the Romanian Mathematical Society.

The special appreciation of his scientific work brought him several prizes, titles and orders, namely: prize of the All-Union Presidium of the ScientificTechnical Societies (1968); prize ”Boris Glăvan” of the Komsomol of Moldova, in Mathematics (1974); title Excellent of the High Education of the USSR (1980); order ”Gloria Muncii” (Glory of Labor) of the Republic of Moldova (2000); State Prize of the Republic of Moldova (2002); Honorary citizen of the Ștefan Vodă county, Republic of Moldova (2005); prize ”Academician Constantin Sibirschi” (2006); Doctor Honorius Causa of the Oradea University (2006); order ”Honor” of the Republic of Moldova (2010); Medal “Dmitrie Cantemir” (2007), Medal ”Nicolae Milescu Spataru” (2012); 70 years since the creation of the first Research Institutions and 55 of the ASM (2016); Researcher of the Year Award (2016); Order of the Republic of Moldova (2020).

The activity within ROMAI.

A few more words should be told about Academician’s Choban relation with ROMAI. In 1992 The Romanian Society for Applied and Industrial Mathematics - ROMAI was founded by Professor Adelina Georgescu, and soon after, with a visionary intuition, Professor Georgescu tried to establish professional relationships with the mathematicians from Republic of Moldova. Soon she succeeded to bring into the Society a number of good and important mathematicians from the sister country. One of them was Academician Mitrofan Choban. In those years that were after the fall of the communism, a vivid relationship was born between the mathematicians from the two sides of the river Prut. It was a period of hope in a

better future, a period when the creative activity of the ROMAI members was effervescent. The Romanian and the Moldavian members of ROMAI met each others with joy at the conferences CAIM (Conferences on Applied and Industrial Mathematics) held each year, either in Romania or in Republic of Moldavia. It was the great pleasure of Professor Adelina Georgescu to organize, at the end of CAIMs, beautiful trips in several zones of Romania, so that the Moldavian colleagues can meet the rich nature and historical relics as well as the contemporary realities of Romania. Also, with great hospitality, the Moldavian members of ROMAI received the Romanian colleagues in Chișinău, at some editions of CAIM, and also shared with them the beauties and the history (with common roots with the Romanian history) of Republic of Moldova. In all the period 1993-2019 Academician Choban was present at CAIMs, with invited lectures. As asserted above, în 1995, he was elected Vice-President of ROMAI. The activity of our Society benefitted a lot from his presence in its Board. He realized the connection between the two main wings of ROMAI. He was always glad to give an advice, to solve any organizing problem, each year delivering high-quality invited lectures in CAIMs, always being a leader of the Moldavian „team" of mathematicians that came in Romania. As a member of the Editorial Board of ROMAI Journal, Academician Choban was very involved either in reviewing or in finding the proper reviewers for the papers received at ROMAI Journal.
We shall always remember him as a kind, gentle, and very pacient person, having a good sense of humour, a person that used his remarquable intelligence in Mathematics as well as in the inter-human relationships.

Academician Mitrofan Choban was a strong pillar of ROMAI, and his departure from this world meant a great loss for our Society. May his soul rest in peace!

Appreciations and recognitions from the academic world.

- "For me there is no doubt that Professor Mitrofan M. Choban is a world-class scholar" - *Professor P. Kenderov, Members of Bulgarian Academy of Sciences.*

- "M.M. Choban is a most talented mathematician, with a great creative force"- *Professor A. Arhangel'skii, Moscow State University.*

- "The results of M. M. Choban gave rise to a whole series of publications in many countries ..." - *Academician A. Fomenko, Russia.*

- "Many experts in the field of topology consider it an honor to carry out scientific research together with M. M. Choban" - *Professor O. Lupanov and Professor V. Fedorchuk, Moscow State University.*

- "Professor M. M. Choban is one of the most famous and recognized topologists in the world. Well known are his substantive research on the theory of multivalued mappings, topological algebras, descriptive theory of sets and function spaces, as well as their numerous applications to other areas of mathematics." - *M. Abel, Professor of the Tartu University, President of the Estonian Mathematical Society*

- ”Mitrofan Choban is in a possesion of an incredible knowledge of the topological phenomena and strong and sophisticated techniques.” - *Professor G. Skordev, Corresponding Member Of the Bulgarian Academy of Sciences*

- "Academician Mitrofan Chobanu is part of the elite of Moldovan scientists. His mathematical, educational and civic work is overwhelming in the field of mathematics and has addressed new and difficult problems in topology, modern algebra and its applications.” - *Academician Radu Miron, ”Alexandru Ioan Cuza” University od Iași*.

- ”Academician Mitrofan Chobanu was and remains a star in the world of mathematicians.” - *Professor Larențiu Calmuțchi, Tiraspol State University*.

- "It is natural to ask: how does academician Mitrofan Choban conceive mathematics? Of course, he sees it in all its complexity, with one small exception - in no way does he perceive it as a form of snobbery. Who but him has tried all the facets: research, teaching, leadership. And every time he succeeded brilliantly, obtaining valuable results, being a talented professor and loved by students, leading the University of Tiraspol and the Mathematical Society.”- *Professor C. Gaindric, Corresponding member of the Moldovan Academy of Sciences, Professor S. Cojocaru, Corresponding member of the Moldovan Academy of Sciences.*

- "I have very many memories of his reign and I am very grateful to him for everything he has done for our countries, for ROMAI, for CAIMs and for me." - *Professor Adelina Georgescu, Romania*

- From whom did Mitrofan M. Choban learn?
  M. Choban: „Alexandr *Arhangel'skii* served me at that time as a model of professionalism and exemplary conduct ... Many other teachers ... contributed to my training as a specialist. For example, I learned from Vladimir Andrunachievich and Pavel Alexandrov the management of the organization of scientific research, from Otto Schmidt, Andrei Kolmogorov and Andrei Tikhonov - the organization of mathematical applications in various fields, from Anatol Maltsev and Alexandr Curosh - universal methods of examining things in depth and at the same time, simple and clear etc. ”

Directory Committee of the
Romanian Society of Applied and Industrial Mathematics – ROMAI

ROMAI J., v.17, no.1(2021), 1–16

130 YEARS OF EFFORT FOR SOLVING THE POINCARÉ'S CENTER-FOCUS PROBLEM

Mitrofan Choban[1], Tatiana Rotaru[2]

[1] *Sciences Academy of Moldova, Chişinău, Republic of Moldova,*

[1] *Tiraspol State University, Chişinău, Republic of Moldova,*

[2] *National Council for Accreditation and Attestation, Chişinău, Republic of Moldova*

Abstract It is well known that many mathematical models use differential equation systems and apply the qualitative theory of differential equations, introduced by Poincaré and Lyapunov. One of the problems that persists in order to control the behavior of systems of this type, is to distinguish between a focus or a center (the Center and Focus Problem).

The solving of this problem goes through the computation of the Poincaré-Lyapunov constants. In the case of polynomial right-hand sides it follows from Hilbert's theorem on the finiteness of bases of polynomial ideals that in this sequence only finitely many are essential and that the remaining ones are consequences of them. Hence, this problem is divided in two parts: in the first, to estimate the number of essential constants; in the second, to determine the minimal upper border of the indexes of a complete system of essential constants. The first part is called the Weak Center and Focus Problem.

The problem of estimation the maximal number of algebraically independent essential constants is called the Generalized Center and Focus Problem. Recently M. N. Popa and V. V. Pricop have solved the Generalized Center and Focus Problem. The present article contains: some moments related to the history of the Center and Focus Problem; the contribution of the Sibirsky's school in the solving of the Center and Focus Problem; methodological aspects of the Popa - Pricop solution of the Generalized Center and Focus Problem.

The problem of the estimation of the minimal upper border of the indexes of a complete system of algebraically independent essential constants is open. Another open problem consists on determining what differential systems are integrable.

Keywords: Poincaré-Lyapunov constants, center and focus problem, generalized center and focus problem.

2020 MSC: 34C60, 37G15.

Editor's Note

The article "130 years of the effort the solving of the Poincaré's center-focus problem", signed by academician Mitrofan Choban and journalist Tatiana Rotaru, was published in Romanian in Journal of Science and Innovation, Culture and Arts of the Academy of Sciences of Moldova "Akademos", 3(30), 2013, pp. 13-21. The work remains current today and had continuity. Based on the examined problem, have been published two monographs by M. N.

Popa V. V. Pricop - one in Russian (2018) and recent - in English: Popa M. N., Pricop V. V. The Center and Focus Problem: Algebraic Solutions and Hypoteheses. Ed. Taylor&Frances Group, 2021, 215 p. The cited above article is reproduced in English in this number of ROMAI Journal to follow the beginning and the stages of solving an old mathematical problem that a troubled the minds of mathematicians for more than a century. The publication of the English version is done with the accord of the Editorial Board of "Akademos".

The much regretted academician Mitrofan Choban (05.01.1942-02.02.2021) was an illustrious international mathematician, patriot and patriarch of science and education in the Republic of Moldova, a great friend and activist of ROMAI, who would have turned 80 in February 2022, if a relentless illness wouldn't have take him from our ranks.

We shall always miss him.

1. FROM THE HISTORY OF MATHEMATICS

Talking about mathematics or mathematicians is a challenge with the risk of being misunderstood or even rejected from the start. Gone are the days when research disciplines were not built strictly, and dialogue between their representatives was a normal way of existence and collaboration. But the nineteenth century brought many surprising discoveries to human civilization. Many of them are the result of logical analysis of phenomena or the mathematical one: Gauss discovered by calculation the asteroids Ceres, Palass, Vesta, Juno; Galle also based on calculations of the identity of the planet Neptune (1846); Mendeleev, starting from the atomic mass, he systematized the chemical elements and anticipated the existence of many new ones; Schliemann, based on Homer's descriptions, determined the location of Troy, etc. It is mathematical research that has helped to solve a number of problems that have plagued the minds of scientists for nearly 2500 years, beginning with Platon, Aristotle, Euclid, Archimedes, as well as the creation of new disciplines in the field.

At the beginning of the twentieth century, mathematics proliferated so much that it became, figuratively speaking, a Kingdom of the Universe of Science, although this word in Greek means "learning", "study", "science". We consider indisputably the fact that science is also an art, an art of human depth and strength of thought. A little earlier, in the 19th century, the brilliant French mathematician Henri Poincaré (1854-1912) created new fields of research, as topology, qualitative theory of dynamical systems, etc.

The development of mathematics in Romania was deeply connected with Poincaré's work and activity: he was a member of the Commission for the defense of doctoral theses for many high-performance mathematicians and physicists such as Nicolae Coculescu, Gheorghe Ţiţeica, Anton Davidoglu, Dragomir Hurmuzescu, Dimitrie Pompeiu, Constantin Nicolau and others. By quantitative methods, Spiru Haret had demonstrated the instability of the Solar System. The qualitative approach, as well as in a much broader framework, led Poincaré to confirm this fact. The results of the KAM theory (Kolmogorov-Arnold-Moser) showed that the Solar System is in a state of relative stability. As a result of this fruitful cooperation with Romanian researchers, Henri Poincaré was awarded the honorary titles of Doctor Honoris Causa of Kolosvar University (Cluj) and Honorary Member of the Romanian Academy (1909).

Henri Poincaré [16] formulated a series of problems, the solution of which determines the further development of science. A sensational news, in this sense, was in the first years of this millennium the solution of the Poincaré Conjecture by the enigmatic Russian mathematician Grigori Perelman. His 2002 demonstration ranked first in the top of the most important scientific discoveries in decades. Poincaré's conjecture or Poincaré's hypothesis, first stated by Poincaré in 1904, states that if S is a compact 3-dimensional variety (a closed, bordered or borderless 3-dimensional surface, immersed in a 4-dimensional space), in which any circle can be continuously deformed until it becomes a point, then this space S is equivalent, from a topological point of view (homeomorphic), to a 3-dimensional sphere. Solving a big problem generates the formulation of other new problems, which determine the further development of science. It is hoped that this famous result of Perelman's will help solve the problem of classifying three-dimensional varieties - another important problem stated by Poincaré in 1904, and especially the study of the Universe.

One of the famous problems of the qualitative theory of differential equations is the Center and Focus Problem, formulated by Poincaré 130 years ago ([15]. In 1881-1899 he studied the periodic and asymptotic solutions of differential equations, developed the method of the small parameter, the method of fixed points, the method of integral invariants, which became classical methods of research not only in mechanics and astronomy, but also in static physics, quantum mechanics. Working on the problems of celestial mechanics, he simultaneously laid the foundation of a new science - topology, which he called "Analysis situs".

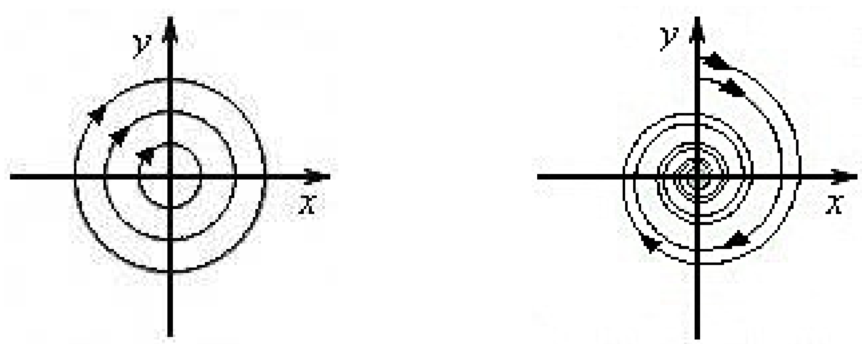
2. THE CENTER AND FOCUS PROBLEM

Let

$$\dot{x} = X(x, y), \ \dot{y} = Y(x, y) \tag{1}$$

be an autonomous system of differential equations. We admit that the functions $X(x, y)$ and $Y(x, y)$ are analytical. The solutions of this system of equations are called integral curves. The qualitative theory has as its starting point the stability theory and the problem of the movement of three and more bodies in the celestial mechanics. Henri Poincaré would say that even if the differential equation is not solved explicitly, it is possible to determine the character of the behavior of the solutions (integral curves) and proposed a classification of the singular points of the solutions: saddle, focus, center, node.

a) saddle b) node
Fig. 1. Singular points of the first type



c) center d) focus
Fig. 2. Singular points of the second type

It is known that if the roots of the characteristic equation of the singular point $O(0,0)$ are imaginary, then it can be center or focus (singular point of the second type). In the case of a center the singular point is surrounded by the closed trajectories and in the case of a focus it is surrounded by spirals. **The Center and Focus Problem** is to determine *the condition* under which a singular point is a center. In general case the Center Problem is algebraically unsolvable [9, 1, 19].

The Center and Focus Problem has deep ties to **David Hilbert's 16th problem**. In 1900, at the Second International Congress of Mathematicians, Hilbert posed 23 important problems for the further development of science. The 16th problem, which remains unsolved at present, concerned algebraic curves and surfaces. Today, this problem is divided into two parts related to different areas. The maximal number of closed branches of an n-order algebraic curves was set by Harnack. The first part of 16th problem is to determine the position of these branches relative to each other. For $n = 6$ are obtained 11 branches and Hilbert assumed that there is one branch that contains another branch, and outside it there are the other nine branches or inverse. However, in 1970 D. A. Gudkov determined that there were cases when five branches fell outside and inside the curve. This has shown that the first part of the problem is much more complicated. Various properties and extraordinary examples have been described by I. G. Petrovski, O. A. Oleinic, V. I. Arnold, V. A. Rohlin, O. Ya. Viro and others. This part of the problem now refers to algebraic geometry (see [15]).

In the second part of 16th problem, which also remains unsolved and completes the Center and Focus Problem, for a polynomial vector field of order n it is required to determine the upper bound $H(n)$ of the number of cycles and their relative position. It is well known that the number of limit cycles is always finite. The number $H(n)$ is called the Hilbert's number.

Researches on the 16th problem has been quite dramatic. In 1923 Henri Dulac [5] proposed a demonstration that the number $H(n, v)$ is finite for any polynomial vector field v of order n. In 1955 Ivan G. Petrovski and Evgeny M. Landis announced the complete solution of the second part of 16th problem, but in 1960 it was determined that their demonstration had serious shortcomings. A great surprise was the work of Yulii Ilyashenko in 1981, in which it was established that the work of Dulac in 1923 also contains gaps, which with great efforts were removed over 10 years by Yulii Ilyashenko and Jean Ecalle (see [6, 8, 15]).

These results have boosted the researches of the polynomial vector fields v of order n. In this case, the second part of 16th problem is a particular case of the **Problem of Global Finiteness**: *In any analytically finite-parametrized family of analytic vector fields on the sphere with the compact parameter space B (from the K-dimensional Euclidean space) the number $H(n, p)$ of limit cycle is uniformly bounded for all values p of the parameter in B.* This problem was formulated by Yu. Ilyashenko in 1994 and is called **the Hilbert-Arnold Problem** (see [8]). In 1986 V. I. Arnold (see [1, 8]), for a smooth vector field denoted on a sphere, introduced the notions of polycycle, the bifurcation number $B(k)$ of maximal cycling of non - trivial polycycle of field, of elementary singular point, of elementary polycycle and of elementary bifurcation number $E(k)$ of maximal cycling of non-trivial elementary polycycle. Thus, **Hilbert-Arnold's Local Problem** was formulated: *to prove that the number $B(k)$ is finite and to estimate this number from above.* The positive solution of the global problem is a consequence of positive answer to the local problem. It is established that $B(1) = 1$ and $B(2) = 2$. For $B(3)$ there is currently only one calculation strategy. V. Yu. Kaloshin established that $E(k) < 25k$ (see [8,9,10,11]).

We examine the case when the functions $X(x, y)$ and $Y(x, y)$ are polynomials. For the Center and Focus Problem to be algebraically solvable, the linear parts of the polynomials $X(x, y)$ and $Y(x, y)$ must not be zero. Under these conditions the system (1) can be written in the form

$$\frac{dx}{dt} = \sum_{i=0}^{\ell} P_{m_i}(x,y), \ \frac{dy}{dt} = \sum_{i=0}^{\ell} Q_{m_i}(x,y) \tag{2}$$

where $P_{m_i}(x,y)$ and $Q_{m_i}(x,y)$ are homogeneous polynomials of degree $m_i \geq 1$ in x, y, and $m_0 = 1$. The set $\{1, m_1, m_2, ..., m_\ell\}$ consists of a finite number ($\ell < \infty$) of distinct natural numbers. The coefficients and variables in polynomials $P_{m_i}(x,y)$ and $Q_{m_i}(x,y)$ take values from the fields of real numbers $\mathbb{R}$. Hereafter we denote system (2) by $s(1, m_1, m_2, ..., m_\ell)$.

The fundamental results on the Center and Focus Pproblem were obtained by **A. M. Lyapunov** (1857-1918) [12]. Henri Poincaré and Aleksandr Lyapunov laid the foundations *of methods of the qualitative theory of differential equations.*

As established, the conditions of a center consists of an infinite sequence of polynomials is equal to zero (focus quantities, Lyapunov's constants, Poincaré-Lyapunov constants)

$$L_1, L_2, ..., L_k, ... \tag{3}$$

which depend on the coefficients of the polynomials on the right sides of system $s(1, m_1, ..., m_\ell)$.

If at least one of quantities (3) is not zero, then origin of coordinates $O(0,0)$ for system $s(1, m_1, ..., m_\ell)$ is a focus. These conditions are necessary and sufficient.

From Hilbert's Theorem on the finiteness of basis of polynomial ideals it follows that *the essential center conditions*, which imply vanishing of an infinite sequence of polynomials (3), consist of a finite number of polynomials, the rest ones are the consequences of them.

Taking into account this result, the Center and Focus Problem can be formulated in the following way: *what finite number of polynomials* (*essential center conditions*)

$$L_{n_1}, L_{n_2}, ..., L_{n_\omega} \quad (n_i \in \{1, 2, ..., k, ...\}; \ i = \overline{1, \omega}; \ \omega < \infty) \tag{4}$$

is necessary for their equality to zero annuls all polynomials from (3)?

Hence the Center and Focus Problem consists of two parts.

The first part relates to finding the number ω that determines the upper bound of the number of focus quantities which constitute the essential center conditions.

The second part consists in finding the set $\Omega = \{n_1, n_2, ..., n_\omega\}$ of indices of essential conditions.

We will consider the first part as *the Weak Center and Focus Problem.*

The Generalized Center and Focus Problem is to determine the upper bound of the number λ of algebraically independent elements from $\Pi = \{L_i : \ i \in \Omega\}$.

The problem of determining essential center conditions (4) with number ω is a rather complicated problem and it is completely solved only for systems $s(1,2)$ and $s(1,3)$, for which we have $\omega = 3$ and $\omega = 5$, respectively (see [3, 23]).

Until now it is not known the number ω for a system $s(1,2,3)$, which seems to be not a complicated system.

There exists a hypothesis formulated by Professor H. Żołądek (Poland), mostly based on intuition, that for system $s(1,2,3)$ the number $\omega \leq 13$. Till now this hypothesis has not been disproved, but there is a recent paper from 2010, which confirms that 12 focus quantities is not enough for solving the Center and Focus Problem in the complex plane for system $s(1,2,3)$ [7].

Lie algebra method and Sibirsky's graded algebras allow us to solve the Generalized Center and Focus Problem.

If the Center and Focus Problem is solved negatively for system (2)*, having at the origin a singular point of second type (center or focus), then solution of the Generalized Center and Focus Problem can be considered as the final solution of this problem.*

3. THE EFFORT OF RESEARCHES BETWEEN CENTER AND FOCUS

Until we move to the basic topic - The Center and Focus Problem - we will mention some theories that, in one way or another, contributed to its appearance and formulation 130 years ago by Poincaré and finding a solution (until its generalized form) only now, in Chişinău. From **Artur Cayley** (1821-1895), Cambridge, England, he started an *invariants theory.* **Marius Sophus Lie** (1842-1899), Christiania, Norway, developed the theory of Lie groups and algebras - a new kind of algebraic structure that bears his name - both being applied in various fields of real science, including geometry and study differential equations. **Constantin Sibirsky** (1928-1990), Chişinău, Republic of Moldova, founded *the theory of algebraic invariants*, which is applied in the qualitative theory of equations knowing that this has to do with Lie's theory.

But who and how established this connection? In 1976, acad. Constantin Sibirsky, head of laboratory at the Institute of Mathematics and Informatics of ASM, founder of the scientific school of differential equations in the Republic of Moldova, published the monograph *Algebraic invariants of differential equations and matrices* (see [21]), which had a great resonance in the world of mathematicians. Three years later, in 1979, the American professor **C. S. Coleman** published a review of this scientific paper, in which he specified that it was written *in the spirit of the research of the Norwegian mathematician Marius Sophus Lie.* What these investigations consisted of it was not clear to Moldavian mathematicians. They only knew that the Norwegian had created a new direction in mathematical research, but the tangent between it and Moldavian research and how Lie's methods could be applied in practice was unknown.

Four years ago, one of the undersigned of this paper (journalist T. Rotaru: n.r.) wrote and prepared for print an article of memoirs, entitled *A troubled life between center and focus*, signed by Ana Sibirsky, wife of the regretted mathematician, acad. Constantin Sibirsky (see [20]). Then, from the first source, she learned about the troubles and researches of a scientist in identifying the scientific truth. At that time, the founder of the Moldavian scientific school in the field of the qualitative theory of differential equations was preoccupied with the elaboration of the theory of algebraic invariants for their application to the solution of the problems related to *the qualitative theory of differential equations*. The respective theory, elaborated by Poincaré in the years 1880-1882, allows to determine the character of the behavior of the solutions (integral curves) in case of differential equation is not solved explicitly. As mentioned above, Poincaré proposed a classification of the singular points of the solutions. But the problem of distinguishing them without explicit knowledge of the solutions proved to be very complicated.

Remembering those time, Prof. Mihail Popa confessed: "I never thought that I would ever deal with the Center and Focus Problem. But, after establishing the connection between Lie algebras and the Sibirsky graded algebras of invariants, I understand that the way is open to solve this problem, formulated by Henri Poincaré 130 years ago".

It should be noted that a large number of works in scientific centers of France, Russia, Belarus, China, Great Britain, Spain, Poland, Slovenia, Canada, USA, etc. are dedicated to the Center and Focus Problem and published in the world literature. Only in the Republic of Moldova their number is more than 100. At different stages the disciples of the academician C. S. Sibirsky (c.m. Nicolae Vulpe, prof. Alexandru Suba, dr. Iurie Calin, dr. Valeriu Baltag, dr. Dumitru Cozma and others), examined various issues of this problem and obtained important results. Some aspects of the development of mathematics in the Republic of Moldova are described in the book [5].

The mathematician Mihail Popa went his own way, starting from establishing the connection between Lie algebras and the Sibirsky graded algebras of invariants - a working tool in further searches. In this context, we will make some clarifications: the way to solve the Center and Focus Problem was initially determined by the Russian mathematician Alexander Lyapunov. But applying this method even for the simplest differential systems, you were faced with some enormous calculations, which could not be overcome even with the help of the most modern computers. That is why the Moldavian researcher took as a basis the Generalized Center and Focus Problem for the mentioned differential systems, avoiding the calculation of the Poincaré-Lyapunov quantities for each system. The Poincaré-Lyapunov quantities sequence (3) was replaced by a series of Lie algebras and a series of linear subspaces of Sibirsky graded algebras of invariants. To estimating the maximal number of algebraically independent focus quantities he applied these algebras. As a result, a finite numerical estimation was obtained for algebraically independent focal quantities that participate in solving the Center and Focus Problem for any system of polynomial differential equations (2). An analysis of the activity of Professor Mihail Popa is contained in the recently published article [4].

Lie algebra of the group $GL(2,\mathbb{R})$ and graded algebra of unimodular comitants and invariants of system $s(1, m_1, ..., m_\ell)$

It is known that the system $s(1, m_1, ..., m_\ell)$ admits the group $GL(2,\mathbb{R})$, to which the reductive Lie algebra $L_4 = \langle X_1, X_2, X_3, X_4\rangle$ corresponds, that consists of operators of linear representation of this group in the space of phase variables and coefficients of polynomials of this system [17].

This algebra generates a graded Sibirsky algebra of invariant polynomials with respect to the unimodular group $SL(2,\mathbb{R}) \subset GL(2,\mathbb{R})$ [16], which we write in the form

$$S_{1,m_1,...,m_\ell} = \sum_{(d)} S^{(d)}_{1,m_1,...,m_\ell}, \tag{5}$$

where (d) is called a type of the space $S^{(d)}_{1,m_1,...,m_\ell}$ and has the form [22, 17]

$$(d) = (\delta, d_0, d_1, ..., d_\ell), \tag{6}$$

and

$$S^{(d)}_{1,m_1,...,m_\ell} \tag{7}$$

is a finite-dimensional linear space of invariant polynomials (homogeneous comitants, invariants) of degree δ with respect to the phase variables x, y and of degree d_i with respect to the coefficients of the polynomials $P_{m_i}(x, y)$ and $Q_{m_i}(x, y)$ $(i = \overline{0, \ell})$ of system (2).

It is known that this algebra is finitely determined.

Using Lie algebra L_4 it can be shown that the maximal number of algebraically independent elements (Krull dimension) of algebra [18] is

$$\varrho(S_{1,m_1,...,m_\ell}) = 2\left(\sum_{i=1}^{\ell} m_i + \ell\right) + 3. \tag{8}$$

It is obvious that if Krull's dimension of algebra $S_{1,m_1,...,m_\ell}$ is $\varrho(S_{1,m_1,...,m_\ell})$ then for any invariant variety $V = \{i_1 = 0, i_2 < 0;\ i_1, i_2 \in S_{1,m_1,...,m_\ell}\}$ where i_1 is the trace of the matrix of the linear part of the system (2), (i_2 does not influence variety [18]) in this algebra $S_{1,m_1,...,m_\ell}$ no more than $\varrho(S_{1,m_1,...,m_\ell})$ algebraically independent elements will be found.

4. SOLVING THE GENERALIZED CENTER AND FOCUS PROBLEM

Remember that the focal quantities of system $s(1, m_1, ..., m_\ell)$ which has at origin of coordinates a singular point of the second type (center or focus), forms an infinite series of polynomials from the coefficients of this system which was written in the form (3).

It can be shown that to each focus quantity L_k $(k = \overline{1, \infty})$ one can associate a finite-dimensional linear spaces of invariant polynomials (unimodular comitants) [18]

$$S^{(d^{(k)})}_{1,m_1,...,m_\ell} \ (k = 1, 2, ...), \tag{9}$$

where

$$(d^{(k)}) = (\delta^{(k)}, d_0^{(k)}, d_1^{(k)}, ..., d_\ell^{(k)}) \tag{10}$$

is a type of a space (9) which was defined above.

The spaces (9) are characterized by the following fact [18]: they contain at least one homogeneous polynomial with respect to x and y (comitant), in which the coefficients are some quantities, named generalized focus pseudo-quantities. They are characterized by the fact that on the invariant variety $V = \{i_1 = 0, i_2 < 0;\ i_1, i_2 \in S_{1,m_1,...,m_\ell}\}$ some of these focus pseudo-quantities, except for a numerical constant, go to the corresponding focus quantity L_k, and the others go to zero. Invariant polynomials i_1 and i_2 does not depend on variables x and y, and i_1 which we will call null focus pseudo-quantity, belongs to the space $S^{(0,1,...,0)}_{1,m_1,...,m_\ell}$. Thus we obtain the sequence of spaces $\mathbb{R} = S^{(0,0,...,0)}_{1,m_1,...,m_\ell}$, $S^{(0,1,...,0)}_{1,m_1,...,m_\ell}$,..., $S^{(\delta^{(k)},d_0^{(k)},d_1^{(k)},...,d_\ell^{(k)})}_{1,m_1,...,m_\ell}$,... from $S_{1,m_1,...,m_\ell}$. Using them, we form a graded subalgebra $S'_{1,m_1,...,m_\ell}$ of algebra $S_{1,m_1,...,m_\ell}$ i.e. $S'_{1,m_1,...,m_\ell} \subset S_{1,m_1,...,m_\ell}$ (see [18]). From this inclusion it follows that between their Krull dimensions the following inequality holds $\varrho(S'_{1,m_1,...,m_\ell}) \le \varrho(S_{1,m_1,...,m_\ell})$. With the help of this inequality and the above formula for Krull's dimension $\varrho(S_{1,m_1,...,m_\ell})$ it can be shown that the following is true:

Lemma 1 [18]. *The maximal number of algebraically independent invariants and comitants, which contain as coefficients null and generalized focus pseudo-quantities of system $s(1, m_1, ..., m_\ell)$, does not exceed the numerical upper bound* $2\left(\sum\limits_{i=1}^{\ell} m_i + \ell\right) + 3$.

Taking into account properties of these comitants from algebra $S'_{1,m_1,...,m_\ell}$ which are related to focal quantities on the variety $V = \{i_1 = 0, i_2 < 0;\ i_1, i_2 \in S_{1,m_1,...,m_\ell}\}$ and Lemma 1, we obtain that there takes place:

Theorem 1 [18]. *The maximal number of algebraically independent focus quantities λ of system $s(1, m_1, ..., m_\ell)$ on the variety $V = \{i_1 = 0, i_2 <$*

$0;\ i_1, i_2 \in S_{1,m_1,...,m_\ell}\}$, *that take part in solving the Center and Focus Problem, does not exceed the the numerical upper bound* $2\left(\sum_{i=1}^{\ell} m_i + \ell\right) + 3$.

Recall that for the differential systems $s(1,2)$ and $s(1,3)$ the number of essential conditions of center is $\omega = 3$ and 5, respectively, and for the differential system $s(1,2,3)$ according to one hypothesis it is $\omega \leq 13$. From Theorem 1 we have that maximal number of algebraically independent focus quantities of the differential system $s(1,2)$ does not exceed 9, for the differential system $s(1,3)$ does not exceed 11, and for the differential system $s(1,2,3)$ does not exceed 17.

These arguments, as well as the fact that the system $s(1, m_1, ..., m_\ell)$ on variety V has at the origin of coordinates a singular point of second type (center or focus), allow us to conclude that the following can be true

General Hypothesis [18]. *If the system* $s(1, m_1, ..., m_\ell)$ *has at the origin of coordinates a singular point of the second type (center or focus), then the number of essential conditions of center* ω *for this system does not exceed the the numerical upper bound* $2\left(\sum_{i=1}^{\ell} m_i + \ell\right) + 3$.

Remark. *The expression* $2\left(\sum_{i=1}^{\ell} m_i + \ell\right) + 3$ *is equal to maximal number of all possible nonzero coefficients of the right-hand sides of the system* $s(1, m_1, ..., m_\ell)$ *minus one.*

5. METHODOLOGICAL CONCLUSIONS

The question arises: how can it be explained that till now we do not have a solution to the Center and Focus Problem for any system $s(1, m_1, ..., m_\ell)$.

First of all, it is obvious that the Center and Focus Problem is a difficult one.

Till now, no general methods have been found for studying the Poincaré-Lyapunov constants in the sequence (3). In particular, there is no general solution strategy. However, the specified way in solving the Center and Focus Problem for system $s(1,2,3)$ is connected with cumbersome computations with application of supercomputers. These difficulties are also insurmountable for other more complicated systems.

From a psychological point of view, there are also impediments in terms of human conservatism to research problems in the traditional, classical way. History confirms that new, unusual methods with great difficulty are approved and appreciated at their fair value. However, according to Kurt Gödel (1906-1978) *incompleteness theorem*, the resources created so far are not sufficient for further studies. Therefore, it is undeniable that subsequent success depends largely on the new means created.

Solving the Center and Focus Problem: traditional aspect is equivalent to determining the essential conditions of the center

$$L_{n_1}, L_{n_2}, ..., L_{n_\omega} \quad (n_i \in \{1, 2, ..., k, ...\};\ i = \overline{1, \omega};\ \omega < \infty)$$

which involves *knowing the number* ω and the set $\Omega = \{n_1, n_2, ..., n_\omega\}$ the finiteness of which results from Hilbert's Theorem on the finiteness of basis of polynomial ideals. This is similar to finiteness theorem of Hilbert's number $H(n)$ in the second part of 16th Problem.

The problem of determining a finite number ω or obtaining for it an argued numerical upper bound (even in the form of a hypothesis), which until now is not known for any system $s(1, m_1, ..., m_\ell)$ is important for the complete solution of the Center and Focus Problem.

Formally the Center and Focus Problem consists in *determining the conditions* that guarantee that a singular point of the second type is a center.

Through counterweights, for example, *matter and antimatter, the world and anti-world*, we penetrate the essence of the universe, constituting amazing symmetries in the world of known phenomenas. From a mathematical point of view, such symmetries are constructed using *the principle of duality.* To construct a duality - means to determine a correspondence between certain types of objects, to which each property of the initial object corresponds a certain property of the respective object to this correspondence. In any duality, *objects* and some of their *properties* have *dual objects and properties.* This method, which is its *reasoning by anti-analogy*, it determines that many objects, different in form and content, are constructed, from the point of view of formal logic, in one and the same way. Any duality between two theories establishes at a certain level an isomorphism between these theories.

Let A and B be two theories, and $\beta : A \to B$ an application to which each object $a \in A$ and properties of objects from A correspond to object $\beta(a)$ and property $\beta(w)$, respectively. It is assumed that this application is logically continuous in the sense: that if the object a possessed the property w, then the object $\beta(a)$ possessed the property $\beta(w)$. The application β can also have a symmetry for certain properties: the properties w and $\beta(w)$ are symmetric (dual), if the object a possessed the property w only and only if object $\beta(a)$ possessed property $\beta(w)$. Under these conditions, for each problem π in theory A is corresponded to a certain problem $\beta(\pi)$ of theory B. From the logical continuity of the application β we obtain:

- If the problem π is solved positively, then the problem $\beta(\pi)$ is solved positively;

- If the problem $\beta(\pi)$ is solved negatively, then the problem π is not solved positively;

- If the properties in the problem π are symmetric, then the problems π and $\beta(\pi)$ are equivalent.

We note that the problem $\beta(\pi)$ is a generalized form of the original problem π. Solving the generalized forms is important if for the initial problem, for a long time, no solutions are found. Moreover, *the solutions to the generalized problem propose strategies and hypotheses for solving the initial problem.* Some estimates from the solution of the generalized problem can serve as working hypotheses for the initial problem.

The study of a new problem or an unsolved problem, applying the methods of solving a known problem is done by various methods: the method of substituting variables; border crossing method etc. They are well known since ancient times. With the boundary crossing method, Hopf, for example, built the solutions of the quasi-linear equations.

Amazing constructions have been proposed by V. I. Arnold in the study of the critical points of the functions defined on varieties with the help of semi-simple Lie groups [2].

In the second half of the last century, the famous Russian mathematician Victor P. Maslov proposed a new theory - *Idempotent Analysis* - based on changing the usual operations $\{+, \times\}$ with two other operations (see [13,14]). By this exceptional method succeeded:

- reduction of Bellman and Hamilton-Jacobi equation theory to linear equation theory;
- studying Fourier transforms using the Legendre transformations;
- research with known methods of many problems in quantum physics, thermodynamics, superconductivity, etc.

Any constructed duality is a valuable event for those theories. The dualities of projective geometry, the Pontryagin duality in the theory of locally compact abelian groups, the Kolmogorov-Gelfand dualism of compact spaces and functional Banach algebras, the dualities of Serre and Alexander in topology, the Radu Miron duality of Cartan and Finsler spaces, the De Morgan duality are well known in sets theory, Stone duality between zero-dimensional compact spaces and Boolean rings, wave-corpuscle duality in theoretical physics, Kramers-Wannier dualism in statistical physics, etc.

All these examples have a simple explanation: many problems, different in form and content, can be studied with a single mathematical method and from a single point of view.

Now let's fix two polynomials $P = P(x, y)$ and $Q = Q(x, y)$ with non-zero linear parts. Then

$$P = \sum_{i=0}^{\ell} P_{m_i}(x, y), \;\; Q = \sum_{i=0}^{\ell} Q_{m_i}(x, y) \tag{11}$$

where $P_{m_i}(x,y)$ and $Q_{m_i}(x,y)$ are homogeneous polynomials of degree $m_1 \geq 1$ in x, y, and $m_0 = 1$. The set $\{1, m_1, ..., m_\ell\}$ determine a system $s(1, m_1, ..., m_\ell)$ of the form (2). An infinite series of polynomials (Poincaré-Lyapunov constants) $p = \{L_k : k = 1, 2, 3, ...\}$ of the form (3) were constructed for this system. For each focal quantities L_k ($k = \overline{1, \infty}$)) it can be corresponded to a finite-dimensional linear spaces of invariant polynomials, unimodular (comitants) $S^{(d^{(k)})}_{1,m_1,...,m_\ell}$ of the form (9), that forming a sequence of the type a. In this way, a correspondence is constructed between the sequence p of polynomials (3) and the sequence of linear spaces of the form (9). The first type of sequences form the P class, and the second type of sequences form class A. Therefore, we have built a correspondence $f : P \to A$. Probably this correspondence is a functor (a symmetry) in some sense. For each sequence $p \in P$ is determined the number $E(p)$ of the essential conditions of center and the set $\Omega = \{n_i : i = \overline{1, \omega}\}$, and for each sequence $a \in A$ is determined the number $gE(a)$ of algebraically independent focal quantities, i.e. the number of generalized essential conditions of the center and the set $g\Omega = \{m_i : i = \overline{1, \lambda}\}$, respectively. Consider $a = f(p)$, if $p = \varphi(s(1, m_1, ..., m_\ell), P, Q)$ and $a = \psi(s(1, m_1, ..., m_\ell), P, Q)$ are respectively determined by system $s(1, m_1, ..., m_\ell)$) and the polynomials P, Q with the respective decompositions.

Theorem 1 says that

$$gE(a) \leq 2\left(\sum_{i=1}^{\ell} m_i + \ell\right) + 3, \text{if} a = \psi(s(1, m_1, ..., m_\ell), P, Q).$$

This inequality is the solution of the Generalized Center and Focus Problem. The General Hypothesis assumes that $E(p) \leq 2\left(\sum\limits_{i=1}^{\ell} m_i + \ell\right) + 3$ for $p = \varphi(s(1, m_1, ..., m_\ell), P, Q)$.

There are simple examples for which inequality $E(p) > gE(f(p))$ holds, but for all known examples $E(p) \leq 2\left(\sum\limits_{i=1}^{\ell} m_i + \ell\right) + 3$. This was the basis for launching the General Hypothesis. Probably, the truth about this inequality is contained in the functor properties of the correspondence f. But to solve the Weak Center and Focus Problem is sufficient a numeric function $h : \mathbb{N} \to \mathbb{N}$ defined on the sequence of natural numbers $\mathbb{N}$ and set to $E(p) \leq h(gE(f(p)))$. In this context, there is another unresolved problem: to study, from a general point of view, the second part of the Center and Focus Problem, that is, to determine the subset of $g\Omega = \{m_i : i = \overline{1, \lambda}\}$ of the set $\Omega = \{n_1, n_2, ..., n_\omega\}$ of indices of algebraically independent focal quantities.

Let's admit that at some point the Center and Focus Problem is solved positively. In this situation, will the study of particular cases be of interest?

We think so. Whether for some system $\omega = 3$ and $1000^{1000} \in \Omega$. Who and by what means will determine the type of the singular point? It is well known that the simplex method, proposed by George Dantzig in 1947, is a general method of solving the linear programming problem formulated by L. V. Kantorovich in 1939. But research in this area continues.

Therefore, the general solution of the initial or generalized problem reflects the ways of examining the different particular cases. Moreover, because the sequence of form (3) is infinite, the solutions of the initial problem and the generalized problem allow to *discover effectively* particular cases for which the number ω and those of Ω are accessible for a deeper study of the properties of integral curves of certain types. Therefore, the role of generalized solutions is enormous in the deep study of different classes of differential systems (1).

References

[1] Arnold V. I., Afrajmovich, V. S., Ilyashenko Yu. S., Shilnikov, L. P. *Bifurcation theory*, Current problems in mathematics. Fundamental directions. Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 5, 1986, pp. 5-218.

[2] Arnold V. I. *Critical points of functions on a manifold with boundary, the simple Lie groups Bk, Ck and F4 and singularities of evolutes*, Russian Mathematical Surveys 33:5 (1978), pp. 99-116.

[3] Bautin N. N. *On the number of limit cycles which appear with the variation of coefficients from equilibrium position of focus or center type*, Math. Sb. 30(72) (1952), 181-196 (Amer. Math. Soc. Transl., 100, 1954, pp. 397-413).

[4] Ciobanu M., Cojocaru S., Şubă A., Rotaru T. *Matematicianul Mihail Popa la ceas de glorie: 130 de ani de zbucium pentru soluţionarea problemei lui Poincaré*, Literatura şi Arta, 1 august, 2013.

[5] Ciobanu M. M., Valuţă I. I. *Elemente de istorie a matematicii şi matematica în Republica Moldova*, Chişinău, Tipogr. AŞM, 2006.

[6] Dulac H. *Sur les cycles limits*, Bull. Soc. Math. France, 51, 1923, pp. 45-188.

[7] Graf V., Bothmer H. C., Kroker J. *Focal Values of Plane Cubic Centers*, Qual. Theory Dyn. Syst. 9, (2010), pp. 319-324.

[8] Ilyashenko Yu. *Centennial History of Hilbert's 16th problem*, Bulletin of the AMS 39(3), (2002), pp. 301-354.

[9] Ilyashenko Yu. *The origin of limit cycles under perturbation of the equation $dw/dz = -Rz/Rw$, where $R(z, w)$ is a polynomial*, Mathematics of the USSR Sbornik 7, (1969), pp. 353-364

[10] Ilyashenko Yu., Kaloshin V. *Bifurcations of planar and spatial polycycles: Arnold's program and its development*, Fields Inst. Commun. 24 (1999), pp. 241-271.

[11] Kaloshin V. *The Existential Hilbert 16-th Problem and an Estimate for Cyclicity of Elementary Polycycles*, Invent, math. 151 (2003), pp. 451-512.

[12] Liapunoff A. *Probléme général de la stabilité du mouvement*, Annales de la Faculté des Sciences de Touluose. Sér 2, 9, (1907), pp. 204-477. Reproduction in Annals of Mathematics Studies 17, Princeton: Princeton University Press, 1947, reprinted, Kraus Reprint Corporation, New York, 1965.

[13] Litvinov G. L. *The Maslov dequantizan, idempotent mathematics: a very brief introduction*, Idempotent Mathematics and Mathematical Phisics. Contemporary Mathematics, 377 (2005), pp 1-17.

[14] Maslov V. R. *On a new principle of superposition for optimization problems*, Russian Mathematical Surveys 42:3, (1987), pp. 43-54.

[15] https://encyclopediaofmath.org/wiki/Centre_and_focus_problem.

[16] Poincaré H. *Mémoire sur les courbes définies par une équation différentielle*, J. Math. Pures et Appl. (Sîr. 3) 7 (1881) pp. 375-422; (Sér. 3) 8 (1882) pp. 251-296; (Sér. 4) 1 (1885) 167-244; (Sér. 4) 2 (1886), pp. 151-217.

[17] Popa M. N. *Algebraic methods for differential systems*, Editura the Flower Power, Universitatea din Piteşti, Seria Matematică Aplicată şi Industrială, 2004, (in Romanian).

[18] Popa M. N., Pricop V. V. *Applications of algebraic methods in solving the center-focus problem*, Buletinul AŞM, Matematica, 1(71), (2013), pp. 45-71.

[19] Schlomiuk D. *New developments based on the mathematical legasy of C. S. Sibirschi*, Buletinul AŞM, Matematica, 1(71), (2013), pp. 3 - 10.

[20] Sibirsky Ana. *O viaţă zbuciumată între centru şi focar*, "Literatura şi Arta" din 11 decembrie 2008.

[21] Sibirsky K. S. *Algebraic invariants of differential equations and matrices*, Kishinev, Ştiinţa, 1976 (in Russian).

[22] Vulpe N. I. *Polynomial bases of comitants of differential systems and their application in qualitative theory*, Kishinev, Ştiinţa, 1986 (in Russian).

[23] Żołądek H. *On certain generalization of the Bautins theorem*, Nonlinearity, 7 (1994), pp. 273-279.

ROMAI J., v.17, no.1(2021), 17–27

ON SEQUENTIALITY AND K-PROPERTY OF FUNCTION SPACES

Mitrofan Choban[1], Radu Dumbrăveanu[2]

[1] *Sciences Academy of Moldova, Chişinău, Republic of Moldova,*

[1] *Tiraspol State University, Chişinău, Republic of Moldova*

[2] *Bălţi State University, Republic of Moldova*

mmchoban@gmail.com, rndumbraveanu@gmail.com

Abstract In the present article the space $C_p(X, E)$ of continuous mappings into a metrizable space E is considered. The conditions for which $C_p(X, E)$ is Fréchet-Urysohn and k-space are determined. Some similar assertions for $C_p(X, \mathbb{R})$ were proved in [9, 2] and for $C_p(X, \mathbb{Z})$ were proved in [5].

Keywords: Fréchet-Urysohn space, k-space, scattered space.

2020 MSC: Primary, 54C35; secondary, 54E35, 54D50, 54G12.

1. INTRODUCTION

Any space is considered to be a completely regular T_1-space. We use the terminology from [2, 6]. Denote by $\mathbb{R}$ the field of the reals in the usual topology, by $\mathbb{Z}$ the discrete ring of integers, by $\mathbb{N}$ the discrete space of positive integers, by $cl_X B$ or clB the closure of the set B in the space X, by $|L|$ the cardinality of the set L, by $w(X)$ the weight of the space X.

A space X is called Fréchet-Urysohn if for any non-empty subset $A \subseteq X$ and any $x \in cl(A)$ there exists a sequence in A which converges to x. A space is called a k-space if $A \subseteq X$ is closed provided $A \cap K$ is closed in K for any compact subset K of X. A space is called a sequential space if $A \subseteq X$ is closed provided $A \cap K$ is closed in K for any metrizable compact subset K of X.The Fréchet-Urysohn property implies the sequentiality and the sequentiality implies the k-property.

A family $\mathcal{A}$ of subsets of X is called an ω-cover (cover for finite subsets [2, 9]) of X if for any finite subset $F \subseteq X$ there exists $U \in \mathcal{A}$ such that $F \subseteq U$. If $\xi = \{A_n : n \in \}$ is a sequence of subsets of X, then the set $\liminf \xi = \cup\{\cap\{A_k : k \geq n\} : n \in \mathbb{N}\}$ is called the lower limit of the sequence ξ.

By E^X we will denote the space of all E-valued mappings on X equipped with the product topology. By $C_p(X, E)$ we will denote the space of all E-valued continuous mappings equipped with the pointwise convergence topology, i.e. the topology with standard basic open sets of the form

$$W(x_1, ..., x_k, U_1, ..., U_k) = \{f \in C(X, E) : f(x_i) = U_i, i \leq k\},$$

where $x_i \in X$, U_i are open subsets from the base of the topology on E and $k \in \mathbb{N}$. In particular, if E is a metric space then the standard basic open neighbourhoods of a mapping $f \in C_p(X, E)$ are of the form

$$W(f, x_1, ..., x_k, n) = \{g \in C_p(X, E) : d(g(x_i), f(x_i)) < \tfrac{1}{n}, i \leq k\},$$

where $n \in \mathbb{N}$.

2. GENERAL CONDITIONS FOR SEQUENTIALITY AND K-PROPERTY

The following result for $E = \mathbb{R}$ was proved by R. A. McCoy and A. V. Arhangel'skii (see [9], Theorem 1; [2], Theorem II.3.2). A similar result for $E = \mathbb{Z}$ was proved by K. M. Drees (see [5], Theorem 4.2.4). We will follow closely the proof scheme from [9, Theorem 1] and [2, Theorem II.3.2].

Proposition 2.1. *Let E be a metrizable space, $|E| \geq 2$ and a space X has the property γ_1: for any sequence $\{\eta_n : n \in \mathbb{N}\}$ of open ω-covers of X there are $U_n \in \eta_n$, $n \in \mathbb{N}$, such that* $\liminf \xi = X$, *where $\xi = \{U_n : n \in \mathbb{N}\}$. Then $C_p(X, E)$ is a Fréchet-Urysohn space;*

Proof. There exists a metrizable Banach algebra F such that E is a subspace og F, 0 is the neutral element and 1 is the unity of the algebra F. Since $C_p(X, E)$ is a subspace of the topological algebra $C_p(X, F)$ it is sufficient to prove that $C_p(X, F)$ is a Fréchet-Urysohn space.

Fix a subset $A \subseteq C_p(X, F)$ and a function $g \in clA \setminus A$. Since $C_p(X, F)$ is a topological group, we can assume that $g(X) = \{0\}$.

Fix an open base $\{O_n : n \in \mathbb{N}\}$ for the space F at the point 0 such that $clO_{n+1} \subseteq O_n$. Then the family $\{W(g, K, O_n) = \{f \in C(X, F) : f(K) \subseteq O_n\} : n \in \mathbb{N}, K \textit{ is a finite subset of } X\}$ is an open base for $C_p(X, F)$ at the point g.

For each $n \in \mathbb{N}$ we put $\eta_n = \{f^{-1}(O_n) : f \in A\}$. We affirm that η_n is an ω-cover of X. Let K be a finite non-empty subset of X. Then the set $W(g, K, O_n)$ is open in $C_p(X, F)$ and $g \in W(g, K, O_n)$. Then the set $A \cap W(g, K, O_n)$ is non-empty. Fix $f \in A \cap W(g, K, O_n)$. Then $K \subseteq f^{-1}(O_n) \in \eta_n$.

By virtue of condition γ_1, there exists a sequence $\{f_n \in A : n \in \mathbb{N}\}$ such that $\liminf \xi = X$, where $\xi = \{f_n^{-1}(O_n) : n \in \mathbb{N}\}$. Obviously, $g = limf_n$. The proof is complete. ∎

Proposition 2.2. *Let E be a non pseudocompact space, a space X is non-empty, $indX = 0$ and $C_p(X, E)$ is a k-space. Then the space X has the property γ_1.*

Proof. We can assume that the ring of integers $\mathbb{Z}$ is a closed discrete subspace of the space E. Since $C_p(X, \mathbb{Z})$ is a closed subspace of the space $C_p(X, E)$,

$C_p(X, \mathbb{Z})$ is a k-space and a topological commutative ring with unity. Analogicaly as in Lemma II.3.8 from [2] one can prove that the space X has the property γ_1. This fact was realized by K. M. Drees in [5]. ∎

Corollary 2.1. *Let E be a non compact metric space, X be a non-empty space and $indX = 0$. Then the following conditions are equivalent:*

(i) $C_p(X, E)$ is a Fréchet-Urysohn space;
(ii) $C_p(X, E)$ is a sequential space;
(iii) $C_p(X, E)$ is a k-space;
(iv) X has the property γ_1;
(v) $C_p(X, E)^{\mathbb{N}}$ is a Fréchet-Urysohn space.

Proof. The implications $(i) \to (ii) \to (iii)$ and $(i) \to (v) \to (i)$ are obvious. The implication $(iv) \to (i)$ follows from the above Proposition 2.1. The implication $(iii) \to (iv)$ follows from the above Proposition 2.2. ∎

Proposition 2.3. *Let E and X be non-empty spaces and the space of reals $\mathbb{R}$ be a closed subspace of E. If $C_p(X, E)$ is a k-space, then the space X has the property γ_1.*

Proof. We can assume that the ring of integers $\mathbb{Z}$ is a closed discrete subspace of the space E. Since $C_p(X, \mathbb{R})$ is a closed subspace of the space $C_p(X, E)$, $C_p(X, \mathbb{R})$ is a k-space and a topological commutative ring with unity. By virtue of Theorem II.3.2 from [2], the space X has the property γ_1. ∎

Corollary 2.2. *Let E be a non trivial metrizable linear space and X be a non-empty space. Then the following conditions are equivalent:*

(i) $C_p(X, E)$ is a Fréchet-Urysohn space;
(ii) $C_p(X, E)$ is a sequential space;
(iii) $C_p(X, E)$ is a k-space;
(iv) X has the property γ_1;
(v) $C_p(X, E)^{\mathbb{N}}$ is a Fréchet-Urysohn space.

Proof. The implications $(i) \to (ii) \to (iii)$ and $(i) \to (v) \to (i)$ are obvious. The implication $(iv) \to (i)$ follows from the above Proposition 2.1. The implication $(iii) \to (iv)$ follows from the above Proposition 2.3. ∎

Corollary 2.3. *Let X be a non-empty space. Then the following conditions are equivalent:*

(i) $C_p(X, E)$ is a Fréchet-Urysohn space for any metrizable space E.
(ii) $C_p(X, E)$ is a sequential space for any metrizable space E.
(iii) $C_p(X, E)$ is a k-space for some non-trivial linear space E.
(iv) X has the property γ_1.
(v) $C_p(X, E)^{\mathbb{N}}$ is a Fréchet-Urysohn space for any metrizable space E.

Corollary 2.4. *Let X be a non-empty space and $indX = 0$. Then the following conditions are equivalent:*

(i) $C_p(X, E)$ is a Fréchet-Urysohn space for any metrizable space E;

(ii) $C_p(X, E)$ is a sequential space for any metrizable space E;

(iii) $C_p(X, E)$ is a k-space for some non-pseudocompact space E;

(iv) X has the property γ_1;

(v) $C_p(X, E)^{\mathbb{N}}$ is a Fréchet-Urysohn space for any metrizable space E;.

The conditions give on E and X in Corollaries 2.1 and 2.2 are essential.

Example 2.1. *Let E be a compact space, $|E| \geq 2$ and X be an uncountable discrete space. Then $C_p(X, E) = E^X$ is a compact space and is non sequential space. Any compact space is a k-space. The space X has not the property γ_1.*

Example 2.2. *Let E be an infinite discrete space and X be an uncountable connected space. Then $C_p(X, E) = E$ is a discrete space. Any discrete space is a Fréchet-Urysohn space. If the space X has not the property γ_1, then $indX = 0$ (see [2, 9]). Hence the space X has not the property γ_1.*

3. TIGHTNESS OF FUNCTIONAL SPACES AND LINDELÖF NUMBER OF SPACES

A tightness $t(X)$ of a space X is the minimal infinite cardinal τ such that for any non-empty set $B \subseteq X$ and any point $x \in cl_X A$ there exists a subset $C \subseteq B$ such that $|C| \leq \tau$ and $x \in cl_X C$ (see [2]). Obviously, $t(X) \leq |X|$. Any sequential space has the countable tightness [2]. Let $\mathbb{D} = \{0, 1\}$ be the discrete two-point space.

A Lindelöf number $l(X)$ of a space X is the minimal infinite cardinal τ such that any open cover of X contains a subcover of the cardinality $\leq \tau$.

Proposition 3.1. *Let X be non-empty spaces, $indX = 0$, E be a space, $|E| \geq 2$, τ be an infinite cardinal and $t(C_p(X, E)) \leq \tau$. Then $l(X^n) \leq \tau$ for any $n \in \mathbb{N}$.*

Proof. We can assume that $E = \mathbb{D} = \{0, 1\}$. Fix $n \in \mathbb{N}$ and an open cover γ of the space X^n. The family ξ of open-and-closed subsets of X is called (γ, n)-small if for any $U_1, U_2, ..., U_n \in \xi$ there exists $G \in \gamma$ such that $\Pi\{V_i : i \leq n\} \subseteq G$. By μ we denote the family of all (γ, n)-small finite families of open-and-closed subsets of X. For any $\xi \in \mu$ we fix $f_\xi \in C_p(X, E)$ such that $f(X \setminus \cup\xi) = 0$ and $f(\cup\xi) = 1$. Let $A = \{f_\xi : \xi \in \mu\}$.

We affirm that the set A is dense in $C_p(X, E)$. Let $g \in C_p(x, E)$, W is open in $C_p(X, E)$ and $g \in W$. There exists a finite subset $K = \{x_1, x_2, ..., x_m\}$ of X such that $m \geq n$ and $W(x_1, ..., x_m, \{g(x_1)\}, ..., \{g(x_)\}) =\{f \in C(X, E) : f(x_i) = g(x_i), i \leq m\} \subseteq W$. There exists a finite family $\nu = \{U_1, U_2, ..., U_m\} \in \mu$ of open-and-closed subsets of X such that $x_i \in U_i$ and $g(U_i) = \{g(x_i)\}$ for each $i \le m$. Let $\xi = \{U \in \nu : g(U) = 1\}$. Then $f_\xi \in \{f \in C(X,E) : f(x_i) = g(x_i), i \le m\} \subseteq W$. Hence the set A is dense in $C_p(X,E)$.

Assume that $g(X) = 1$. If $g \in A$, then γ contains a finite subcover. Suppose that $g \notin A$. Then there exists a subset $B \subseteq A$ such that $g \in clB$ and $|B| \le \tau$. For each $\xi \in \mu$ we put $\eta_\xi = \{U_1 \times U_2 \times ... \times U_n : U_1, U_2, ..., U_n \in \xi\}$. Any family η_ξ is finite and for each $V \in \eta_\xi$ there exists $G \in \gamma$ such that $V \subseteq G$. Now we put $\eta = \cup\{\eta_\xi : \xi \in \mu, f_\xi \in B\}$. By construction, $|\eta| \le \tau$. Fix a point $(x_1, x_2, ..., x_n) \in X^n$. The set $H = \{f \in C_p(X,E) : f(x_i) = 1, i \le n\}$ is open in $C_p(X,E)$ and $g \in H$. Fix $f_\xi \in H \cap B$. For each $i \le n$ fix $U_i \in \xi$ such that $x_i \in U_i$. Then $(x_1, x_2, ..., x_n) \in U_1 \times U_2 \times ... \times U_n \in \eta_\xi \subseteq \eta$. Therefore η is a cover of X^n and a refinement of γ. The proof is complete. ∎

Corollary 3.1. *Let X be non-empty spaces, $indX = 0$, E be a metrizable space and $|E| \ge 2$. Then $t(C_p(X,E)) = sup\{l(X^n) : n \in \mathbb{N}\}$.*

Proof. From Proposition 3.1 it follows that $t(C_p(X,E)) \ge sup\{l(X^n) : n \in \mathbb{N}\}$. In [9], Corollary 1, was proved that $t(C_p(X,E)) \le sup\{l(X^n) : n \in \mathbb{N}\}$. ∎

Corollary 3.2. *Let X be non-empty spaces, $indX = 0$ and τ be an infinite cardinal. The following assertions are equivalent:*

1. $l(X^n) \le \tau$ for any $n \in \mathbb{N}$.
2. $t(C_p(X,E)) \le \tau$ for each metrizable space E.
3. $t(C_p(X,\mathbb{D})) \le \tau$.

4. SPECIAL CONDITIONS FOR SEQUENTIALITY AND K-PROPERTY

Let X be a space. Denote by PX the set X with the topology generated by the G_δ-subsets of X. The topology of the space PX is called the Baire topology or the G_δ-topology of the space X. If $X = PX$, then X is called a P-space.

A space X is functionally countable if for any continuous mapping f of X into a metrizable space Y the image $f(X)$ is countable (see [7, 3]). Any Lindelöf P-space is functionally countable. Moreover, if X is a Lindelöf P-space, then the space X^n is Lindelöf for each $n \in \mathbb{N}$.

A space X is scattered if any non-empty subspace $A \subseteq X$ contains an isolated point in A. If X is a Lindelöf scattered space, then:

- PX is a Lindelöf scattered P-space (see [7, 2]);
- the space X^n is Lindelöf for each $n \in \mathbb{N}$.

A Lindelöf Čech-complete space is functionally countable if and only if X is scattered (see [4]).

Lemma 4.1. *Let K be the Cantor perfect set. Then the space $C_p(K,\mathbb{D})$ contains a non-discrete subspace Y with properties:*

- the space Y has a unique non-isolated point;
- the space Y is countable and closed in $C_p(K,\mathbb{D})$;
- any compact subset of Y is finite and Y is not a k-space.
In particular, $C_p(K,\mathbb{D})$ is not a k-space.

Proof. We can assume that $K = \mathbb{D}^{\mathbb{N}}$. For each $n \in \mathbb{N}$ and $i_1, i_2, ..., i_n \in \mathbb{D}$ we put $O(i_1, i_2, ..., i_n) = \{j = (j_1, j_2, ...) \in K : j_1 = i_1, j_2 = i_2, ..., j_n = i_n\}$. The sets $O(i_1, i_2, ..., i_n)$ are open-and-closed and forms an open base of the compact K.

On K there exists a σ-additive measure μ with the followin properties:
1. $\mu(K) = 1$ and $\mu(O(i_1, i_2, ..., i_n)) = 2^{-n}$ for each $n \in \mathbb{N}$ and $i_1, i_2, ..., i_n \in \mathbb{D}$.
2. $\mu(U) > 0$ for each non-empty open subset U of K.
3. $\mu(L) = 0$ for each finite subset L of K.
Now we mention that on K there exists an open base $\mathcal{B} = \{V_n : n \in \mathbb{N}\}$ of open-and-closed subsets of K such that:
4. $\mu(V_{n+1}) \leq \mu(V_n)$ for each $n \in \mathbb{N}$.
5. $lim_{n\to\infty}\mu(V_n) = 0$.
6. For any finite subset $L \subseteq K$, any open subset U which contains L and for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ for which $m \geq n$ and $L \subseteq V_m \subseteq U$.

Let $B = \{b_n : n \in \mathbb{N}\}$ be a dense subset of the space K. Now for each $n \in \mathbb{N}$ fix a continuous function $f_n \in C_p(K,\mathbb{D})$ such that $V_n \cup \{b_1, ..., b_n\} \subseteq f_n^{-1}(0)$ and $\mu(f_n^{-1}(0)) \leq \mu(V_n) + 2^{-n}$.
Let $g(x) = 0$ for each $x \in K$. We put $Y = \{g\} \cup \{f_n : n \in \mathbb{N}\}$.

Claim 1. *Y is a closed subspace of the space $C_p(K,\mathbb{D})$.*
Assume that $f \in clY \setminus Y$. Then $f(b_n) = 0$ for each $n \in \mathbb{N}$. Since f is continuous and B is a dense subset of K, we have $f(x) = g(x) = 0$ for each $x \in K$. Hence $f = g$, a contradiction.

Claim 2. *g is not an isolated point of Y.*
Let W be an open subset of $C_p(K,\mathbb{D})$ and $g \in W$. Then for some finite subset L of K we have $g \in \{f \in C_p(K,\mathbb{D}) : f(L) = 0\} \subseteq W$. There exists $n \in \mathbb{N}$ such that $L \subseteq V_n$. By construction, $f_n \in \{f \in C_p(K,\mathbb{D}) : f(L) = 0\} \subseteq W$. Thus $g \in cl(Y \setminus \{g\})$.

Claim 3. *g is the unique non-isolated point of Y.*
If $f \in cl(Y \setminus \{f\})$, then $f(b_n) = 0$ for each $n \in \mathbb{N}$. Hence $f = g$.

Claim 4. *Any compact subset of Y is finite.*
Assume that F is an infinite compact subset of Y. Since the compact F is countable, F is metrizable. From Claim 3 it follows that $g \in F$ and $F \setminus \{g\}$ is a convergent sequence. Then in F there exists a convergent to g sequence $\{g_i = f_{n_i} : i \in \mathbb{N}\}$ such that:

- $g = lim_{i \to \infty} g_i$;
- $\mu(g_i^{-1}(0) \leq 2^{-i-1}$ for each $i \in \mathbb{N}$.

Let $V = \{g_i^{-1}(0) : i \in \mathbb{N}\}$. By construction, $\mu(V) \leq \Sigma\{2^{-i-1} : i \in \mathbb{N}\} = 2^{-1}$. Hence $\mu(K \setminus V) \geq 2^{-1}$ and $g_i(x) = 1$ for all $x \in K \setminus V$ and $i \in \mathbb{N}$, a contradiction.

The proof is complete. ∎

Theorem 4.1. *Let X be a functionally countable space and the space X^n is Lindelöf for each $n \in \mathbb{N}$. Then $C_p(X, E)$ is a Fréchet-Urysohn space for any metrizable space E.*

Proof. Since X is a functionally countable space, we have $indX = 0$. Fix a metrizable space E. From Corollary 3.2 it follows that $t(C_p(X, E)) = \aleph_0$.

Let $A \subseteq C_p(X, E)$ and $g \in clA \setminus A$. Since $t(C_p(X, E)) = \aleph_0$, there exists a countable subset $B_1 \subseteq A$ such that $g \in clB_1$. Let $B = \{g\} \cup B_1$. By construction, the set B is countable, the space E^B is metrizable and the mapping $\varphi : X \longrightarrow M \subseteq E^B = \Pi\{E_f = E : f \in B\}$, where $M = \varphi(X)$ and $\varphi(x) = (f(x) : f \in B) \in E^B$ for each $x \in X$, is continuous. Obviously, the space M is countable. Hence $C_p(M, E)$ is a metrizable subspace of the space E^M. For each $f \in B$ consider the projection $pf : M \longrightarrow E_f = E$. Then $f = pf \circ \varphi$ for each $f \in B$. Let $C_1 = \{pf : f \in B_1\}$. By construction, $pg, pf \in C_p(M, E)$ for any $f \in B_1$ and $pg \in clC_1$. Since $C_p(M, E)$ is metrizable, there exists a sequence $\{pf_n : n \in \mathbb{N}\}$ which converge to pg. Therefore $g = lim_{n \to \infty} f_n$. The proof is complete. ∎

Corollary 4.1. *Let X be a Lindelöf P-space. Then $C_p(X, E)$ is a Fréchet-Urysohn space for any metrizable space E.*

Corollary 4.2. *Let X be a Lindelöf scattered space. Then $C_p(X, E)$ is a Fréchet-Urysohn space for any metrizable space E.*

Corollary 4.3. *Let X be a Lindelöf Čech-complete space, $indX = 0$, E be a metrizable space and $|E| \geq 2$. The following assertions are equivalent:*

1. *$C_p(X, E)$ is a Fréchet-Urysohn space.*
2. *$C_p(X, E)$ is a sequential space.*
3. *$C_p(X, E)$ is a k-space.*
4. *X is a scattered space.*

Proof. Implications $1 \to 2 \to 3$ are obvious. Implication $4 \to 1$ follows from Corollary 4.2.

Assume that $C_p(X, E)$ is a k-space and X is not scattered. Then X contains a compact G_δ-subset Y and a continuous mapping $\varphi : Y \longrightarrow K$ of Y onto the Cantor perfect set K. Since $indX = 0$ there exists a continuous mapping $\psi : X \longrightarrow Y$ such that $\varphi = \psi|Y$. Then, since ψ is a quotient mapping, the dual mapping $\Psi : C_p(K, E) \longrightarrow C_p(X, E)$, where $\Psi(f) = f \circ \psi$ for each

$f \in C_p(K, E)$, is a closed embedding of $C_p(K, E)$ in $C_p(X, E)$. By virtue of Lemma 4.1, $C_p(K, E)$ is not a k-space. Since $\Psi(C_p(K, E))$ is a closed subspace of the space $C_p(X, E)$, $C_p(X, E)$ is not a k-space, a contradiction. Implication $3 \to 4$ is proved. The proof is complete. ∎

5. ON SEQUENTIALITY OF FUNCTIONAL SPACES INTO FIRST-COUNTABLE SPACES

Proposition 5.1. *Let E be a first-countable space and a space X has the property γ_1. Assume that the set $f(X)$ is countable for each function $f \in C_p(X, E)$. Then $C_p(X, E)$ is a Fréchet-Urysohn space.*

Proof. Since X has the property γ_1, $indX = 0$ and the space X^n is Lindelöf for each $n \in \mathbb{N}$ (see [2]).

The assertion of Proposition follows from Proposition 2.1 for a metrizable space E. Assume that the space E is not metrizable.

Claim 1. *Let Y be a subspace of E, $\varphi : E \longrightarrow F$ be a continuous mapping of E onto a metrizable space F, $\mathcal{B}$ be an open base for E at each point $x \in Y$, the set $\varphi(V)$ is open in F and $V = \varphi^{-1}(\varphi(V))$ for any $V \in \mathcal{B}$. Then if $A \subseteq C_p(X, E)$, $g \in clA$ and $g(X) \subseteq Y$, then there exists a sequence $\{f_n \in A : n \in \mathbb{N}\}$ such that $g = lim_{n\to\infty} f_n$, i. e. g is a Fréchet-Urysohn point of the space $C_p(X, E)$.*

Consider the continuous mapping $\Phi : C_p(X, E) \longrightarrow C_p(X, F)$, where $\Phi(f) = \varphi \circ f$ for each $f \in C_p(X, E)$. Fix a subset $A \subseteq C_p(X, E)$ and a function $g \in clA \setminus A$ such that $g(X) \subseteq Y$. Then $\Phi(g) \in cl\Phi(A)$. Since F is metrizable, by virtue of Proposition 2.1, there exists a sequence $\{f_n \in A : n \in \mathbb{N}\}$ such that $\Phi(g) = lim_{n\to\infty}\Phi(f_n)$. Since $\mathcal{B}$ is an open base for E at each point $x \in Y$, the set $\varphi(V)$ is open in F and $V = \varphi^{-1}(\varphi(V))$ for any $V \in \mathcal{B}$, we have $g = lim_{n\to\infty} f_n$.

Claim 2. *For any countable subspace Y of E there exist a metric space F, a continuous mapping φ of E onto F, an open base $\mathcal{B}$ for E at each point $x \in Y$ such that the set $\varphi(V)$ is open in F and $V = \varphi^{-1}(\varphi(V))$ for any $V \in \mathcal{B}$.*

Since the space E is first-countable, for each point $a \in E$ there exist a continuous function $h_a : E \longrightarrow I_a = [0, 1]$ such that $h_a(a) = 0$ and $\mathcal{B}(a) = \{h_a^{-1}([0, 2^{-n})) : n \in \mathbb{N}\}$ is an open base for E at the point a. Consider the mapping $\varphi : E \longrightarrow F$, where $F = \varphi(E)$ is a subspace of the Cartesian product $\Pi\{I_y : y \in Y\}$ and $\varphi(x) = (h_y(x) : y \in Y)$ for each $x \in E$. The open family $\mathcal{B} = \cup\mathcal{B}(y) : y \in Y\}$, the space F and the mapping φ are constructed.

Now, the assertion of Proposition follows from Claims 1 and 2. The proof is complete. ∎

From Corollary 2.1 and Proposition 5.1 it follows;

Corollary 5.1. *Let E be a non compact metric space, X be a non-empty space, $f(X)$ is countable for each function $f \in C_p(X, E)$ and $indX = 0$. Then the following conditions are equivalent:*
(i) $C_p(X, E)$ is a Fréchet-Urysohn space;
(ii) $C_p(X, E)$ is a sequential space;
(iii) $C_p(X, E)$ is a k-space;
(iv) X has the property γ_1.

Corollary 5.2. *Let X be a non-empty Lindelöf P-space. Then $C_p(X, E)$ and $C_p(X, E)^{\mathbb{N}}$ are Fréchet-Urysohn spaces for each first-countable space E.*

Proof. From Corollaries 2.4 and 4.2 the space X has property γ_1. We have $C_p(X, E^{\mathbb{N}}) = C_p(X, E)^{\mathbb{N}}$ and $E^{\mathbb{N}}$ is a first-countable space. Let $f \in C_p(X, E^{\mathbb{N}}$. Since X is a P-space and any point of $E^{\mathbb{N}}$ is a G_δ-subset, the set $f^{-1}(y)$ is open-and-closed for each $y \in E^{\mathbb{N}}$. Hence $\{f^{-1}(y) : y \in E^{\mathbb{N}}\}$ is a countable discrete cover of the Lindelöf space X. Therefore the set $f(X)$ is countable. Proposition 5.1 completes the proof. ∎

Corollary 5.3. *Let X be a non-empty Lindelöf scattered space. Then $C_p(X, E)$ and $C_p(X, E)^{\mathbb{N}}$ are Fréchet-Urysohn spaces for each first-countable space E.*

Proof. Since PX is a Lindelöf P-space and $C_p(X, E)$ is a subspace of the space $C_p(PX, E)$, Corollary 5.2 completes the proof. ∎

In the distinct topological constructions apear the Σ-product of spaces. Moreover, is important the problem of classification of the compact subspaces and their subspaces of Σ-product of metrizable spaces (see [2, 6, 8, 10]).

Example 5.1. *Fix an uncountable cardinal τ and a discrete space $\mathbb{D}(\tau)$. By $\mathbb{L}(\tau) = \mathbb{D}(\tau) \cup \{c\}$ we demote the one-point lindelöfication of $\mathbb{D}(\tau)$, where $c \notin \mathbb{D}(\tau)$, $\mathbb{D}(\tau)$ is an open discrete subspace of $\mathbb{L}(\tau)$ and $\{\mathbb{D}(\tau) \setminus H : H \subseteq \mathbb{D}(\tau), |H| \leq \aleph_0\}$ is a base for $\mathbb{L}(\tau)$ at the point c.*

Let $\{E_\mu : \mu \in \mathbb{D}(\tau)\}$ be a non-empty family of topological first-countable spaces and $B = \{b_\mu \in E_\mu : \mu \in \mathbb{D}(\tau)\}$ be a fixed set.

By $\Sigma_B\{E_\mu : \mu \in \mathbb{D}(\tau)\}$ or Σ_B denote the subspace of the Cartesian product $\Pi\{E_\mu : \mu \in \mathbb{D}(\tau)\}$ consisting of all points $x = (x_\mu : \mu \in \mathbb{D}(\tau)) \in \Pi\{E_\mu : \mu \in \mathbb{D}(\tau)\}$ such that the set $\mathbb{D}_x(\tau) = \{\mu \in \mathbb{D}(\tau) : x_\mu \neq b_\mu\}$ is countable (see [6]).

For each $\mu \in \mathbb{D}(\tau)$ fix an open base $\{U_n(\mu) : n \in \mathbb{N}\}$ for a space E_μ at the point b_μ. Assume that the sets $U_{n+1}(\mu)$ and $E_\mu \setminus U_n(\mu)$ are functionally separated for all $n \in \mathbb{N}$ and $\mu \in \mathbb{D}(\tau)$. If in the discrete sum $E = \oplus\{E_\mu : \mu \in \mathbb{D}(\tau)\}$ we identify the closed discrete set B, then we obtain the set $F = \oplus_B\{E_\mu : \mu \in M\}$ and the natural projection $\pi_B : E \longrightarrow F$. Assume that $\pi_B(B) = b \in F$. On F we consider the topology relatively to which:

- $\pi_\lambda = \pi_B|E_\mu$ is an embedding of E_μ into F for each $\mu \in \mathbb{D}(\tau)$;

- $\{U_n = \cup\{U_n(\mu) : \mu \in \mathbb{D}(\tau)\} : n \in \mathbb{N}\}$ is a base for F at the point b.

We identify E_μ with $\pi_B(E_\mu) \subseteq F$. By construction, F is a completely regular first-countable space. If any space is (complete) metrizable, then the spaces E and F are (complete) metrizable.

Property 1. *The Cartesian product $\Pi\{E_\mu : \mu \in \mathbb{D}(\tau)\}$ is a closed subspace of the τ power $F^{\mathbb{D}(\tau)}$ of the space F.*

By $\Sigma_{(b,\tau)}F$ denote the subspace of the τ power $F^{\mathbb{D}(\tau)}$ consisting of all points $y = (y_\mu : \mu \in \mathbb{D}(\tau)) \in F^{\mathbb{D}(\tau)}\}$ such that the set $\mathbb{D}_y(\tau) = \{\mu \in \mathbb{D}(\tau) : y_\mu \neq b\}$ is countable.

From Property 1 it follows:

Property 2. *The Σ-product $\Sigma_B\{E_\mu : \mu \in \mathbb{D}(\tau)\}$ is a closed subspace of the τ-Σ-power $\Sigma_{(b,\tau)}F$.*

Now, as in Example IV.2.15 from [2], we mention the following property:

Property 3. *There exists a closed embedding Φ of the τ-Σ-power $\Sigma_{(b,\tau)}F$ in the space $C_p(\mathbb{L}(\tau), F)$.*

Realy, for each point $y = (y_\mu : \mu \in \mathbb{D}(\tau)) \in \Sigma_{(b,\tau)}F\}$ we consider the function $h_y : \mathbb{L}(\tau) \longrightarrow F$, where $h_y(c) = b$ and $h_y(\mu) = y_\mu$ for each $\mu \in \mathbb{D}(\tau)$. The mapping Φ is an embedding and the set $\Phi(\Sigma_{(b,\tau)}F) = \{f \in C_p(\mathbb{L}(\tau), F) : f(c) = b\}$ is a closed subset of the space $C_p(\mathbb{L}(\tau), F)$.

Property 4. *(N. Noble, [10]). The τ-Σ-power $\Sigma_{(b,\tau)}F$ and the Σ-product $\Sigma_B\{E_\mu : \mu \in \mathbb{D}(\tau)\}$ are Fréchet-Urysohn spaces.*

Since $\mathbb{L}(\tau)$ is a Lindelöf scattered P-space and F is a first-countable space, from Corollary 5.2 it follows that $C_p(\mathbb{L}(\tau), F)$ is a Fréchet-Urysohn space. By virtue of Property 3, $\Sigma_{(b,\tau)}F$ is a Fréchet-Urysohn space too. Property 2 completes the proof.

References

[1] A. V. Arhangel'skii, *Functional tightness, Q-spaces and τ-embeddings*, Comment. Math. Univ. Carol. 24:1 (1983), 105-120.

[2] A. V. Arhangel'skii, *Topological Function Spaces*, Mathematics and its Applications (Soviet Series), vol. 78, Kluwer Academic Publishers Group, Dordrecht, 1992.

[3] M. M. Choban, *Functionally countable spaces and Baire functions*, Serdica Math J. 23(1997), 233-242. (Uspehi Matem. Nauk 37:1 (1982), 157-158.

[4] M. M. Choban, *Baire sets in complete topological spaces*, Ucrainskii Matem. Jurnal 22:3 (1970), 330-342 (English translation: Ukrainian Math. Journal 22, no. 3 (1970), 286-295).

[5] K. M. Drees, $C_p(X, \mathbb{Z})$, Electronic Thesis of Dissertation. Bowling Green State University, 2009. https://etd.ohiolink.edu/

[6] R. Engelking, *General Topology*, PWN, Warszawa, 1977.

[7] R. Levy, N. D. Rice, *Normal p-spaces and the G_δ-topology*, Colloq. Math. 44:1 (1981), 227-240.

[8] J.Lindenstrauss, *Weakly compact sets - their topological properties and the Banach spaces they generate*, Ann. Math. Stud. 69 (1972), 235-273.

[9] R. A. McCoy, *K-space function spaces*, Internat. J. Math. & Math. Sci. 3(4) (1980), 701-711.

[10] N. Noble, *The continuity of functions on Cartesian products*, Trans. Amer. Math. Soc. 149 (1970), 187-198.

[11] E. G. Pytkeev, *On the tightness of spaces of continuous functions*, Russian Math. Surveys 37:1 (1982), 176-177 (Uspehi Matem. Nauk 37:1 (1982), 157-158.

[12] E. G. Pytkeev, *On sequentiality of spaces of continuous functions*, Russian Math. Surveys 37:5 (1982), 190-191 (Uspehi Matem. Nauk 37:5 (1982), 197-188.

[13] V. V. Tkachuk *A C_p-Theory Problem Book: Special Features of Function Spaces*, Problem Books in Mathematics, Springer International Publishing Switzerland 2014.

ROMAI J., v.17, no.1(2021), 29–39

COUPLES DES SOUSCATÉGORIES BISEMIRÉFLEXIVES ET LES THÉORIES DE TORSION RELATIVE

Dumitru Botnaru
L'Université d'Etat de Tiraspol, Chişinău, Republic of Moldova
dumitru.botnaru@gmail.com

Abstract (**Résumé**) On introduit la notion de couples de souscatégories bisemiréflexives, on construit des exemples, et on étudie certaines de leurs propriétés.
Toute théorie de torsion relative est un couple de souscatégories bisemiréflexives, Sont indiquées les conditions quand un couple de souscatégories bisemiréflexives est une théorie de torsion relative.

Mots clés: réflective, c-réflective, $\mathcal{L}$-semiréflexive souscatégories.
2020 MSC: 46 M 15; 18 F 60.

1. INTRODUCTION

Dans la catégorie $\mathcal{C}_2\mathcal{V}$ des espaces localement convexes topologiques vectoriaux Hansdorff, on introduit la notion des couples de souscatégories bisemiréflexives. Le Théorème 2.2 permet de construire de tels couples, et toute théorie de torsion relative (TTR) est un tel couple (Proposition 2.5). Cette Proposition et sa duale soulignent certaines propriétés de ces couples.

Le Théorème 2.7 décrit les couples bisemiréflexifs dans le cas où le second component est une souscatégorie c-réflective.

Dans ce cas tout couple de souscatégories bisemiréflexives est une TTR.

On utilise dans l'article les notions suivantes:

Les structures de factorisation:

$(\mathcal{E}pi, \mathcal{M}_f)$ = (la classe des épimorphismes, la classe des noyaux) = (la classe de tous les épimorphismes, la classe de toutes les inclusions topologiques avec l'image fermée);

$(\mathcal{E}_u, \mathcal{M}_p)$=(la classe des épimorphismes universaux, la classe des monomorphismes précis)=(la classe des morphismes surjectifs, la classe des inclusions topologiques);

$(\mathcal{E}_p, \mathcal{M}_u)$ = la classe des epimorhismes précis, la classe de monomorphismes universaux) [7];

$(\mathcal{P}''(\mathcal{R}), \mathcal{I}''(\mathcal{R})) = ((\varepsilon\mathcal{R}) \circ \mathcal{E}_p, ((\varepsilon\mathcal{R}) \circ \mathcal{E}_p)^{\llcorner}), \mathcal{R} \in \mathbb{R}$ (voir [3]);

$(\mathcal{E}'(\mathcal{K}), \mathcal{M}'(\mathcal{K})) = ((\mathcal{M}_p \circ (\mu\mathcal{K}))^{\urcorner}, \mathcal{M}_p \circ (\mu\mathcal{K})), \mathcal{K} \in \mathbb{K}$ (voir [3]);

$\mathbb{R}(\mathbb{K})$ = la classe de toutes les souscatégories réflectives (coréflectives) non nulles.

$\widetilde{\mathcal{M}}$ = la souscatégorie coréflective des espaces avec la topologie Mackey;

$\mathcal{S}$ = la souscatégorie réflective des espaces avec la topologie faible;

Π = la souscatégorie réflective des espaces complets avec la topologie faible;

Γ_0 = la souscatégorie réflective des espaces complets.

$l\Gamma_0$ = la souscatégorie des espaces localement complets (vor [9]);

$q\Gamma_0$ = la souscatégorie des espaces quasicomplets;

$s\mathcal{R}$ = la souscatégorie des espaces semiréflexifs;

$i\mathcal{R}$ = la souscatégorie des espaces inductivement semiréflexifs [1];

$\mathcal{B}$-$i\mathcal{R}$ = la souscatégorie des espaces $\mathcal{B}$-inductivement semiréflexifs [11];

$\mathcal{S}h$ = la souscatégorie des espaces Schwartz (voir [5]);

$u\mathcal{N}$ = la souscatégorie des espaces ultranucléaires (voir [8]);

Dans la catégorie $\mathcal{C}_2\mathcal{V}$ sont vraies les relations (voir [3]):

$\mathbb{R} \subset \mathbb{R}(\mathcal{E}pi \cap \mathcal{M}_u)$,

$\mathbb{K} \subset \mathbb{K}(\mathcal{E}_u \cap \mathcal{M}ono)$.

Si $\mathcal{A}$ est une souscatégorie et $\mathcal{B} \subset \mathcal{E}pi \cap \mathcal{M}ono$, alors $\mathcal{S}_{\mathcal{B}}(\mathcal{A})$ (respectivement: $\mathcal{Q}_{\mathcal{B}}(\mathcal{A})$) est la souscatégorie pleine de tous les $\mathcal{B}$-sousobjets (respectivement: $\mathcal{B}$-facteurbjets) des objets de la catégorie $\mathcal{A}$.

1.1. Soit $\mathcal{R}(\mathcal{K})$ une souscatégorie réflective (coréflective) de la catégorie $\mathcal{C}_2\mathcal{V}$ avec le foncteur réflectif $r : \mathcal{C}_2\mathcal{V} \to \mathcal{R}$ (coréflectif $k : \mathcal{C}_2\mathcal{V} \to \mathcal{K}$). Notons

$$\varepsilon\mathcal{R} = \{e \in \mathcal{E}pi | r(e) \in \mathcal{I}so\} \quad \mu\mathcal{K} = \{m \in \mathcal{M}ono | k(m) \in \mathcal{I}so\}.$$

Soit $b : X \to Y \in \mathcal{C}_2\mathcal{V}, r^X : X \to rX$ $\mathcal{R}$-réplique de $X, b \in \varepsilon\mathcal{R}$, alors et seulement alors quand $b \in \mathcal{E}pi$ et

$$l^X = f \cdot b \tag{1}$$

pour un f (voir[4]).

1.2. Les couples de souscatégories $(\mathcal{K}, \mathcal{L})$ qui verifient la condition $\mu\mathcal{K} = \varepsilon\mathcal{L}$ s'appellent les couples de souscatégories conjuguées et forment la classe $\mathbb{P}_c$, les soucatégories réflectives (coréflectives) s'appellent c-réflectives (c-coréflectives) et forment la classe $\mathbb{R}_c(\mathbb{K}_c)$.

Si $(\mathcal{K}, \mathcal{L}) \in \mathbb{P}_c$, alors chaque componente est de manière unique définie. $(\mathcal{C}_2\mathcal{V}, \mathcal{C}_2\mathcal{V})$ est le plus grand élément de la classe $\mathbb{P}_c$ et $(\widetilde{\mathcal{M}}, \mathcal{S})$ le plus petit élément. A la classe $\mathbb{R}_c$ appartiennent la souscatégorie $\mathcal{S}h$ des espaces Schwartz (voir [5]), $u\mathcal{N}$ des espaces ultranucléaires (voir [8]), les souscatégories générées des objets injectifs (voir [5]).

1.3. Définition [4]. *Soit $\mathcal{L} \in \mathbb{R}$. La classe des souscatégories réflectives fermées par rapport à $(\varepsilon\mathcal{L})$-sousobjets (respectivement: $(\varepsilon\mathcal{L})$-facteurobjets) est* *noté* $\mathbb{R}^s(\varepsilon\mathcal{L})$ *(respectivement:* $\mathbb{R}_f(\varepsilon\mathcal{L})$*), et* $\mathbb{R}^s_f(\varepsilon\mathcal{L}) = \mathbb{R}^s(\varepsilon\mathcal{L}) \cap \mathbb{R}_f(\varepsilon\mathcal{L})$*. Les éléments de la classe* $\mathbb{R}^s_f(\varepsilon\mathcal{L})$ *s'appellent souscatégories* $\mathcal{L}$*-semiréflexives.*

Pour $\mathcal{T} \in \mathbb{K}$ dualement, on définit les classes $\mathbb{K}^s(\mu\mathcal{T})$, $\mathbb{K}_f(\mu\mathcal{T})$ et la classe de souscatégories $\mathcal{T}$-semicoréflexives $\mathbb{K}^s_f(\mu\mathcal{T})$.

1.4. Définition [4]. *Soit* $\mathcal{A}$ *une souscatégorie, et* $\mathcal{L}$ *une souscatégorie réflective de la catégorie* $\mathcal{C}_2\mathcal{V}$*. L'objet* X *se nomme* $(\mathcal{L}, \mathcal{A})$*-semiréfleexifs, si sa réplique appartient à la souscatégorie* $\mathcal{A}$*. La souscatégorie pleine de tous les objets* $(\mathcal{L}, \mathcal{A})$*-semiréflexifs se nomme produit semiréflexif des souscatégories* $\mathcal{L}$ *et* $\mathcal{A}$*, notée*

$$\mathcal{R} = \mathcal{L} *_{sr} \mathcal{A}.$$

1.5. Corollaire ([6], Corollaire 2.5). *Soit* $\mathcal{L} \in \mathbb{R}_c$ *et* $\mathcal{A} \in \mathbb{R}$*. Alors* $\mathcal{L} *_{sr} \mathcal{A} \in \mathbb{R}^s_f(\varepsilon\mathcal{L})$*.*

1.6. Lemme ([4], Lemme 1.8). *Soit* $\mathcal{R} \in \mathbb{R}$*, et* $\mathcal{K} \in \mathbb{K}$*. Les affirmations suivantes sont équivalentes:*

1. $r(\mathcal{K}) \subset \mathcal{K}$.

2. $k(\mathcal{R}) \subset \mathcal{R}$.

1.7. Foncteurs commutatifs. On examinera deux foncteurs t_1, t_2, tous les deux coréflecteurs, tous les deux réflecteurs, ou l'un coréflecteur et l'autre réflecteur. Dans la catégorie $\mathcal{C}_2\mathcal{V}$ si $t_1t_2A \sim t_2t_1A$ pour tout $A \in |\mathcal{C}_2\mathcal{V}|$, alors on peut facilement vérifier que les $t_1 \cdot t_2$ et $t_2 \cdot t_1$ sont isomorphes.

1.8. Définition [2]. *Soit* $\mathcal{K} \in \mathbb{K}$*, et* $\mathcal{R} \in \mathbb{R}$*.*$(\mathcal{K}, \mathcal{R})$ *est nommée une théorie de torsion relative (TTR), si les foncteurs* k *et* r *commutent:* $k \cdot r = r \cdot k$*, et pour tout objet* $X \in |\mathcal{C}_2\mathcal{V}|$ *le carré*

$$r^X \cdot k^X = k^{rX} \cdot r^{kX}$$

est cartésien et aussi cocartésien.

Une TTR $(\mathcal{K}, \mathcal{R})$ est relative en rapport avec la souscatégorie $\mathcal{K} \cap \mathcal{R}$. Une théorie de torsion dans une catégorie abélienne est relative en rapport avec la souscatégorie nulle (voir [2]).

1.9. Définition [6]. *Soit* $\mathcal{K} \in \mathbb{K}$ *et* $\mathcal{R} \in \mathbb{R}$*. La souscatégorie* $\mathcal{Q}_{\varepsilon\mathcal{R}}(\mathcal{K})$ *se nomme le produit de gauche des souscatégories* $\mathcal{K}$ *et* $\mathcal{R}$*, et est noté* $\mathcal{W} = \mathcal{K} *_s \mathcal{R}$*.*

Soit $X \in |\mathcal{C}_2\mathcal{V}|, k^X : kX \to X$ $\mathcal{K}$-coréplique de X et $rX : X \to rX$ $r^{kX} : kX \to rkX$ $\mathcal{R}$-réplique des objets respectifs. Alors

$$r^X \cdot w^X = r(k^X) \cdot r^{kX}. \tag{1}$$

Soit

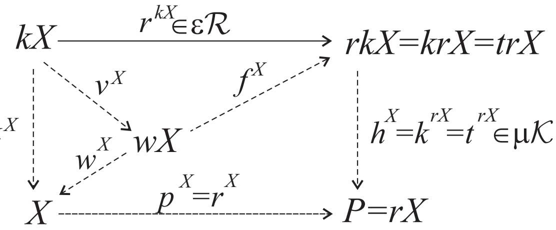
$$r^X \cdot w^X = k^{rX} \cdot f^X \tag{2}$$

le carré cartésien construit sur les morphismes r^X et r^{kX}. Alors

$$k^X = w^X \cdot v^X, \tag{3}$$

$$r^{kX} = f^X \cdot v^X \tag{4}$$

pour un $v^X : kX \to wX$,



Notion duale: le produit de droite $\mathcal{V} = \mathcal{K} *_d \mathcal{R}$ des catégories $\mathcal{K}$ et $\mathcal{R}$.

1.10. Mentionnons les propriétés suivantes du produit de gauche (voir [6]).

1. $v^X \in \varepsilon\mathcal{R}$ et $wX \in |\mathcal{W}|$.

2. La correspondance $X \mapsto (wX, w^X)$ indique $\mathcal{W}$ comme une souscatégorie faiblement coréflective (tout morphisme $f : Y \to X$ avec $X \in |\mathcal{W}|$ est facteurisé par w^X, mais en général pas uniquement).

3. w^X est $\mathcal{W}$-coréplique de X dans les cas suivants:
a) $\mathcal{S} \subset \mathcal{R}$; b) $\tilde{\mathcal{M}} \subset \mathcal{K}$.

4. L'egalité

$$u^X = w^X \cdot v^X$$

est $(\varepsilon\mathcal{R}, (\varepsilon\mathcal{R})^\top)$-factorisation du morphisme k^X.

2. Couples de souscatégories bisemiréflexives

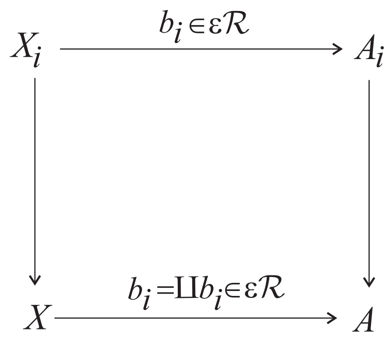
2.1. Definition. *Soit $\mathcal{K} \in \mathbb{K}$ et $\mathcal{R} \in \mathbb{R}$. $(\mathcal{K}, \mathcal{R})$ se nomme un couple de souscatégories bisemiréflexives, si $\mathcal{K} \in \mathbb{K}^s_f(\varepsilon\mathcal{R})$, et $\mathcal{R} \in \mathbb{R}^s_f(\mu\mathcal{K})$, c'est-à-dire $\mathcal{K}$ est une souscatégorie $\mathcal{R}$-semicoréflexive, et $\mathcal{R}$ est une souscatégorie $\mathcal{K}$-semiréflexive.*

2.2. Théorème. *1. Soit $\mathcal{R}$ une souscatégorie $\mathcal{K}$-semiréflexive. Alors $(\mathcal{S}_{\varepsilon\mathcal{R}}(\mathcal{K}), \mathcal{R})$ est un couple de souscatégories bisemiréflexives.*

1. Soit $\mathcal{K}$ une souscatégorie $\mathcal{R}$-semiréflexive. Alors $(\mathcal{K}, \mathcal{Q}_{\mu\mathcal{K}}(\mathcal{R}))$ est un couple de souscatégories bisemiréflexives.*

$\downarrow$ Notons $\mathcal{S}_{\varepsilon\mathcal{R}}(\mathcal{K})$ par $\mathcal{T}$.

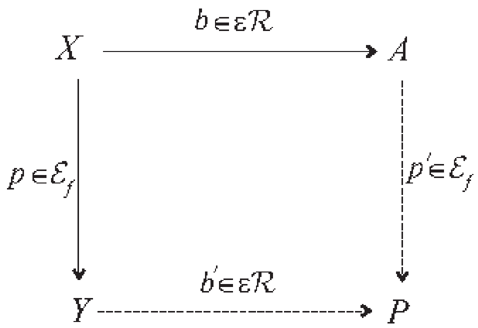
$\mathcal{T} \in \mathbb{K}$. Vérifions que $\mathcal{T}$ est fermé par rapport avec les sommes. Soit $A_i \in |\mathcal{K}|, b_i : X_i \to A_i \in \varepsilon\mathcal{R}, X = \sqcup X_i.A = \sqcup A_i$ et $b = \sqcup b_i, i \in \mathcal{I}$. Alors $A \in |\mathcal{K}|$ et $b \in \varepsilon\mathcal{R}$, puisque $(\varepsilon\mathcal{R}, (\varepsilon\mathcal{R})^\perp)$ est une structure de factorisation de droite. Ainsi $X \in |\mathcal{T}|$.



Vérifions que $\mathcal{T}$ est fermé par rapport à $\mathcal{E}_f$-facteurobjets. Soit $A \in |\mathcal{K}|, b : X \to A \in \varepsilon\mathcal{R}$ et $p : X \to Y \in \mathcal{E}_f$. Examinons le carré cocartésien

$$p' \cdot b = b' \cdot p \tag{1}$$

construit sur les morphismes p et b. Alors $p' \in \mathcal{E}_f$ et $b' \in \varepsilon\mathcal{R}$. Donc $P \in |\mathcal{K}|$ et $Y \in |\mathcal{T}|$. Ainsi on a démontré que $\mathcal{T} \in \mathbb{K}$.

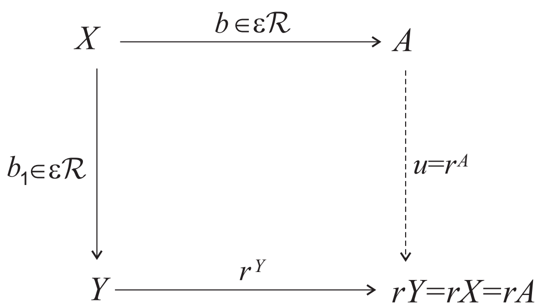


$\mathcal{T} \in \mathbb{K}^s(\varepsilon\mathcal{R})$. Evidemment.

$\mathcal{T} \in \mathbb{K}_f(\varepsilon\mathcal{R})$. Soit $A \in |\mathcal{K}|, b : X \to A \in \varepsilon\mathcal{R}$ et $b_1 : X \to Y \in \varepsilon\mathcal{R}$. Soit encore $r^Y : Y \to rY$ $\mathcal{R}$-réplique de Y. Alors $r^Y \cdot b_1 : X \to rY$ est $\mathcal{R}$-réplique de X et

$$r^Y \cdot b_1 = u \cdot b \tag{2}$$

pour un u. Ainsi $u \cdot b \in \varepsilon\mathcal{R}$ et $b \in \varepsilon\mathcal{R}$. Donc $u \in \varepsilon\mathcal{R}$ et u est $\mathcal{R}$-réplique de $A : u = r^A$.



Comme $\mathcal{R} \in \mathbb{R}^s_f(\mu\mathcal{K})$, on déduit que $k(\mathcal{R}) \subset \mathcal{R}$, et en vertu du Lemme 1.6 $r(\mathcal{K}) \subset \mathcal{K}$. Ainsi $rA = |\mathcal{K}|$, c'est-à-dire $rY \in |\mathcal{K}|$, ou $Y \in |\mathcal{T}|$.

$\mathcal{R} \in \mathbb{R}^s_f(\mu\mathcal{T})$. Puisque $\mathcal{K} \subset \mathcal{T}$, il résulte que $\mu\mathcal{T} \subset \mu\mathcal{K}$. Ainsi $\mathcal{R} \in \mathbb{R}^s_f(\mu\mathcal{T})$.

2.3. Corollaire. *Soit* $\mathcal{K}, \mathcal{P} \in \mathbb{K}, \mathcal{K} \subset \mathcal{P}, \mathcal{R} \in \mathbb{R}$ *et* $\mathcal{R} \in \mathbb{R}^s_f(\mu\mathcal{K})$. *Alors* $(\mathcal{S}_{\varepsilon\mathcal{R}}(\mathcal{P}), \mathcal{R})$ *est un couple de souscatégories bisemiréflexives.*

2.4. Examinons certains exemples.

1. Puisque $l\Gamma_0 \in \mathbb{R}^s_f(\varepsilon\mathcal{S})$ et $\mu\tilde{\mathcal{M}} = \varepsilon\mathcal{S}$, déduisons que pour tout $\mathcal{K} \in \mathbb{K}$, si $\tilde{\mathcal{M}} \subset \mathcal{K}$, alors $(\mathcal{S}_{\varepsilon l\Gamma_0}(\mathcal{K}), l\Gamma_0)$ est un couple de souscatégories bisemiréflexives.

2. La souscatégorie des espaces semiréflexifs $\mathcal{S}r$ est $\mathcal{S}$-semiréflexive, $\mathcal{S}r = \mathcal{S} *_{sr} q\Gamma_0$. Ainsi pour $\mathcal{K} \in \mathbb{K}$ et $\tilde{\mathcal{M}} \subset \mathcal{K}$ $(\mathcal{S}_{\varepsilon\mathcal{S}r}(\mathcal{K}), \mathcal{S}r)$ est un couple de souscatégories bisemiréflexives, et $t(i\mathcal{R}) \subset q\Gamma_0$, où $t : \mathcal{C}_2\mathcal{V} \to \mathcal{S}_{\varepsilon i\mathcal{R}}(\mathcal{K})$ est le foncteur coréflecteur.

3. La souscatégorie des espaces inductivement semiréflexifs $i\mathcal{R}$ est $\mathcal{S}h$-semiréflexive, où $\mathcal{S}h$ est la souscatégorie des espaces Schwartz, et $i\mathcal{R} = \mathcal{S}h *_{sr} \Gamma_0$. $(\mathcal{C}h, \mathcal{S}h)$ forment un couple de souscatégories conjuguées. Ainsi si $\mathcal{P} \in \mathbb{K}$ et $\mathcal{C}h \subset \mathcal{P}$, alors $(\mathcal{S}_{\varepsilon\mathcal{R}}(\mathcal{P}), i\mathcal{R})$ est un couple de souscatégories bisemiréflxives, et $t(i\mathcal{R}) \subset \Gamma_0$, où $t : \mathcal{C}_2\mathcal{V} \to \mathcal{S}_{\varepsilon i\mathcal{R}}(\mathcal{P})$ est le foncteur coréflecteur.

2.5 Proposition. *Soit* $(\mathcal{T}, \mathcal{R})$ *une TTR. Alors* $(\mathcal{T}, \mathcal{R})$ *est un couple de souscatégories bisemiréflexives.*

$\downarrow$ $\mathcal{T} \in \mathbb{K}^s(\varepsilon\mathcal{R})$. Soit $X \in |\mathcal{T}|$ et $b : Y \to X \in \varepsilon\mathcal{R}$. Puisque $X \in |\mathcal{T}|$ et $(\mathcal{T}, \mathcal{R})$ est une TTR, alors $t^X \in \mathcal{I}so$ et $t^{rX} \in \mathcal{I}so$. Ainsi $rX \in |\mathcal{T}|$. Et $rX = rY$, et $t^Y \in \mathcal{I}so$.

$\mathcal{T} \in \mathbb{K}_f(\varepsilon\mathcal{R})$. Soit $X \in |\mathcal{T}|$ et $b : X \to Y \in \varepsilon\mathcal{R}$. Alors $rX = rY \in |\mathcal{T}|$ et $Y \in |\mathcal{T}|$ en vertu de précédentes démonstrations: $\mathcal{T} \in \mathbb{R}^s(\varepsilon\mathcal{R})$.

$\mathcal{R} \in \mathbb{R}^s_f(\mu\mathcal{T})$. Démonstration duale. $\uparrow$

2.6 Proposition. *Soit* $\mathcal{K} \in \mathbb{K}, \mathcal{R} \in \mathbb{R}^s_f(\mu\mathcal{K})$ *et* $\mathcal{T} = \mathcal{S}_{\varepsilon\mathcal{R}}(\mathcal{K})$. *Les affirmations suivantes sont vraies:*

1. Si $X \in |\mathcal{R}|$, *alors* $kX = tX$.

2. $\mathcal{T} = \mathcal{S}_{\varepsilon\mathcal{R}}(\mathcal{K} \cap \mathcal{R})$.

3. $\mathcal{Q}_{\varepsilon\mathcal{R}}(\mathcal{K}) \subset \mathcal{S}_{\varepsilon\mathcal{R}}(\mathcal{K})$.

4. Soit $\tilde{\mathcal{M}} \subset \mathcal{K}, X \in |\mathcal{C}_2\mathcal{V}|$ *et*

$$k^X = p^X \cdot b^X \tag{1}$$

$(\mathcal{P}''(\mathcal{R}), \mathcal{I}''(\mathcal{R}))$*-factorisation de la* $\mathcal{K}$*-coréplique* $k^X : kX \to X$.

Alors $p^X : pX \to X$ *est* $\mathcal{T}$*-coréplique de l'objet* X.

5. Soit $\mathcal{S} \subset \mathcal{R}, X \in |\mathcal{C}_2\mathcal{V}|$ *et*

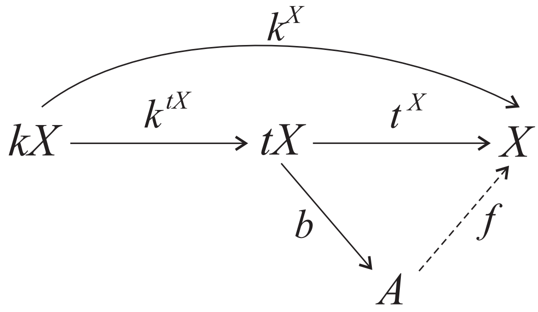
$$k^X = p^X \cdot b^X$$

$(\varepsilon\mathcal{R}, (\varepsilon\mathcal{R})^\perp)$*-factorisation* k^X. *Alors* $p^X : pX \to X$ *est* $\mathcal{T}$*-coréplique de l'objet* X.

↓ 1. Soit $X \in |\mathcal{R}|, k^X : kX \to X$ et $t^X : tX \to X$ $\mathcal{K}$- et $\mathcal{T}$-corépliques. Il existe $A \in |\mathcal{K}|$ et $b : tX \to A \in \varepsilon\mathcal{R}$. Alors

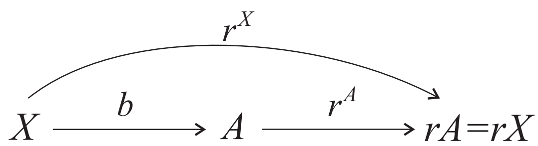
$$t^X = f \cdot b \tag{2}$$

pour un f, et on a le diagramme suivant commutatif

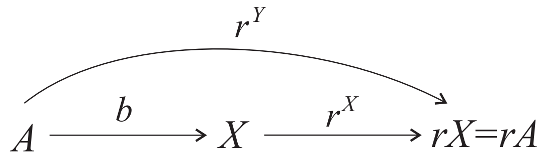


Ainsi $X \in |\mathcal{R}|$ et $t^X \in \mu\mathcal{K}$. Donc $tX \in |\mathcal{R}|$ et $b \in \mathcal{I}so$, ou $tX \in |\mathcal{K}|$. Il résulte que $tX = kX$.

2. Vérifions l'inclusion $\mathcal{T} \subset \mathcal{S}_{\varepsilon\mathcal{R}}(\mathcal{K} \cap \mathcal{R})$. Soit $X \in |\mathcal{T}|$. Il existe $A \in |\mathcal{K}|$ et $b : X \to A \in \varepsilon\mathcal{R}$. Si $r^A : A \to rA$ est $\mathcal{R}$-réplique de A, alors $r^X \cdot b : X \to rX$ est $\mathcal{R}$-réplique de X et $r^A \cdot b \in \varepsilon\mathcal{R}$. Ainsi $rX = rA \in |\mathcal{K}|$ ou $rX \in |\mathcal{K} \cap \mathcal{R}|$. Donc, $X \in |\mathcal{S}_{\varepsilon\mathcal{R}}(\mathcal{K} \cap \mathcal{R})|$.



3. Soit $A \in |\mathcal{K}|$, et $b : A \to X \in \varepsilon\mathcal{R}$. Si $r^X : X \to rX$ est $\mathcal{R}$-réplique de X, alors $r^X \cdot b : A \to rX$ est $\mathcal{R}$-réplique de A. Donc $rA = rA \in |\mathcal{K}|$, et $X \in |\mathcal{S}_{\varepsilon\mathcal{R}}(\mathcal{K})|$.

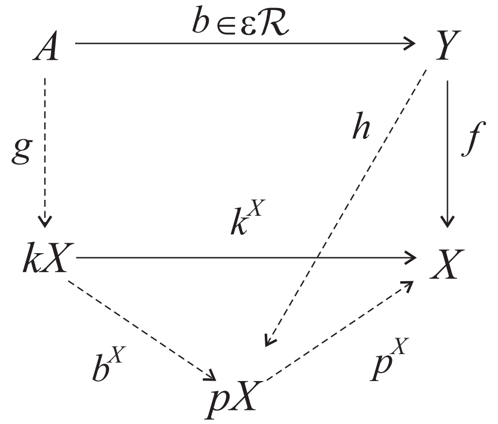


4. Puisque $p^X \in \mathcal{I}''(\mathcal{R})$ et $p^X \in \mathcal{E}_u$, il résulte que $k^X, p^X \in \mathcal{E}_u \cap \mathcal{M}ono$. Soit $\tilde{\mathcal{M}} \subset \mathcal{K}$. Alors $k^X, b^X \in \mathcal{M}_u$. Ainsi $b^X \in \mathcal{P}''(\mathcal{R}) \cap \mathcal{M}_u = ((\varepsilon\mathcal{R}) \circ \mathcal{E}_p) \cap \mathcal{M}_u = \varepsilon\mathcal{R}$ et $pX \in |\mathcal{T}|$.

Soit $Y \in |\mathcal{T}|$ et $f : Y \to X \in \mathcal{C}_2\mathcal{V}$. Il existe l'objet $A \in |\mathcal{K}|$ et le morphisme $b : A \to Y \in \varepsilon\mathcal{R}$. Alors

$$f \cdot b = k^X \cdot g \tag{3}$$

pour un g. Ou
$$f \cdot b = p^X(b^X \cdot g) \tag{4}$$
avec $b \in \varepsilon\mathcal{R} \subset \mathcal{P}''(\mathcal{R})$ et $p^X \in \mathcal{I}''(\mathcal{R})$. Ainsi $b \perp p^X$ et
$$b^X \cdot g = h \cdot b, \tag{5}$$
$$f = p^X \cdot h \tag{6}$$
pour un h. Donc f se factorise par p^X



5. Puisque $\mathcal{S} \subset \mathcal{R}$, il résulte que $k^X, p^X \in \mathcal{E}_u \cap \mathcal{M}ono$. Donc $p^X \in \mathcal{E}_u \cap \mathcal{M}ono$ et $pX \in |\mathcal{T}|$. Et on répète la démonstration p.4 ↑

2.6*. Proposition. *Soit* $\mathcal{R} \in \mathbb{R}$, $\mathcal{K} \in \mathbb{K}^s_f(\varepsilon\mathcal{R}))$ *et* $\mathcal{F} = \mathcal{Q}_{\mu\mathcal{K}}(\mathcal{R})$. *Les affirmations suivantes sont vraies:*

1. Si $X \in |\mathcal{K}|$, *alors* $rX = fX$.

2. $\mathcal{F} = \mathcal{Q}_{\mu\mathcal{K}}(\mathcal{K} \cap \mathcal{R})$.

3. $\mathcal{S}_{\mu\mathcal{K}}(\mathcal{R}) \subset \mathcal{Q}_{\mu\mathcal{K}}(\mathcal{R})$.

4. Soit $\mathcal{S} \subset \mathcal{R}, X \in |\mathcal{C}_2\mathcal{V}|$ *et*
$$r^X = p^X \cdot b^X$$
$(\mathcal{E}'(\mathcal{K}), \mathcal{M}'(\mathcal{K}))$*-factorisation de la* $\mathcal{R}$*-coréplique de l'objet* X. *Alors* $b^X : X \to pX$ *est* $\mathcal{F}$*-réplique de l'objet* X.

5. Soit $\tilde{\mathcal{M}} \subset \mathcal{K}, X \in |\mathcal{C}_2\mathcal{V}|$ *et*
$$r^X = p^X \cdot b^X$$
est $((\mu\mathcal{K})^\top, \mu\mathcal{K})$*-factorisation de la* $\mathcal{R}$*-réplique de* $r^X : X \to rX$. *Alors* $b^X : X \to pX$ *est* $\mathcal{F}$*-réplique de l'objet* X. ↑

2.7. Théorème. *Soit* $(\mathcal{K}, \mathcal{L}) \in \mathbb{P}_c, \mathcal{R} \in \mathbb{R}^s_f(\mu\mathcal{K})$ *et* $\mathcal{T} = \mathcal{S}_{\varepsilon\mathcal{R}}(\mathcal{K})$. *Alors:*

1. $(\mathcal{T}, \mathcal{R})$ *est une TTR.En particulier,* $(\mathcal{T}, \mathcal{R})$ *est un couple de souscatégories bisemiréflexives.*

2. $\mathcal{S}_{\varepsilon\mathcal{R}}(\mathcal{K}) = \mathcal{Q}_{\varepsilon\mathcal{R}}(\mathcal{K})$.

3. $t \cdot r = r \cdot t = k \cdot r = r \cdot k$.

$\downarrow$ 1. Soit $X \in |\mathcal{C}_2\mathcal{V}|, k^X : kX \to X$ $\mathcal{K}$-coréplique, et $rkX : kX \to rkX$ $\mathcal{R}$-réplique des objets respectifs. Sur le morphisme k^X et r^{kX}on construit le carré cocartésien

$$p^X \cdot k^X = h^X \cdot r^{kX}. \tag{1}$$

Puisque $r^{kX} \in \varepsilon\mathcal{R}$, il résulte que $p^X \in \varepsilon\mathcal{R}$. De même $k^X \in \mu\mathcal{K}$, donc $h^X \in \mu\mathcal{K}$. De plus, $rkX \in |\mathcal{K}|$. Ainsi $h^X : rkX \to P$ est $\mathcal{K}$-coréplique de P et $P \in |\mathcal{R}|$. Donc $p^X : X \to P \in \varepsilon\mathcal{R}$ et $P \in |\mathcal{R}|$. Donc p^X est $\mathcal{R}$-réplique de X. On a $krX = kP = rkX$, et les foncteurs k et r commutent.

Soit

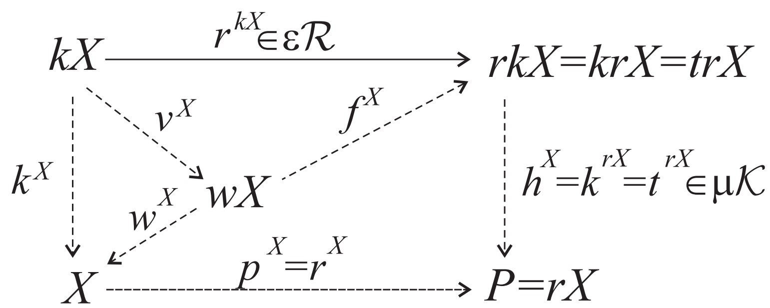
$$r^X \cdot w^X = k^{rX} \cdot f^X \tag{2}$$

le carré cartésien construit sur les morphismes $r^X (= p^X)$, et $k^{rX} (= h^X)$. Alors

$$k^X = w^X \cdot v^X, \tag{3}$$

$$r^{kX} = f^X \cdot v^X \tag{4}$$

pour un v^X, qui appartient à la classe $\varepsilon\mathcal{R}$. Donc $f^X \in \varepsilon\mathcal{R}$ et $wX \in |\mathcal{T}|$. De plus, $v^X \in \mathcal{E}pi$, et le carré (1) est cocartésien, de même le carré (2) est aussi cocartésien.



On vérifie facilement que l'égalité (3) est $(\varepsilon\mathcal{R}, (\varepsilon\mathcal{R})^\perp)$-factorisations du morphisme k^X En vertu de la Proposition 2.6 p.5 $w^X : wX \to X$ est $\mathcal{T}$-coréplique de l'objet X.

Et $k^{rX} : rkX \to rX \in (\varepsilon\mathcal{R})^\perp$. Ainsi k^{rX} est $\mathcal{T}$-coréplique de l'objet rX et $f^X : wX \to rkX$ est $\mathcal{R}$-réplique de l'objet wX.

Soit $w : \mathcal{C}_2\mathcal{V} \to \mathcal{T}$ foncteur coréflecteur. Alors $rwX = rkX, wrX = rkX$. Ainsi les foncteurs w et r commutent, et $(\mathcal{T}, \mathcal{R})$ est une TTR.

2. En vertu du p.3 de la Proposition 2.6, il est suffisant de vérifier que $\mathcal{S}_{\varepsilon\mathcal{R}}(\mathcal{K}) \subset \mathcal{Q}_{\varepsilon\mathcal{R}}(\mathcal{K})$. Soit $A \in |\mathcal{K}|$ et $b : X \to A \in \varepsilon\mathcal{R}$. Alors

$$r^X = f \cdot b \tag{8}$$

pour un f,

$$f = k^{rX} \cdot g \tag{9}$$

pour un g. On a

$$k^{rX} \cdot g \cdot b = f \cdot b = r^X \tag{10}$$

i.e.

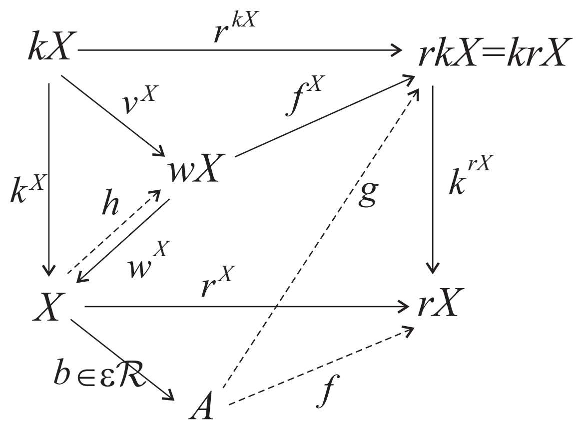
$$k^{rX} \cdot g \cdot b = r^X \cdot 1. \tag{11}$$

Il existe $h : X \to wX$ ainsi que

$$1 = w^X \cdot h, \tag{12}$$

$$g \cdot b = f^X \cdot h, \tag{13}$$

d'où il résulte que $w^X = h^{-1}$. Donc $v^X, k^X \in \varepsilon\mathcal{R}$ et $X \in |\mathcal{Q}_{\varepsilon\mathcal{R}}(\mathcal{K})|$.



Puisque $rkX \in |\mathcal{K} \cap \mathcal{R}| \subset \mathcal{T}|$ et $h^X \in \mu\mathcal{K}$, on déduit que h^X est $\mathcal{T}$-coréplique de $rX : h^X = t^{rX}$. Et $f^X \in \varepsilon\mathcal{R}$ et $rkX \in |\mathcal{K}|$, donc $X \sim wX \in |\mathcal{T}|$.

Il résulte 3 du p.1. $\uparrow$

2.7*. Théorème. *Soit* $(\mathcal{K}, \mathcal{L}) \in \mathbb{P}_c, \mathcal{T} \in \mathbb{K}_f^s(\mu\mathcal{K})$ *et* $\mathcal{T} = \mathcal{Q}_{\mu\mathcal{T}}(\mathcal{L})$. *Alors:*

1. $(\mathcal{T}, \mathcal{F})$ *est une TTR. En particulier,* $(\mathcal{T}, \mathcal{Q}_\mu\mathcal{T}(\mathcal{L}))$ *est un couple des souscatégories bisemiréflexives.*

2. $\mathcal{F} = \mathcal{S}_{\mu\mathcal{T}}(\mathcal{L})$.

3. $f \cdot t = t \cdot f = l \cdot t = t \cdot l$. $\uparrow$

2.8. Soit $\mathcal{R} \in \mathbb{R}$. Notons $\mathbb{B}(\mathcal{R})$ la classe des souscatégories coréflectives $\mathcal{K}$ pour qui $\mathcal{R} \in \mathbb{R}_f^s(\mu\mathcal{K})$. Alors:

1. $\mathcal{C}_2\mathcal{V} \subset \mathbb{B}(\mathcal{R})$.
2. Si $\mathcal{K}_1, \mathcal{K}_2 \in \mathbb{K}$, $\mathcal{K}_1 \subset \mathcal{K}_2$ et $\mathcal{K}_1 \in \mathbb{B}(\mathcal{R})$, alors et $\mathcal{K}_2 \in \mathbb{B}(\mathcal{R})$.
3. Soit $\mathcal{K}_\mathcal{R} = \cap\{\mathcal{T} | \mathcal{T} \in \mathbb{B}(\mathcal{R})\}$. Alors $\mathcal{K}_\mathcal{R} \in \mathbb{B}(\mathcal{R})$.

4. Soit $\overline{\mathcal{K}_{\mathcal{R}}} = \mathcal{S}_{\varepsilon\mathcal{R}}(\mathcal{K}_{\mathcal{R}})$. Alors $(\overline{\mathcal{K}_{\mathcal{R}}}, \mathcal{R})$ est une paire des souscatégories bisemiréflexives et $\overline{\mathcal{K}_{\mathcal{R}}}$ est le plus petite souscatégorie coréflective avec cette propriété.

2.9. Problèmes. 1. Examinons le couple de souscatégories conjuguées $(\tilde{\mathcal{M}}, \mathcal{S})$ et $\mathcal{R}$ une des souscatégories $\mathcal{S}$-semiréflexives: $\mathcal{B}$-$i\mathcal{R}, s\mathcal{R}$ ou $l\Gamma_0$.

Décrire les souscatégories $\mathcal{S}_{\varepsilon\mathcal{R}}(\tilde{\mathcal{M}}), \mathcal{K}(\mathcal{R})$ et $\overline{\mathcal{K}}(\mathcal{R})$.

2. Pour le couple des souscatégories conjugées $(\mathcal{C}h, \mathcal{S}h)$ el faut décrire les souscatégories $\mathcal{S}_{\varepsilon i\mathcal{R}}(\mathcal{C}h), \mathcal{K}(\mathcal{R})$ et $\overline{\mathcal{K}}(\mathcal{R})$, où $i\mathcal{R}$ est la souscatégorie des espaces inductivement semiréflexifs [1].

References

[1] Berezansky J.A., *Les espaces inductivement réflexifs localement convexes*, Dokl. Ak. Nauk. SSSR, **182**, 1(1966), 20-22 (en russe).

[2] Botnaru D., *La théorie des torsions relatives dans la catégorie des espaces localement convexes*, Mat. issled., 90(1986), 28-40 (en russe).

[3] Botnaru D., *Structures bicatégorielles complémentaires*, Rev. Roumaine Math. Pures Appl., **55**, 2(2010), 97-119.

[4] Botnaru D., *Groupoïd des souscatégories $\mathcal{L}$-semi-réflexives*, Rev.Roumaine Math. Pures Appl., **63**, 1(2018), 61-71.

[5] Botnaru D., *Couples des souscatégories conjugées*, ROMAI J., **14**, 1(2018), 23-41.

[6] Botnaru D., Ţurcanu A., *On the right product of two subcatregories,* ROMAI J., **12**, 2(2016), 2, 9-18

[7] Botnaru D., Gysin V.B., *Monomorphismes stables de la catégorie des espaces localement convexes*, Bulletin. Acad. Sc. R.S.S.Moldova., 1(1973), 3-7 (en russe).

[8] Brudovsky B.S., *La topologie nucléaire associée, application du type s et espaces ultra-nucléaires*, Dokl. Ak. Nauk SSSR, **178**, 2(1968), 271-273 (en russe).

[9] Raïcov D.A., *L'exponentielle pour les espaces des applications linéaires continues*, Mat. sb., **7(109)**, 2(1965), 279-302 (en russe).

[10] Robertson A.P., Robertson W.J., *Topological vectors spaces,* Cambridge Tracts in Mathematics and Mathematical Phisics no 53, Cambridge University Press, 1964.

[11] Sekevanov V.S., *Espaces localement convexes $\mathcal{B}$-inductifs réflexifs*, Func. an., Mejvouz. sb., Oulianosk, 14(1980), 128-131 (en russe).

ROMAI J., v.17, no.1(2021), 41–46

DATA DEDUPLICATION

Ivan A. Budanaev
Institute of Mathematics and Computer Sciences of ASM, Chişinău, Republic of Moldova
ivan.budanaev@gmail.com

Abstract Each day an abundance of new data is generated. With that comes the necessity of a data deduplication process within each data project. This need arises due to multiple reasons, some stronger than others: storage efficiency, data linkage, information representation, etc. This article describes the application of distance functions in the process of data deduplication which covers the aforementioned uses cases.

Keywords: metric, Hamming distance, Levenshtein distance, Jaccard Index.
2020 MSC: 32F45, 68P30, 94A55.

1. INTRODUCTION

Every data related project be it in data science, data engineering or data analysis project, sooner or later faces the problem of duplicated data - it is to no surprise, in the year of 2020, the units of data generated has reached measures in the quintillions (10^{18} bytes). In many cases the duplications generated are exact duplicates, i.e. they have a one-to-one match with previously observed data. This happens, for example, during batch or real time data processing when the source generates data repetitively or when a failed ETL process appends duplicates after re-run. In many other cases the duplication is not so obvious and the reasons behind this are specific to each use case. For instance, the records generated by a faulty sensor are identical in a long sequence of measures but the timestamp when generated is different. Another frequent use case is human error, for example a typo in client's name during data entry process.

Data duplication can lead to unpredictable results - from insignificant like extra server logs to dangerous errors in financial or healthcare records. To avoid these issues, data deduplication comes to the rescue - a process of detecting and/or removing duplicates from a data set. The benefits it brings to the table are numerous. The first benefit is storage efficiency. It used to be a very important aspect decades ago but nowadays with low storage prices it is more and more an afterthought. The second benefit is data linkage - a process by which you create a logical link between similar or almost identical data. This is important for many domains, one being US political campaign donations analysis. In the USA, data on individual contributions for political campaigns is publicly accessible, and contains information like the contributor's name, occupation, city, state, address, date of transaction, amount of contribution, and name of committee disclosing the contribution. Over the years, contributors change their occupation, address and even name. Nevertheless, a thorough analysis of contributions requires the linkage of these data points which are slightly distinct records but attributable to one person or organization. Another benefit of deduplication is information representation. For example, having dozens of contributions' information from a single person, you may want to select the most descriptive information representing that cluster of contributions - maybe based on the most recent transaction or maybe recovered information in case there was a typo in that record. There might be other reasons but they all serve one goal - retrieve one record that best describes the cluster it resides in.

The next chapter introduces the notion of distance functions, one of the fundamental building blocks in information theory. I would like to express a special note of gratitude to my doctorate advisor who left us this year - professor, academician Mitrofan M. Cioban. Unbounded knowledge, wisdom and patience with which he introduced me to the wonderful world of metrics spaces and distance function deserve deepest admiration and gratitude.

2. DISTANCE FUNCTIONS

Let X be a non-empty set and $d : X \times X \to \mathbb{R}$ be a mapping such that for all $x, y \in X$ we have:
(i_m) $d(x, y) \geq 0$;
(ii_m) $d(x, y) + d(y, x) = 0$ if and only if $x = y$,
(iii_m) $d(x, z) \leq d(x, y) + d(y, z)$.

Then (X, d) is called a *quasimetric space* and d is called a *quasimetric* on X. A function d with properties (i_m) and (ii_m) is called a distance on X. If d is a quasimetric on X with property (iv_m) $d(x, y) = d(y, x)$, then (X, d) is called a *metric space* and d is called a *metric*.

For a more in-depth description of distance and metric spaces properties please consult [3, 4, 5, 6].

The symmetry property (iv_m) plays an important role in information theory as the invariance of distance between records is more important than their order. The following metrics are most popular distances used in data deduplication and they all satisfy the symmetry condition.

We begin with the Hamming Distance [10, 12] - if not the most important information distance then without any doubts one of the most important:

$$d_H(a_1 a_2 \cdots a_n, b_1 b_2 \cdots b_n) = |\{i \leq n : a_i \neq b_i\}|$$

is the Hamming distance two strings $a = a_1a_2 \cdots a_n$ and $b = b_1b_2 \cdots b_m$, i.e. the number of distinct characters with the same index. Usually, the Hamming distance is calculated over the strings of the same length, but it can be generalized on strings of any length by padding the shortest one with special characters to meet the length of the longest string. Since a character here can be a digit or a letter, Hamming distance can be used on both numerical and text columns [2].

The *Levenshtein distance* [11] (also known as the *Edit distance*) between two strings a and b is defined as the minimum number of insertions, deletions, and substitutions required to transform one string to the other. A formal definition of Levenshtein's distance $d_L(a, b)$ is given by the following recurrent formula:

$$d_L(a_1 \cdots a_i, b_1 \cdots b_j) = \begin{cases} i, & \text{if j=0,} \\ j, & \text{if i=0,} \\ \min \begin{cases} d_L(a_1 \cdots a_{i-1}, b_1 \cdots b_j) + 1 \\ d_L(a_1 \cdots a_i, b_1 \cdots b_{j-1}) + 1 \\ d_L(a_1 \cdots a_{i-1}, b_1 \cdots b_{j-1}) + 1_{(a_i \neq b_j)}, \end{cases} \end{cases}$$

where $1_{(a_i \neq b_j)}$ equals to 0 if $a_i = b_j$ and to 1 otherwise. We can see that by default Levenshtein distance expects a pair of strings of any length as its input. Although these strings can be numbers as well, a standard usage of edit distance is on text type columns. Additionally, the above definition can be generalize to the *Weighted Levenshtein* distance when distance between different characters takes different values - some predefined vocabulary distance matrix $1_{(a_i \neq b_j)} = d_{vocabulary}(a_i, b_j)$.

Next on our list is the *Jaccard Index* distance, also known as the *Jaccard similarity coefficient*. This distance measures the similarity between finite sets and can be used on strings as the ratio between the number of characters simultaneously appearing in both strings (intersection) over the total number of distinct characters appearing in the concatenation of these strings(union):

$$J(A, B) = \frac{|A \cap B|}{|A \cup B|}$$

where A and B are sets of characters appearing in strings a and b. It is noticeable that this metric does not take into account the position of the symbols in strings, thus it is less penalizing in cases when data is erroneously permutated during write/update operation.

The presented metrics are some of the more popular and frequently used in data deduplication. Additional distances worth mentioning are *Jaro Winkler*, *Longest Common Substring*, *QGram*, *Cosine Similarity* and *Soundex*. For an exhaustive list please consult [7, 8]. For some data types standard metrics are not as useful as expected. For example calculating the distance between two

US zip codes is not a trivial task, and such problems are tackled with third party tools.

It is worth mentioning that when we have a measure of similarity, to obtain a measure of dissimilarity, we subtract its value from 1.

3. DATA DEDUPLICATION

Deduplication is the process of of identifying extra copies or similar versions of data. There are many deduplication tools available online - some are open source, others are commercial software. The approach behind all of them heavily relies on distance functions described in the previous chapter.

A typical deduplication project setup looks like the following - given an input dataset it is required to detect all pairs of duplicates, preferably along with a metric value describing their similarity. The input is usually a tabular dataset of shape (n, m) - n rows of records, each having m fields. These fields can be of different data types but usually they contain text data like a person's name, address, city, and numerical data like phone number, zip code, contribution amount. Usually, the text data is passed though a pre-processing step where it is cleaned and transformed to a desired format. For example, an address string is first cleaned from redundant spacing or punctuation symbols and then the information is transformed to a structured data with city name, street name, house number and zip code. Usually these types of operations generate additional columns of distinct data types making the process of deduplication more accurate. This is achieved by using different distance functions on different field data types in contrast to using a single metric on all concatenated fields.

Once all data cleaning and pre-processing is finished, deduplication can start. The main idea behind deduplication is finding pairs of strings that have a very high similarity coefficient, usually higher than a given threshold. The similarity of two records is the complementary of dissimilarity of those records, which is the same concept as the distance between them. Therefore, the problem of deduplication can be reduced to the task of finding the distance between a pair of records.

Given two records $x = (x_1, x_2, ..., x_m)$ and $y = (y_1, y_2, ..., y_m)$, the distance between them can be calculated using the formula:

$$d(x, y) = \frac{\sum_i^m d_i(x_i, y_i)}{m}$$

where d_i is the distance function of choice for the specific data types of fields x_i and y_i. For instance, if the field are zip code values, then the metric should be used as the distance between US zip codes. If the field however is of text type, the most feasible distance function of use would be Levenshtein distance. In certain cases a combination of various distance functions can be used on the same data types. In practice it is shown that many inconsistencies in data entry occur due to typos or other human error. In such cases, for example with zip code values, both numerical and text distance functions can be used. For ease of use when comparing distance values, all values must be in range $[0,1]$. To achieve this, each distance function d_i goes through the normalization process when its value is greater than 1. For example the Levenshtein distance can be normalized if divided by the length of the string.

Special attention must be paid to missing values during data deduplication. There are different approaches to tackling such scenarios and the best practices depend from case to case. The method used most often is to drop these fields. Other approaches impute data with most common element in the group thus reducing the error during distance calculation.

Calculating the distance between each pair x and y out of n records has a computational complexity of $O(n^2)$, not mentioning the space complexity. This is usually a heavy processing task, even with today's cloud computing resources. To improve the performance of data deduplication process, indexing and caching tools can be used. One such tool is *Apache Lucene* [1]. An example of an open source deduplication software that successfully applied *Apache Lucene* is *Duke* [9].

Once the distance matrix for the entire dataset is calculated, the duplicates are easily identified - pairs x and y for which the distance is above a certain threshold. This threshold varies from case to case, but a value within the range $[0, 0.1]$ is usually a good candidate.

In conclusion, there are many automated deduplication software and services available today. Each of them have their advantages and disadvantages. However, if the context of the project your team is working on, requires to build an in-house solution, the above described methodology can confidently be taken as starting points for the deduplication system.

References

[1] Apache Lucene Engine. Accessed on: Nov. 7, 2021, https://lucene.apache.org/.

[2] I. Budanaev, *On Hamming type Distance Functions*, ROMAI J., **12**, 2(2016), 25-32.

[3] M. M. Choban, *The theory of stable metrics*, Math. Balkanica, 2 (1988), 357-373.

[4] M. M. Choban, I. A. Budanaev, *Distances on Monoids of Strings and Their Applications*, Proceedings of the Conference on Mathematical Foundations of Informatics MFOI2016, July 25-29, 2016, Chisinau, Republic of Moldova, 2016, 144-159.

[5] M.M. Choban, I.A. Budanaev, *About Applications of Distances on Monoids of Strings*. Computer Science Journal of Moldova, **24**, 3(2016), 335-356.

[6] M. M. Choban, L. L. Chiriac, *On free groups in classes of groups with topologies*, Bul. Acad. Ştiinţe Repub. Moldova, Matematica 2-3(2013), 61–79.

[7] M. M. Deza, E. Deza, *Dictionary of Distances*, Elsevier, Amsterdam, 2006.

[8] M. M. Deza, E. Deza, *Encyclopedia of Distances*, Springer, Berlin, 2014.

[9] *Duke - a fast and flexible deduplication*, Accessed on: Nov. 7, 2021, https://github.com/larsga/Duke.

[10] R. W. Hamming, *Error Detecting and Error Correcting Codes,* Bell System Technical Journal **29**, 2(1952), 147-160.

[11] V. I. Levenshtein, *Binary codes capable of correcting deletions, insertions, and reversals*, Dokl. AN SSSR, **163**, 4(1965), 845–848 (in Russian). (English translation: Soviet Physics – Doklady **10**, 8(1966), 707–710.

[12] C. Shannon, *"A Mathematical Theory of Communication"*, The Bell System Technical Journal, **27**, pp. 279-423, pp. 623–656, 1948.

ROMAI J., v.17, no.1(2021), 47–51

METHODS OF CONSTRUCTION OF HAUSDORFF EXTENSIONS

Laurenţiu I. Calmutchi
Tiraspol State University, Chişinău, Republic of Moldova
lcalmutchi@gmail.com

Abstract In this paper we study the extensions of Hausdorff spaces generated by discrete families of open sets.

Keywords: extension, P-space, remainder.
2020 MSC: 54D35, 54B10, 54C20, 54D40.

1. INTRODUCTION

Any space is considered to be a Hausdorff space. We use the terminology from [3]. For any completely regular space X denote by βX the Stone-Čech compactification of the space X.

Fix a space X. A space eX is an extension of the space X if X is a dense subspace of eX. If eX is a compact space, then eX is called a compactification of the space X. The subspace $eX \setminus X$ is called a remainder of the extension eX.

Denote by $Ext(X)$ the family of all extensions of the space X. If X is a completely regular space, then by $Ext_\rho(X)$ we the family of all completely regular extensions of the space X. Obviously, $Ext_\rho(X) \subset Ext(X)$. Let $Y, Z \in Ext(X)$ be two extensions of the space X. We consider that $Z \leq Y$ if there exists a continuous mapping $f : Y \longrightarrow Z$ such that $f(x) = x$ for each $x \in X$. If $Z \leq Y$ and $Y \leq Z$, then we say that extensions Y and Z are equivalent and there exists a unique homeomorphism $f : Y \longrightarrow Z$ of Y onto Z such that $f(x) = x$ for each $x \in X$. We identify the equivalent extensions. In this case $Ext(X)$ and Ext_ρ are partial ordered sets.

Let τ be an infinite cardinal.

Denote by $O(\tau)$ the set of all ordinal numbers of cardinality $< \tau$. We consider that τ is the first ordinal number of the cardinality τ. For any $\alpha \in O(\tau)$ we put $O(\alpha) = \{\beta \in O(\tau) : \beta < \alpha\}$. In this case $O(\tau)$ is well ordered set such that $|O(\tau)| = \tau$ and $|O(\alpha)| < \tau$ for every $\alpha \in O(\tau)$.

A point $x \in X$ is called a $P(\tau)$-point of the space X if for any non-empty family γ of open subsets of X for which $x \in \cap\gamma$ and $|\gamma| < \tau$ there exists an open subset U of X such that $x \in U \subset \cap\gamma$. If any point of X is a $P(\tau)$-point, then we say that $P(\tau)$-space.

Any point is an $\aleph_0$-point. If $\tau = \aleph_1$, then the $P(\tau)$-point is called the P-point.

2. HAUSDORFF EXTENSIONS OF DISCRETE SPACES

Let τ be an infinite cardinal. Let E be a discrete space of the cardinality $\geq \tau$.

A family η of subsets of E is called a τ-centered if the family η is non-empty, $\cap\eta = \emptyset$, $\emptyset \notin \eta$ and any subfamily $\zeta \subset \eta$, with cardinality $|\zeta| < \tau$, there exists $l \in \eta$ such that $L \subset \cap\zeta$.

Two families η and ζ of subsets of the space E are almost disjoint if there exist $L \in \eta$ and $Z \in \zeta$ such that $L \cap Z = \emptyset$.

Any family of subsets is ordered by the following order: $L \preceq H$ if and only if $H \subset L$. Relatively to this oder some families of sets are well-ordered.

Proposition 2.1. *Let $k = |E| \geq \tau$ and $\Sigma\{k^m : m < \tau\} = k$. Then on E there exists a set Ω of well-ordered almost disjoint τ-centered families such that $|\Omega$ $= k^\tau$ and $|\eta| = \tau$ for each $\eta \in \Omega$.*

Proof. We fix an element $0 \in E$. For every $\alpha \in O(\tau)$ we put $E_\alpha = E$ and $0_\alpha = 0$. Then $E^\tau = \Pi\{E_\alpha : \alpha \in O(\tau)\}$. For each $x = (x_\alpha : \alpha \in O(\tau)) \in E^\tau$ we put $\phi(x) = sup\{0, \alpha : x_\alpha \neq 0_\alpha\}$. Obviously, $0 \leq \phi(x) \leq \tau$. Let $D = \{x = (x_\alpha : \alpha \in O(\tau)) \in E^\tau : \phi(x) < \tau\}$. By construction, $|D| = \Sigma\{k^m : m < \tau\} = k$ and $|E^\tau| = k^\tau$. Since $|E| = |D|$, we can fix a one-to-one mapping $f : E \longrightarrow D$. Fix a point $x = (x_\alpha : \alpha \in O(\tau)) \in E^\tau$. For any $\beta \in O(\tau)$ we put $V(x, \beta) = \{y = (y_\alpha : \alpha \in O(\tau)) \in E^\tau : y_\alpha = x_\alpha \textit{ for every } \alpha \leq \beta\}$ and $\eta_x = \{L(x, \beta) = f^{-1}(D \cap V(x, \beta) : \beta \in 0(\tau)\}$. Then $\Omega = \{\eta_x : x \in E^\tau\}$ is the desired set of τ-centered families. ∎

Remark 2.1. *Let $|E| = k \geq \tau$. Since on E there exists k mutually disjoint subsets of cardinality τ, on E there exists a set Φ of well-ordered almost disjoint τ-centered families such that $|\Phi| \geq k$ and $|\eta| = \tau$ for each $\eta \in \Phi$.*

Fix a set Φ of almost disjoint τ-centered families of subsets of the set E. We put $e_\Phi E = E \cup \Phi$. On $e_\Phi E$ we construct two topologies.

Topology $T^s(\Phi)$. The basis of the topology $T^s(\Phi)$ is the family $\mathcal{B}^s(\Phi) = \{U_L = L \cup \{\eta \in \Phi : H \subset L \textit{ for some } H \in \eta\} : L \subset E\}$.

Topology $T_m(\Phi)$. For each $x \in E$ we put $B_m(x) = \{\{x\}\}$. For every $\eta \in \Phi$ we put $B_m(\eta) = \{V_{(\eta,L)} = \{\eta\} \cup L : L \in \eta\}$. The basis of the topology $T_m(\Phi)$ is the family $\mathcal{B}_m(\Phi) = \cup\{B_m(x) : x \in e_\Phi E\}$.

Theorem 2.1. *The spaces $(e_\Phi E, T^s(\Phi))$ and $(e_\Phi E, T_m(\Phi))$ are Hausdorff zero-dimensional extensions of the discrete space E, and $T^s(\Phi) \subset T_m(\Phi))$. In particular, $(e_\Phi E, T^s(\Phi)) \leq (e_\Phi E, T_m(\Phi))$.*

Proof. The inclusion $T^s(\Phi) \subset T_m(\Phi))$ follows from the constructions of the topologies $T^s(\Phi)$ and $T_m(\Phi))$. If $L \in \eta \in \Phi$, then $\eta \in clL$. Hence the set E is dense in the spaces $(e_\Phi E, T^s(\Phi))$ and $(e_\Phi E, T_m(\Phi))$. If the families $\eta, \zeta \in \Phi$ are distinct, then there exist $L \in \eta$ and $Z \in \zeta$ such that $L \cap Z = \emptyset$. Then $U_L \cap U_Z = \emptyset$. If $L \subset E$ and $|L| < \tau$, then L is an open-and-closed subset of the spaces $(e_\Phi E, T^s(\Phi))$ and $(e_\Phi E, T_m(\Phi))$. Hence the topologies $T^s(\Phi)$ and $T_m(\Phi)$ are discrete on E and the spaces $(e_\Phi E, T^s(\Phi))$ and $(e_\Phi E, T_m(\Phi))$ are Hausdorff extensions of the discrete space E. Since the sets U_L and $V_{(\eta,L)}$ are open-and-closed in the topologies $T^s(\Phi)$ and $T_m(\Phi))$, respectively, the spaces $(e_\Phi E, T^s(\Phi))$ and $(e_\Phi E, T_m(\Phi))$ are zero-dimensional. ∎

Theorem 2.2. *The spaces $(e_\Phi E, T^s(\Phi))$ and $(e_\Phi E, T_m(\Phi))$ are $P(\tau)$-spaces.*

Proof. Fix $\eta \in \Phi$. If $\zeta \subset \eta$ and $|\zeta| < \tau$, then there exists $L(\zeta) \in \eta$ such that $L(\zeta) \subset \cap\zeta$. From this fact immediately follows that $(e_\Phi E, T_m(\Phi))$ is a $P(\tau)$-space. Assume that $\{L_\mu : \mu \in M\}$ is a family of subsets of E, $|M| < \tau$, $\eta \in \Phi$ and $\eta \in \cap\{L_\mu : \mu \in M\}$. Then there exists $L \in \eta$ such that $L \subset \cap\{L_\mu : \mu \in M\}$. Thus $\eta \in U_L \in \cap\{U_{L_\mu} : \mu \in M\}$. From this fact immediately follows that $(e_\Phi E, T^s(\Phi))$ is a $P(\tau)$-space. ∎

Corollary 2.1. *If $T^s(\Phi) \subset \mathcal{T} \subset T_m(\Phi))$, then $(e_\Phi E, \mathcal{T})$ is a Hausdorff extension of the discrete space E, and $(e_\Phi E, T^s(\Phi)) \leq (e_\Phi E, \mathcal{T}) \leq (e_\Phi E, T_m(\Phi))$.*

Theorem 2.3. *The space $(e_\Omega E, T^s(\Omega))$, where Ω is the set of well-ordered almost disjoint τ-centered families from Proposition 2.1, is a zero-dimensional paracompact space with character $\chi(e_\Omega E, T^s(\Omega)) = \tau$ and weight $\Sigma\{|E|^m : m < \tau\}$.*

Proof. We consider that $E = D$. The family $\mathcal{B} = \{\{x\} : x \in D\} \cup \{V(x, \beta) : x \in E^\tau, \beta \in O(\tau)\}$ is a base of the topology $T^s(\Omega)$. If $U, V \in \mathcal{B}$, then or $U \subset V$, or $V \subset U$, or $U \cap V = \emptyset$. From the A. V. Arhangel'skii theorem [1] it follows that $(e_\Omega E, T^s(\Omega))$ is a zero-dimensional paracompact space. ∎

3. CONSTRUCTION OF HAUSDORFF EXTENSIONS

Let τ be an infinite cardinal. Fix a $P(\tau)$-space X. Let $\gamma = \{H_\mu : \mu \in M\}$ be a discrete family of non-empty open subsets of the space X and $\tau \leq |M|$. For any $\mu \in M$ we fix a point $e_\mu \in U_\mu$ and a family $\xi_\mu = \{H_{(\mu,\alpha)} : \alpha \in O(\tau)\}$ of open subsets of X such that $e_\mu \in \cap\xi_\mu$ and $H_{(\mu,\beta)} \subset H_{(\mu,\alpha)} \subset H_\mu$ for all $\alpha \in O(\tau)$ and $\beta \in O(\alpha)$. Then $E = \{e_\mu : \mu \in M\}$ is a discrete closed subspace of the space X.

Consider the Hausdorff extension rE of the space E. We put $e_{rE}X = X \cup (rE \setminus E)$. In $e_{rE}X$ we construct the topology $\mathcal{T} = T(\gamma, E, \xi_\mu, \tau)$ as follows:
- we consider X as an open subspace of $e_{(E,Y)}X$;
- let T_X be the topology of X and T_{rE} be the topology of the space rE;
- if $V \in T_{rE}$, then $e_\alpha V = V \cup (\cup\{H_{(\mu,\alpha)} : e_\mu \in V\}$;
- $\mathcal{B} = T_X \cup (\cup\{e_\alpha V : V \in T_{rE}\}$ is an open base of the topology $\mathcal{T} = T(\gamma, E, \xi_\mu, \tau)$.

Theorem 3.1. *The space* $(e_{(E,Y)}X, T(\gamma, E, \xi_\mu, \tau))$ *is a Hausdorff extension of the space* X.

Proof. If $V, W \in T_{rE}$, then:
- $e_\alpha W \subset e_\alpha V$ if and only if $W \subset V$;
- $e_\alpha W \cap e_\alpha V = \emptyset$ if and only if $W \cap tV = \emptyset$;
- $e_\alpha V \cap rE = V$.

This facts and Theorem 2.3 completes the proof. ∎

Theorem 3.2. *If* rE *is a* $P(\tau)$*-space, then* $(e_{(E,Y)}X, T(\gamma, E, \xi_\mu, \tau))$ *is a* $P(\tau)$*-space too. Moreover,*

$$\chi(e_{(E,Y)}X, T(\gamma, E, \xi_\mu, \tau)) = \chi(X) + \chi(rE)$$

and

$$w(e_{(E,Y)}X, T(\gamma, E, \xi_\mu, \tau)) = w(X) + w(rE).$$

Proof. Follows immediately from the construction of the sets $e_\alpha V$. ∎

Theorem 3.3. *Assume that the spaces* rE *and* X *are zero-dimensional, and the sets* $H_{(\mu,\alpha)}$ *are open-and-closed in* X. *Then: 1.* $(e_{(E,Y)}X, T(\gamma, E, \xi_\mu, \tau))$ *is a zero-dimensional space.*

2. The space $(e_{(E,Y)}X, T(\gamma, E, \xi_\mu, \tau))$ *is paracompact if and only if the spaces* rE *and* X *are paracompact.*

Proof. If the set V is open-and-closed in rE and the sets $H_{(\mu,\alpha)}$ are open-and-closed in X, then the sets $e_\alpha V$ are open-and-closed in $(e_{(E,Y)}X, T(\gamma, E, \xi_\mu, \tau))$. If $\{V_\lambda : \lambda \in L\}$ is a discrete cover of rE, and $\alpha(\lambda) \in O(\tau)$, then $\{e_{\alpha(\lambda)}V_\lambda : \lambda \in L\}$ is a discrete family of open-and-closed sets. This fact completes the proof. ∎

References

[1] A. V. Arhangel'skii and V. V. Filippov, *Spaces with bases of finite rank*, Matem. Sbornik 87 (1972), no. 2, 147-158 (Russian, English translation: Math. USSR Sbornik, 16 (1972), 147-158.

[2] L. I. Calmutchi, *Hausdorff extensions*, The 26^{th} Conference on Applied and Industrial Matematic, $20^{th} - 23^{th}$ september 2018, Chişinău,(85).

[3] R. Engelking, *General Topology*, PWN, Warszawa, 1977.

ROMAI J., v.17, no.1(2021), 53–63

ON TOPOLOGICAL PARAMEDIAL QUASIGROUPS

Liubomir Chiriac, Natalia Bobeică

Tiraspol State University, Chişinău, Republic of Moldova

llchiriac@gmail.com, nbobeica1978@gmail.com

Abstract This paper studies some properties of paramedial groupoids and topological paramedial quasigroups. We proved some characterizations of the abelian groups as paramedial groupoids. A new method of constructing non-associative paramedial and medial topological quasigroups is given. It is also proved that if $(G, \cdot)$ is a locally compact paramedial quaisigroup, then G is endowed with a unique invariant Haar measure.

Keywords: paramedial groupoids, abelian groups,topological paramedial quasigroups, (n, m)-identity, (n, m)-homogeneous isotope, Haar measure on the quasigroups.
2020 MSC: 22A99.

1. INTRODUCTION

The results of this paper can be summarized as follows. In Section 3 we show some characterizations of the abelian groups as paramedial groupoids, inclusive by using the notion of a center associative element. Thus, we prove that if in paramedial groupoid G there is some center associative element a such that, for every $b \in G$, the equations $ax = b$, $ya = b$ and $bz = a$ soluble in G, then the groupoid G is a commutative group. Section 4 presents some results and constructions which can be used to produce examples of medial and paramedial quasigroups that are not associative. Finally, in Section 5, using the concept of the (n, m)-identities, we show that if $(G, \cdot)$ is a locally compact paramedial quasigroup, then there exists a unique invariant Haar measure on G. In order to facilitate the study of topological quasigroups with (n, m)-identities, we expand on the notions of multiple identities and (n, m)-homogeneous isotopies. The concept of (n, m)-identities was introduced by M.M. Choban and L.L. Chiriac in [1, 2].

We dedicate this paper to the memory of Professor Mitrofan Choban who brought many important contributions to general topology and topological algebra.

2. BASIC NOTIONS

In this section we recall some fundamental definitions and notations.

A non-empty set G is said to be a *groupoid* with respect to a binary operation denoted by $\{\cdot\}$, if for every ordered pair (a, b) of elements of G there is a unique element $ab \in G$.

If the groupoid G is a topological space and the multiplication operation $(a, b) \to a \cdot b$ is continuous, then G is called a *topological groupoid.*

A groupoid G is called a primitive groupoid with divisions, if there exist two binary operation $l : G \times G \to G$, $r : G \times G \to G$ such that $l(a, b) \cdot a = b, a \cdot r(a, b) = b$ for all $a, b \in G$. Thus a primitive groupoid with divisions is a universal algebra with three binary operations.

A primitive groupoid G with divisions is called a quasigroup if the equations $ax = b$ and $ya = b$ have unique solutions. In a quasigroup G the divisions l, r are unique. If the multiplication operation in a quasigroup $(G, \cdot)$ with a topology is continuous, then G is called a semitopoligical quasigroup. If in a semitopological quasigroup G the divisions l and r are continuous, then G is called a topological quasigroup.

An element $e \in G$ is called an *identity* if $ex = xe = x$ every $x \in X$.

A quasigroup with an identity is called a *loop.* A groupoid G is called *medial* if it satisfies the law $xy \cdot zt = xz \cdot yt$ for all $x, y, z, t \in G$.

A groupoid G is called *paramedial* if it satisfies the law $xy \cdot zt = ty \cdot zx$ for all $x, y, z, t \in G$.

A groupoid G is called *bicommutative* if it satisfies the law $xy \cdot zt = tz \cdot yx$ for all $x, y, z, t \in G$.

If a paramedial guasigroup G contains an element e such that $e \cdot x = x (x \cdot e = x)$ for all x in G, then e is called a *left (right) identity* element of G and G is called a *left (right) paramedial loop.*

A groupoid G is called a *groupoid Abel-Grassmann* or *AG-groupoid* if it satisfies the left invertive law $(a \cdot b) \cdot c = (c \cdot b) \cdot a$ for all $a, b, c \in G$.

A groupoid G is called *GA-groupoid* if it satisfies the law $xy \cdot z = z \cdot yx$ for all $x, y, z, t \in G$.

A groupoid G is called *AD-groupoid* if it satisfies the law $x \cdot yz = z \cdot yx$ for all $x, y, z, t \in G$.

Let $\mathbb{N} = \{1, 2, ...\}$ and $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$. We shall use the notations and terminology from [1, 2, 3, 4, 5].

3. CHARACTERIZATION OF THE ABELIAN GROUPS AS PARAMEDIAL GROUPOIDS

In this Section we proved some characterizations of the abelian groups as paramedial groupoids.

Theorem 3.1. *Let G be a paramedial groupoid satisfying the following conditions:*

1 For two any elements $x, y \in G$, there are elements $e, f \in G$ such that $ex = x = xf$ and $ey = y = yf$;

2 For each element $x \in G$ and each element $e \in G$ such that $ex = x$, there is an element x'_e such that $x'_e \cdot x = e$.

Then G is an abelian group.

Proof. Let $x, y \in G$ and let e, f be elements of G satisfying condition (1). By paramedial law and condition (1) we have $xy = xf \cdot ey = yf \cdot ex = yx$. Hence, groupoid G is commutative.

If t denotes any element of G such that $tx = x$ and $tz = z$, then by commutativity and paramedial law we obtain

$$xy \cdot z = xy \cdot tz = zy \cdot tx = zy \cdot x = x \cdot zy = x \cdot yz.$$

Therefore, every paramedial groupoids satisfying condition (1) is associative and commutative.

Now, we show that, under the condition of the theorem, for each element $x \in G$, there is only one local identity, $e \in G$ such that $ex = xe = x$.

Let e and f be elements of G such that $ex = fx = x$. We consider two elements x'_e and x'_f from G such that $x'_e \cdot x = e$ and $x'_f \cdot x = f$. Since G is commutative and associative we have

$$e = x'_e x = x'_e \cdot fx = x'_e f \cdot x = f x'_e \cdot x = f \cdot x'_e x = fe = x'_f x \cdot e = e \cdot x'_f x =$$
$$= e \cdot x x'_f = ex \cdot x'_f = x x'_f = x'_f x = f.$$

From condition (1) and by the fact above $f = e$ we have that $ey = ye = y$ for every $y \in G$. Hence e is an identity element of G.

By conditions (2) and commutativity we obtain that every element of G has an inverse element and G is an abelian group.

Conversely, if G is an abelian group then G is paramedial groupoid satisfying the conditions (1) and (2). Thus an abelian group may be characterized as an paramedial groupoid satisfying these conditions. The proof is complete. ∎

Using the notion of a center associative element we can obtain another characterization of the commutative groups as paramedial groupoids.

Definition 3.1. *An element a of the groupoid G is said to be a center associative element, if and only if $x \cdot ay = xa \cdot y$ for all $x, y \in G$ [6].*

Theorem 3.2. *Let G be a paramedial groupoid. If in G there is some center associative element a such that, for every $b \in G$ the equation $ax = b$ and $ya = b$ are soluble in G, then the groupoid G is a commutative monoid.*

Proof. Examine the equation $ax = a$. Let e be a solution of this equation and $ae = a$ (3).

Let $b, a \in G$ and a is center associative element. For some $y \in G$ we have $b = ya$ (4). Using (3) and (4) we obtain that $be = ya \cdot e = y \cdot ae = ya = b$. Hence, e is a right identity of the groupoid G.

Now, examine the equation $ya = a$. Let e' be a solution of the equation and $e'a = a$ (5).

Let $b, a \in G$ and a is a center associative element. For some $x \in G$ we obtain that $b = ax$ (6).

Using (5) and (6) we have $e'b = e' \cdot ax = e'a \cdot x = ax = b$. Hence, e' is a left identity of G. From $e = e'e = e'$ one concludes that e is an identity element in G.

Since G is a paramedial groupoid, it results $xy = xe \cdot ey = ye \cdot ex = yx$. Therefore G is paramedial commutative groupoid. On the other hand, we have

$$x \cdot yz = xe \cdot yz = ze \cdot yx = z \cdot xy = xy \cdot z.$$

Hence, G is a commutative monoid. The proof is complete. ∎

From Theorem 3.2 it follows the next Proposition.

Proposition 3.1. *If in groupoid G there is some center associative element a such that, for every $b \in G$ the equations $ax = b$ and $ya = b$ are soluble in G, then G has an identity element.*

Theorem 3.3. *Let G be a paramedial groupoid. If in G there is some center associative element a such that, for every $b \in G$, the equations $ax = b$, $ya = b$ and $bz = a$ soluble in G, then the groupoid G is a commutative group.*

Proof. By Theorem 3.2 paramedial groupoid G is a commutative monoid. To prove that G is an abelian group it is suffices to show that for each $b \in G$, there is in G some element b' such that $b'b = e$ or $bb' = e$, where e is the identity element.

Examine the equation $y \cdot ba = a$. Let b' be a solution of the equation and $a = b' \cdot ba = b'e \cdot ba = ae \cdot bb' = a \cdot bb'$.

Now, we examine the equation $ax = e$. If a' is a solution of the equation, then

$$e = aa' = (a \cdot bb') \cdot a' = (a \cdot bb') \cdot (a' \cdot e) = (e \cdot bb') \cdot (a' \cdot a) = (bb') \cdot (aa') = (bb') \cdot e = bb'.$$

Hence $bb' = e$ and G is an abelian group. The proof is complete. ∎

Some Theorems about characterizations of the commutative groups as medial groupoids were proved in [7].

4. ON A METHOD OF CONSTRUCTING PARAMEDIAL AND MEDIAL TOPOLOGICAL QUASIGROUPS

In Section 4 we prove a new method of constructing non-associative paramedial and medial topological quasigroups.

Theorem 4.1. *Let $(G, +, \tau)$ be a commutative topological group. For (x_1, y_1) and (x_2, y_2) in $G \times G$ define*

$$(x_1, y_1) \circ (x_2, y_2) = (x_1 - y_2 - x_2, -x_1 + y_2 - y_1).$$

Then $(G \times G, \circ, \tau_G)$, relative to the product topology τ_G, is a paramedial, non-medial and non-associative topological quasigroup. Moreover, if (G, τ) is $T_i -$ space, then $(G \times G, \tau_G)$ is $T_i -$ space too, where $i = 1, 2, 3, 3.5$.

Proof. We will prove that $(G \times G, \circ)$ is a quasigroup. To this end, we will show that the equations $y \circ a = b$ and $x \circ a = b$ have unique solutions in $(G \times G, \circ)$. Let $y = (y_1, y_2)$, $x = (x_1, x_2)$, $a = (a_1, a_2)$ and $b = (b_1, b_2)$. Since $y \circ a = b$ we have

$$(y_1, y_2) \circ (a_1, a_1) = (b_1, b_2). \tag{7}$$

According to the conditions of the Theorem

$$(y_1, y_2) \circ (a_1, a_2) = (y_1 - a_1 - a_2, a_2 - y_1 - y_2). \tag{8}$$

From (7) and (8) obtain that $y_1 = b_1 + a_1 + a_2$ and $y_2 = -b_1 - b_2 - a_1$.
In this case

$$l((a_1, a_2), (b_1, b_2)) = (b_1 + a_1 + a_2, -b_1 - b_2 - a_1)$$

and $l((a_1, a_2), (b_1, b_2)) \circ (a_1, a_2) = (b_1, b_2)$.

It is easy to show that any other solutions of that equation coincide with y_1 and y_2.

Similarly it is shown that the equation $a \circ x = b$ or

$$(a_1, a_2) \circ (x_1, x_2) = (b_1, b_2) \tag{9}$$

has a unique solution $x_1 = -b_2 - b_1 - a_2$ and $x_2 = a_1 - a_2 - b_1$.
It is clear that

$$r((a_1, a_2), (b_1, b_2)) = (-b_2 - b_1 - a_2, a_1 - a_2 - b_1)$$

and $(a_1, a_2) \circ r((a_1, a_2), (b_1, b_2)) = (b_1, b_2)$.

Any other solution of the equation $a \circ x = b$ coincides with x_1 and x_2. Thus $(G \times G, \circ)$ is a quasigroup.

We now prove that $(G \times G, \circ)$ is a paramedial quasigroup, that is, the property $(x \circ y) \circ (z \circ t) = (t \circ y) \circ (z \circ x)$ holds. Let $x = (x_1, y_1), y = (x_2, y_2), z = (x_3, y_3), t = (x_4, y_4)$ be in $(G \times G, \circ)$, then

$$((x_1,y_1)\circ(x_2,y_2))\circ((x_3,y_3)\circ(x_4,y_4)) = ((x_4,y_4)\circ(x_2,y_2))\circ((x_3,y_3)\circ(x_1,y_1)) \quad (10).$$

Simplifying relation (10) we obtain that both sides are equal to

$$(x_1 + y_3 - x_2 - y_2 + x_4, x_2 + y_4 - x_3 - y_3 + y_1).$$

Therefore, the quasigroup $(G \times G, \circ)$ is paramedial.
Similarly, it is shown that associativity $((x_1, y_1) \circ (x_2, y_2)) \circ (x_3, y_3) = (x_1, y_1) \circ ((x_2, y_2) \circ (x_3, y_3))$ does not hold in $(G \times G, \circ)$. Indeed, $((x_1, y_1) \circ (x_2, y_2)) \circ (x_3, y_3) = (x_1 - y_2 - x_2 - y_3 - x_3, x_2 + y_3 + y_1)$ and $(x_1, y_1) \circ ((x_2, y_2) \circ (x_3, y_3)) = (x_3 + y_2 + x_1, -x_1 - x_2 - y_1 - y_2 + y_3)$. The same applies for mediality.

Multiplication $(\circ)$ and divisions $l(a, b)$ and $r(a, b)$ are jointly continuous relative to the product topology. Consequently, $(G \times G, \circ, \tau_G)$ is a topological paramedial quasigroup.

If (G, τ) is T_i-space, then according to Theorem 2.3.11 in [4], a product of T_i-spaces is a T_i-spaces, where $i = 1, 2, 3, 3.5$. The proof is complete. ∎

Theorem 4.2. *Let $(G, +, \tau)$ be a commutative topological group. For (x_1, y_1) and (x_2, y_2) in $G \times G$ define*

$$(x_1, y_1) \circ (x_2, y_2) = (x_1 - y_1 + x_2 - y_2, y_1 + y_2).$$

Then $(G \times G, \circ, \tau_G)$, relative to the product topology τ_G, is a non-associative, medial, paramedial, bicommutative and GA-topological quasigroup. Moreover, if (G, τ) is $T_i -$space, then $(G \times G, \tau_G)$ is $T_i -$space too, where $i = 1, 2, 3, 3.5$.

Proof. The proof is analogous to that of Theorem 4.1. ∎

Example 4.1. *Let $G = \{0, 1, 2\}$. We define the binary operation " + ".*

(+)	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Then $(G, +)$ is a commutative group. Define a binary operation $(\circ)$ on the set $G \times G$ by $(x_1, y_1) \circ (x_2, y_2) = (x_1 - y_2 - x_2, -x_1 + y_2 - y_1)$, for all $x_1, y_1, x_2, y_2 \in G \times G$ .If we label the elements as follows $(0, 0) \leftrightarrow 0$, $(0, 1) \leftrightarrow 1$,

$(0,2) \leftrightarrow 2$, $(1,0) \leftrightarrow 3$, $(1,1) \leftrightarrow 4$, $(1,2) \leftrightarrow 5$, $(2,0) \leftrightarrow 6$, $(2,1) \leftrightarrow 7$, $(2,2) \leftrightarrow 8$, *then obtain:*

$(\circ)$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
0	*0*	*7*	*5*	*6*	*4*	*2*	*3*	*1*	*8*
1	*2*	*6*	*4*	*8*	*3*	*1*	*5*	*0*	*7*
2	*1*	*8*	*3*	*7*	*5*	*0*	*4*	*2*	*6*
3	*5*	*0*	*7*	*2*	*6*	*4*	*8*	*3*	*1*
4	*4*	*2*	*6*	*1*	*8*	*3*	*7*	*5*	*0*
5	*3*	*1*	*8*	*0*	*7*	*5*	*6*	*4*	*2*
6	*7*	*5*	*0*	*4*	*2*	*6*	*1*	*8*	*3*
7	*6*	*4*	*2*	*3*	*1*	*8*	*0*	*7*	*5*
8	*8*	*3*	*1*	*5*	*0*	*7*	*2*	*6*	*4*

Then $(G \times G, \circ)$ *is a non-associative, non-medial, paramedial quasigroup.*

5. TOPOLOGICAL PARAMEDIAL QUASIGROUPS WITH HAAR MEASURES

In this section, using the concept of the (n,m)-identities, we prove that if $(G,\cdot)$ is a locally compact paramedial quasigroup, then there exists a unique invariant Haar measure on G.

We recall some important definitions and notations.

Consider a groupoid $(G,+)$. For every two elements a,b from $(G,+)$ we denote:

$$1(a,b,+) = (a,b,+)1 = a+b, \text{ and } n(a,b,+) = a + (n-1)(a,b,+),$$
$$(a,b,+)n = (a,b,+)(n-1) + b$$

for all $n \geq 2$.

If a binary operation $(+)$ is given on a set G, then we shall use the symbols $n(a,b)$ and $(a,b)n$ instead of $n(a,b,+)$ and $(a,b,+)n$.

Definition 5.1. *Let* $(G,+)$ *be a groupoid and let* $n,m \geq 1$. *The element* e *of the groupoid* $(G,+)$ *is called:*

- *an* (n,m)*-zero of* G *if* $e+e=e$ *and* $n(e,x) = (x,e)m = x$ *for every* $x \in G$;
- *an* $(n,\propto)$*-zero if* $e+e=e$ *and* $n(e,x) = x$ *for every* $x \in G$;
- *an* $(\propto,m)$*-zero if* $e+e=e$ *and* $(x,e)m = x$ *for every* $x \in G$.

Clearly, if $e \in G$ is both an $(n, \propto)$-zero and an $(\propto, m)$-zero, then it is also an (n, m)-zero. If $(G, \cdot)$ is a multiplicative groupoid, then the element e is called an (n, m)-identity.

Example 5.1. *Let $(G, \cdot)$ be a paramedial groupoid, $e \in G$ and $xe = x$ for every $x \in G$. Then $(G, \cdot)$ is a paramedial groupoid with $(2, 1)$-identity e in G. Indeed, if $x \in G$, then $e \cdot ex = ee \cdot ex = xe \cdot ee = xe \cdot e = x \cdot e = x$.*

Definition 5.2. *Let $(G, +)$ be a topological groupoid. A groupoid $(G, \cdot)$ is called a homogeneous isotope of the topological groupoid $(G, +)$ if there exist two topological automorphisms $\varphi, \psi : (G, +) \to (G, +)$ such that $x \cdot y = \varphi(x) + \psi(y)$, for all $x, y \in G$.*

For every mapping $f : X \to X$ we denote $f^1(x) = f(x)$ and $f^{n+1}(x) = f(f^n(x))$ for any $n \geq 1$.

Definition 5.3. *Let $n, m \leq \infty$. A groupoid $(G, \cdot)$ is called an (n, m)-homogeneous isotope of a topological groupoid $(G, +)$ if there exist two topological automorphisms $\varphi, \psi : (G, +) \to (G, +)$ such that:*
1. $x \cdot y = \varphi(x) + \psi(y)$ for all $x, y \in G$;
2. $\varphi\varphi = \psi\psi$;
3. If $n < \infty$, then $\varphi^n(x) = x$ for all $x \in G$;
4. If $m < \infty$, then $\psi^m(x) = x$ for all $x \in G$.

Definition 5.4. *A groupoid $(G, \cdot)$ is called an isotope of a topological groupoid $(G, +)$, if there exist two homeomorphisms $\varphi, \psi : (G, +) \to (G, +)$ such that $x \cdot y = \varphi(x) + \psi(y)$ for all $x, y \in G$.*

Under the conditions of Definition 5.4 we shall say that the isotope $(G, \cdot)$ is generated by the homeomorphisms φ, ψ of the topological groupoids $(G, +)$ and write $(G, \cdot) = g(G, +, \varphi, \psi)$.

Example 5.2. *Denote by $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} = \{0, 1, ..., p-1\}$ the cyclic group of order p. Let $(G, +) = (\mathbb{Z}_7, +)$, $\varphi(x) = 6x$, $\psi(x) = x$ and $x \cdot y = 6x + y$. Then $(G, \cdot) = g(G, +, \varphi, \psi)$ is a medial, paramedial, bicommutative and AG-quasigroup and the zero of $(G, +)$ is a $(1, 2)$-identity in $(G, \cdot)$.*

Example 5.3. *Denote by $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} = \{0, 1, ..., p-1\}$ the cyclic group of order p. Let $(G, +) = (\mathbb{Z}_{11}, +)$, $\varphi(x) = x$, $\psi(x) = 10x$ and $x \cdot y = x + 10y$. Then $(G, \cdot) = g(G, +, \varphi, \psi)$ is a medial, paramedial, bicommutative and AD-quasigroup and the zero of $(G, +)$ is a $(2, 1)$-identity in $(G, \cdot)$.*

Example 5.4. *Denote by $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} = \{0, 1, ..., p-1\}$ the cyclic group of order p. Let $(G, +) = (\mathbb{Z}_{11}, +)$, $\varphi(x) = 10x$, $\psi(x) = x$ and $x \cdot y = 10x + y$. Then $(G, \cdot) = g(G, +, \varphi, \psi)$ is a medial, paramedial, bicommutative and AG-quasigroup and the zero of $(G, +)$ is a $(1, 2)$-identity in $(G, \cdot)$.*

Example 5.5. *Let* $(G,+) = (\mathbb{Z}_{11},+)$*,* $\varphi(x) = 9x$*,* $\psi(x) = 2x$ *and* $x \cdot y = 9x+2y$*. Then* $(G,\cdot) = g(G,+,\varphi,\psi)$ *is a medial, paramedial and bicommutative quasigroup and the zero of* $(G,+)$ *is a* $(10,5)$*-identity in* $(G,\cdot)$*.*

Let $(G,\cdot)$ be a topological medial or paramedial quasigroup. A theorem of Toyoda [8, 9] asserts that there exist a binary operation $(+)$ on G, two elements $0, a \in G$ and two topological automorphisms $\varphi, \psi : (G,+) \to (G,+)$, such that

$$x \cdot y = \varphi(x) + \psi(y) + a$$

for all $x, y \in G$ such that $(G,+)$ is a topological commutative group, 0 is the zero of $(G,+)$, and $(G,\cdot) = g(G,+,\varphi,\psi,0,a)$ is a homogeneous isotope of $(G,+)$. Moreover, $\varphi\varphi = \psi\psi$ for paramedial quasigroups and $\varphi\psi = \psi\varphi$ for medial quasigroups.

By $B(X)$ denote the family of Borel subsets of the space X. A non-negative real-valued function μ defined on the family $B(X)$ of Borel subsets of a space X is said to be a Radon measure on X if it has the following properties:

- $\mu(H) = \sup\{\mu(F) : F \subseteq H,\ F$ is a compact subset of $H\}$ for every $H \in B(X)$;

- for every point $x \in X$ there exists an open subset V_x such that $x \in V_x$ and $\mu(V_x) < \infty$.

Definition 5.5. *Let* $(A,\cdot)$ *be a topological quasigroup with the divisions* r, l*. A Radon measure* μ *on* A *is called:*

- *a* **left invariant Haar measure** *if* $\mu(U) > 0$ *and* $\mu(xH) = \mu(H)$ *for every non-empty open set* $U \subseteq A$*, a point* $x \in A$ *and a Borel set* $H \in B(A)$*;*

- *a* **right invariant Haar measure** *if* $\mu(U) > 0$ *and* $\mu(Hx) = \mu(H)$ *for every non-empty open set* $U \subseteq A$*, a point* $x \in A$ *and a Borel set* $H \in B(A)$*;*

- *an* **invariant Haar measure***, if* $\mu(U) > 0$ *and* $\mu(xH) = \mu(Hx) = \mu(l(x,H)) = \mu(r(H,x)) = \mu(H)$ *for every non-empty open set* $U \subseteq A$*, a point* $x \in A$ *and a Borel set* $H \in B(A)$*.*

Definition 5.6. *We say that on a topological quasigroup* $(A,\cdot)$ ***there exists a unique left(right) invariant Haar measure****, if for every two left (right) unvariant Haar measures* μ_1*,* μ_2 *on* A *there exists a constant* $c > 0$ *such that* $\mu_2(H) = c \cdot \mu_1(H)$ *for every Borel set* $H \in B(A)$*.*

If $(G,+)$ is a locally compact commutative group, then on G there exists a unique invariant Haar measure μ_G [5]. We will follow closely the proof scheme from [1] to prove the next Theorems.

We consider on the abelian topological group $(G,+)$ the invariant measure μ_G.

Theorem 5.1. *Let* $(G,\cdot)$ *be a locally compact paramedial quasigroup. Then:*

1. There is a commutative topological group $(G, +)$ and $\varphi, \psi : G \to G$ continuous automorphism of $(G, +)$, $a \in G$, $\varphi^2 = \psi^2$ and $(G, \cdot) = g(G, +, \varphi, \psi, 0, a)$;

2. If on the Abelian topological group $(G, +)$ consider the invariant Haar measure μ_G, then on $(G, \cdot)$ the right (left) invariant Haar measure is unique;

3. If μ is a left (right) measure on $(G, \cdot)$, then μ is a left (right) invariant Haar measure on $(G, +)$ too;

4. On $(G, \cdot)$ there exists some left (right) invariant Haar measure if and only if $\mu_G(\psi(H)) = \mu_G(H)$ $(\mu_G(\varphi(H)) = \mu_G(H))$ for every $H \in B(A)$;

5. If $n < +\infty$, and on G there exists some $(n, +\infty)$-identity, then on $(G, \cdot)$ the measure μ_G is a unique right invariant Haar measure;

6. If $m < +\infty$, and on G there exists some $(+\infty, m)$-identity, then on $(G, \cdot)$ the measure μ_G is a unique left invariant Haar measure;

7. If $n, m < +\infty$, and on G there exists some (n, m)-identity, then on $(G, \cdot)$ the measure μ_G is a unique invariant Haar measure.

Proof. The assertion 1 follows from Toyoda's Theorem for paramedial topological quasigroups [8, 9]. Let μ be a left invariant Haar measure on $(G, \cdot)$. As $x \cdot y = \varphi(x) + \psi(y) + a$ for all $x, y \in G$ follows that $xH = \varphi(H) + \psi(H) + a$. Therefore μ is an invariant Haar measure on Abelian topological group $(G, +)$. Since on the loccally compact commutative group $(G, +)$ we have the invariant measure μ_G there exist a constant $c > 0$ such that $\mu(H) = c \cdot \mu_G(H)$. Hence μ_G is a left invariant Haar measure on $(G, \cdot)$. The assertion 2, 3 and 4 are proved.
Consider some topological automorhism $r : (G, +) \to (G, +)$. Hence $\mu_r(H) = \mu_G(r(H))$ is an invariant Haar measure on $(G, +)$. Then there exists a constant $c_r > 0$ such that $\mu_r(H) = \mu_G(r(H)) = c_r \cdot \mu_G(H)$ for every Borel subset $H \in B(G)$. In particular, $\mu_G(r^k(H)) = c_r^k \cdot \mu_G(H)$ for every $k \in N$. If $m < +\infty$ and 0 is an $(+\infty, m)$-identity, then $\psi^n = x$ for every $x \in G$ and $c_\psi^n = 1$. Thus $c_\psi = 1$, $\mu_G(H) = \mu_G(r(H))$ and μ_G is a left invariant measure on $(G, \cdot)$ Similarly we can prove that μ_G is a right invariant measure on $(G, \cdot)$. The assertions 5, 6 and 7 are proved. The proof is complete. ∎

In this way we can prove the following result.

Theorem 5.2. *On a compact paramedial quasigroup G there exists a unique Haar measure μ for which $\mu(G) = 1$.*

Theorem 5.1 and 5.2 for topological medial quasigroups were proved in [1]. Other properties of topological paramedial quaisgroups were proved in [10, 11, 12].

Example 5.6. *Let $(\mathbb{R}, +)$ be the topological Abelian group of real numbers.*

1. If $\varphi(x) = x$, $\psi(x) = -x$ and $x \cdot y = x - y$, then $(\mathbb{R}, \cdot) = g(\mathbb{R}, +, \varphi, \psi)$ is a commutative locally compact paramedial quasigroup. By Theorem 5.1 there exists a left and a right invariant Haar measure on $(\mathbb{R}, \cdot)$.

2. If $\varphi(x) = x$, $\psi(x) = 7x$ and $x \cdot y = x + 7y$, then $(\mathbb{R}, \cdot) = g(\mathbb{R}, +, \varphi, \psi)$ is a commutative locally compact medial quasigroup. By Theorem 7 from [1] there exists a left (but no right) invariant Haar measure on $(\mathbb{R}, \cdot)$.

3. If $\varphi(x) = 3x$, $\psi(x) = 3x$ and $x \cdot y = 3x + 3y$, then $(\mathbb{R}, \cdot) = g(\mathbb{R}, +, \varphi, \psi)$ is a commutative locally compact paramedial quasigroup and on $(\mathbb{R}, \cdot)$. As above, by Theorem 5.1, there does not exist any left or right invariant Haar measure.

Example 5.7. *Consider the commutative group $(G, +) = (\mathbb{Z}, +)$, $\varphi(x) = x$, $\psi(x) = x - 1$ and $x \cdot y = x + y - 1$. Then $(G, \cdot) = g(G, +, \varphi, \psi, 0, 1)$ is a medial and paramedial quasigroup and $(G, \cdot)$ does not contain (n, m)-identities. There exists an invariant Haar measure on $(G, \cdot)$.*

References

[1] M. M. Choban, L. L. Kiriyak, *The topological quasigroups with multiple identities.* Quasigroups and Related Systems, **9**, (2002), p.19-31.

[2] M. M. Choban, L. L. Kiriyak, *The Medial Topological Quasigroup with Multiple identities*, The 4 th Conference on Applied and Industrial Mathematics, Oradea-CAIM, 1995, p.11.

[3] V. D. Belousov, *Foundation of the theory of quasigroups and loops.* Moscow, Nauka, 1967.

[4] R. Engelking, *General topology.* Polish Scientific Publishers, Warszawa, 1977.

[5] E. Hewitt, K. A. Ross, *Abstract harmonic analysis. Vol. 1: Structure of topological groups. Integration theory. Group representation.* Springer-Verlag. Berlin-Gottingen-Heidelberg, 1963.

[6] G.N. Garrison, *Quasigroups,* Ann. of Math., 41 (1940), p. 474-487.

[7] Jose Morgedo, *Entropic groupoids and abelian groups,* Gazeta de Matematica, ANO XXVII, N 103-104, Julho-Dez, 1966, p. 8-10.

[8] K. Toyoda, *On axioms of linear functions.* In: Proc.Imp.Acad. Tokyo, 1941, 17, p.221-227.

[9] M. Polonijo, *On medial-like identities.* Quasigroup and Related System, 2005, no13, p.281-288

[10] N. Bobeica, L. Chiriac, *On topological AG-groupoids and paramedial quasigroups wiht multiple identities.* Romai Journal, vol.6, nr.1, 2010, p.5-14.

[11] L. L. Chiriac, L. Chiriac Jr, N. Bobeica, *On topological groupoids and multiple identities*, Buletinul Academiei de Ştiinţe a RM, Matematica, **1(59)**, 2009, p.67-78.

[12] L. L. Chiriac, *Topological Algebraic System*, Chisinau, Stiinta, 2009.

ROMAI J., v.17, no.1(2021), 65–86

INTEGRABILITY OF A CUBIC SYSTEM WITH AN INVARIANT STRAIGHT LINE AND AN INVARIANT CUBIC

Dumitru Cozma, Angela Matei

Tiraspol State University, Chişinău, Republic of Moldova

dcozma@gmail.com, angela.matei@tsu.md

Abstract For a cubic differential system with an invariant straight line and an invariant cubic curve we find conditions for the origin to be a center by constructing Darboux integrating factors.

Keywords: cubic differential system, center-focus problem, invariant algebraic curve, Darboux integrability.

2020 MSC: 34C05.

1. INTRODUCTION

We consider the cubic system of differential equations

$$\begin{aligned} \dot{x} &= y + p_2(x,y) + p_3(x,y) \equiv P(x,y), \\ \dot{y} &= -x + q_2(x,y) + q_3(x,y) \equiv Q(x,y), \end{aligned} \tag{1}$$

where $p_j(x,y)$, $q_j(x,y)$ are real homogeneous polynomials of degree j and $P(x,y)$, $Q(x,y)$ are coprime polynomials. The origin $O(0,0)$ is a singular point for (1) with purely imaginary eigenvalues, i.e. a focus or a center.

Although the problem of the center dates from the end of the 19th century, it is completely solved only for: quadratic systems $\dot{x} = y + p_2(x,y)$, $\dot{y} = -x + q_2(x,y)$; cubic symmetric systems $\dot{x} = y + p_3(x,y)$, $\dot{y} = -x + q_3(x,y)$; Kukles system $\dot{x} = y$, $\dot{y} = -x + q_2(x,y) + q_3(x,y)$ and a few particular cases in families of polynomial systems of higher degree.

If the cubic system (1) contains both quadratic and cubic nonlinearities, then the problem of finding a finite number of necessary and sufficient conditions for the center is still open. An approach to the problem of the center for cubic system (1) is to study the local integrability of the system in some neighborhood of the singular point $O(0,0)$.

The integrability conditions for some families of cubic differential systems having invariant algebraic curves were found in [4], [9], [8], [16], [20], [22].

Darboux integrability conditions for a cubic differential system (1) having at least two parallel invariant straight lines were obtained in [5], for some reversible cubic differential systems in [2] and for a few families of the complex cubic system in [15].

The goal of this paper is to obtain the center conditions for a cubic differential system (1) with two algebraic solutions by using the method of Darboux integrability. The paper is organized as follows. In Section 2 we present the known results concerning relation between invariant algebraic curves and Darboux integrability. In Section 3 we find twenty eight sets of conditions for the existence of one invariant straight line and one irreducible invariant cubic curve. In Section 4 we determine the integrability conditions for cubic differential system (1) with one invariant straight line and one invariant cubic by constructing Darboux integrating factors. Finally we obtain twenty sets of conditions for a singular point $O(0,0)$ to be a center.

2. INVARIANT ALGEBRAIC CURVES AND INTEGRATING FACTORS

We study the problem of integrability for cubic system (1) assuming that the system has irreducible invariant algebraic curves.

Definition 2.1. *An algebraic curve* $\Phi(x,y)=0$ *in* $\mathbb{C}^2$ *with* $\Phi \in \mathbb{C}[x,y]$ *is said to be an invariant algebraic curve of system (1) if*

$$\frac{\partial \Phi}{\partial x}P(x,y)+\frac{\partial \Phi}{\partial y}Q(x,y)=\Phi(x,y)K(x,y), \tag{2}$$

for some polynomial $K(x,y)\in\mathbb{C}[x,y]$ *called the cofactor of the invariant algebraic curve* $\Phi(x,y)=0$.

It is a very hard problem to calculate the invariant algebraic curves for a given differential system because, in general, we do not have any evidence on the number of invariant algebraic curves and on the degree of a curve.

The problem of the existence for cubic system (1) of invariant algebraic curves was studied when the curves are: straight lines [13]; straight lines and conics [7],[8]; straight lines and cubic curves [10], [11]; cubic curves [12].

We are interested in the algebraic integrability of cubic differential system (1), called *the Darboux integrability* [3]. It consists in constructing of a first integral or an integrating factor of the form

$$\Phi_1^{\alpha_1}\Phi_2^{\alpha_2}\cdots\Phi_q^{\alpha_q}, \tag{3}$$

where $\Phi_j=0$, $j=\overline{1,q}$ are invariant algebraic curves of (1) and $\Phi_j\in\mathbb{C}[x,y]$, $\alpha_j\in\mathbb{C}$.

Definition 2.2. *An integrating factor for system (1) on some open set* U *of* $\mathbb{R}^2$ *is a* C^1 *function* μ *defined on* U*, not identically zero on* U *such that*

$$P(x,y)\frac{\partial \mu}{\partial x}+Q(x,y)\frac{\partial \mu}{\partial y}+\mu\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)\equiv 0. \tag{4}$$

Conditions for the existence of an integrating factor of the form $\mu = \Phi^\beta$ for system (1), where $\Phi = 0$ is an invariant cubic were obtained in [12]. In this paper we find the conditions under which the cubic differential system (1) has a Darboux integrating factor of the form

$$\mu = l_1^{\alpha_1} \Phi^{\alpha_2} \tag{5}$$

composed of one invariant straight line $l_1 \equiv 1 + a_1 x + b_1 y = 0$ and one invariant cubic $\Phi(x, y) \equiv x^2 + y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 = 0$.

It is known from Poincaré and Lyapunov [1], [18], [19] that a singular point $O(0, 0)$ is a center for (1) if and only if the system has a nonconstant analytic first integral $F(x, y) = C$ in the neighborhood of $O(0, 0)$ or an analytic integrating factor of the form

$$\mu(x, y) = 1 + \sum_{k=1}^{\infty} \mu_k(x, y),$$

where F_k and μ_k are homogeneous polynomials of degree k.

The conditions for a singular point $O(0, 0)$ of a center or a focus type to be a center in a cubic differential system (1) with two distinct invariant straight lines were obtained in [9] and with two parallel invariant straight lines were determined in [20]. The problem of the center was solved for system (1) with: four invariant straight lines [14], [17]; three invariant straight lines [8], [21]; two invariant straight lines and one irreducible invariant conic [6], [8]; two invariant straight lines and one irreducible invariant cubic [10]. The presence of a center in these papers was proved by using the method of Darboux integrability and the rational reversibility.

3. CUBIC SYSTEMS WITH TWO ALGEBRAIC SOLUTIONS

We consider the cubic differential system (1) in the form

$$\begin{aligned} \dot{x} &= y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x, y), \\ \dot{y} &= -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x, y), \end{aligned} \tag{6}$$

where $P(x, y), Q(x, y)$ are coprime polynomials in $\mathbb{R}[x, y]$. The origin $O(0, 0)$ is a singular point which is a center or a focus (a fine focus) for (6).

Let the cubic system (6) have a real invariant straight line $1 + a_1x + b_1y = 0$, $(a_1, b_1) \neq 0$. Then by rotating the system of coordinates ($x \to x\cos\varphi - y\sin\varphi, y \to x\sin\varphi + y\cos\varphi$) and rescaling the axes of coordinates ($x \to \alpha x,\ y \to \alpha y$), we can make the line to be $1 - x = 0$.

Lemma 3.1. *The cubic system* (6) *has an invariant straight line* $1 - x = 0$ *if and only if the following set of conditions holds*

$$k = -a,\ m = -c - 1,\ p = -f,\ r = 0. \tag{7}$$

Proof. By Definition 2.1, a straight line $1 - x = 0$ is an invariant line for system (6) if there exist numbers $c_{20}, c_{11}, c_{02}, c_{10}, c_{01} \in \mathbb{R}$ such that

$$P(x,y) \equiv (x-1)(c_{20}x^2 + c_{11}xy + c_{02}y^2 + c_{10}x + c_{01}y). \tag{8}$$

Identifying the coefficients of the monomials $x^i y^j$ in (8), we find that $c_{10} = 0$, $c_{01} = -1$, $c_{20} = -a$, $c_{11} = -c - 1$, $c_{02} = -f$ and $k = -a$, $m = -c - 1$, $p = -f$, $r = 0$. We obtain the conditions (7) and the straight line $1 - x = 0$ has the cofactor $K(x,y) = -(ax^2 + cxy + fy^2 + xy + y)$. ∎

Suppose the set of conditions (7) is fulfilled. We shall find the conditions on the coefficients of system (6) under which the system has one real irreducible invariant cubic curve of the form

$$\Phi(x,y) \equiv x^2 + y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 = 0, \tag{9}$$

where $a_{ij} \in \mathbb{R}$ and $(a_{30}, a_{21}, a_{12}, a_{03}) \neq 0$.

Theorem 3.1. *The cubic system* (6) *has the straight line* $1 - x = 0$ *and the cubic curve* (9) *as invariant curves if and only if one of the following sets of conditions holds:*

(c_1) $d = 2a$, $k = -a$, $l = [f(2b - c - 1)]/3$, $m = -c - 1$, $n = [(2b - c - 2)(c + 1)]/2$, $p = -f$, $q = (2b - c - 3)a$, $r = 0$, $s = [(2g + 3 + c - 2b)(2b - c - 4)]/6$;

(c_2) $c = [\sqrt{3}(5f - 3a) - 12]/6$, $d = f - a$, $g = [6b - \sqrt{3}(a + f) - 12]/6$, $k = -a$, $l = [4(3bf + f + a - ab) + \sqrt{3}(a + f)(a - 3f)]/16$, $m = [\sqrt{3}(3a - 5f) + 6]/6$, $n = [4\sqrt{3}(4f + 9bf - 3ab) + 9a^2 - 18af - 48b - 27f^2]/48$, $p = -f$, $q = [(a - 3f)(4 - 4b + \sqrt{3}(a + f)) + 16a]/16$, $r = 0$, $s = [4\sqrt{3}(6f + 3bf + 10a - ab) + 3a^2 - 6af - 144b - 9f^2 + 144]/144$;

(c_3) $c = [\sqrt{3}(3a - 5f) - 12]/6$, $d = f - a$, $g = [6b + \sqrt{3}(a + f) - 12]/6$, $k = -a$, $l = [4(3bf + f + a - ab) - \sqrt{3}(a + f)(a - 3f)]/16$, $m = [\sqrt{3}(5f - 3a) + 6]/6$, $n = [9a^2 - 18af - 48b - 27f^2 + 4\sqrt{3}(3ab - 9bf - 4f)]/48$, $p = -f$, $q = [4(3bf - 3f - ab + 5a) + \sqrt{3}(a + f)(3f - a)]/16$, $r = 0$, $s = [4\sqrt{3}(ab - 6f - 3bf - 10a) + 3a^2 - 6af - 144b - 9f^2 + 144]/144$;

(c_4) $a = [16(a_{12}+1)^2 - 3u^2 + 24fu]/(8u)$, $d = 2a - 8f + u$, $c = [2(12u - 4fa_{12} - 4f)]/u$, $g = (16a_{12}^3 + 48a_{12}^2 - 16fua_{12} + a_{12}u^2 + 48a_{12} + 2bu^2 - 16fu - 3u^2 + 16)/(2u^2)$, $k = -a$, $l = (bu - 12fa_{12} + a_{12}u)/12$, $m = -c - 1$, $n = (a_{12}^2 u - 12fa_{12}^2 + a_{12}bu - 20fa_{12} + 2a_{12}u - 8f + u)/u$, $p = -f$, $q = [32a_{12}^3(u - 12f) + 16a_{12}^2(2bu - 72f + 5u) + 64a_{12}(bu - 18f + u) + 32bu + 24fu^2 - 384f - 3u^3 + 16u]/(8u^2)$, $r = 0$, $s = [(16a_{12}^3 + 48a_{12}^2 + 48a_{12} - 3u^2 + 16)(bu - 12fa_{12} + a_{12}u - 12f)]/(3u^3)$;

(c_5) $a = [16a_{12}^2 - 3u^2 + 24fu]/(8u)$, $d = 2a - 8f + u$, $c = (2ua_{12} - 8fa_{12} - u)/u$, $g = (16a_{12}^3 - 16fua_{12} + a_{12}u^2 + 2bu^2 - 2u^2)/(2u^2)$, $k = -a$, $l = (bu - 12fa_{12} + a_{12}u)/12$, $m = -c - 1$, $n = [a_{12}(a_{12}u - 12fa_{12} + bu - 12f + u)]/u$, $p = -f$, $q = (16a_{12}^3u - 192fa_{12}^3 + 16bua_{12}^2 - 16ua_{12}^2 + 8fu^2 - u^3)/(4u^2)$, $r = 0$, $s = [a_{12}(3u^3 - 384fa_{12}^3 + 32a_{12}^3u + 32bua_{12}^2 - 48ua_{12}^2 - 24fu^2)]/(6u^3)$;

(c_6) $a = 3f$, $d = (1 - 18fa_{03})/(9a_{03})$, $g = b + c$, $k = -3f$, $l = (2ba_{03} - a_{03} + f)/2$, $m = -c - 1$, $n = (f - 6ba_{03} + 6ca_{03} + 9a_{03})/(12a_{03})$, $p = -f$, $q = (18ba_{03} + 24ca_{03} + 15a_{03} + 5f)/2$, $r = 0$, $s = (f - 6ba_{03} - 6ca_{03} - 3a_{03})/(12a_{03})$, $108a_{03}^2 - 1 = 0$;

(c_7) $d = (ft - at - 4)/t$, $g = b + c$, $k = -a$, $l = (36f - t(2b - 1))/72$, $m = -c - 1$, $n = (4a - 6bt + 6ct - 48f + 9t)/(12t)$, $p = -f$, $q = (24a - 6bt - 8ct - 12f - 5t)/24$, $r = 0$, $s = -(4a + 2bt + 2ct + t)/(4t)$, $t^2 = 12$;

(c_8) $c = [-(18(at^2 + 36f)(t^2 + 4)^2 + t(t^4 + 168t^2 + 144)(t^2 + 12))]/[108t(t^2 + 4)^2]$, $d = [18(f - a)(t^2 + 4)^2 - t(3t^4 + 56t^2 + 432)]/[18(t^2 + 4)^2]$, $g = [36(6(bt - 6f) - at^2)(t^2 + 4)^2 - t(5t^4 + 264t^2 + 720)(t^2 + 12)]/[216t(t^2 + 4)^2]$, $k = -a$, $l = [-4(2(9(2b - 1)(t^2 + 4)^2 + (t^2 - 12)^2)t - 27(t^2 + 12)(t^2 + 4)^2 f)t^2]/[243(t^2 + 4)^4]$, $m = -c - 1$, $n = [-72at^4(t^2 + 4)^3 - 432bt^3(t^2 + 12)(t^2 + 4)^2 - 27f(7t^6 + 220t^4 + 1296t^2 + 5184)(t^2 + 4)^2 - 4t^3(t^8 + 40t^6 + 1056t^4 + 5760t^2 + 20736)]/[972t(t^2 + 4)^4]$, $p = -f$, $q = [((7t^4 + 184t^2 + 1008)(t^4 + 72t^2 + 144) - 432(t^2 + 12)(t^2 + 4)^2 b)(t^2 + 12)t + 18(7t^2 + 36)(t^2 + 36)(t^2 + 4)^3 a - 54(t^4 - 56t^2 - 624)(t^2 + 12)(t^2 + 4)^2 f)/[3888(t^2 + 4)^4]$, $r = 0$, $s = [((((t^4 + 72t^2 + 144)(t^2 + 12)^3 + 18(t^2 + 4)^3(t^2 - 12)at)t - 72(bt - 6f)(t^2 + 12)^2(t^2 + 4)^2)(t^2 + 12)]/[7776t(t^2 + 4)^4]$;

(c_9) $a = [27c(v^2 + 1)^2 - v^6 + 36v^4 + 63v^2 + 54]/[9(v^2 + 1)^2 v]$, $d = [-(27c(v^2 + 1)^2 + 2v^6 + 54v^4 + 90v^2 + 54)]/[9(v^2 + 1)^2 v]$, $f = (-4v^3)/[9(v^2 + 1)^2]$, $g = [18(b + c)(v^2 + 1)^2 + (v^2 + 3)(v^2 - 3)^2]/[18(v^2 + 1)^2]$, $k = -a$, $l = [-4v^3(18b(v^2 + 1)^2 + 5v^4 + 6v^2 + 9)]/[243(v^2 + 1)^2]$, $m = -c - 1$, $n = [9(v^2 + 1)^2((v^2 + 1)(5v^2 + 9)c - b(5v^4 + 6v^2 + 9)) + 2(8v^6 + 31v^4 + 42v^2 + 27)(5v^2 + 3)]/[81(v^2 + 1)^4]$, $p = -f$, $q = [36bv^2(v^2 + 3)^2(v^2 + 1)^2 + 9c(7v^4 - 36v^2 - 27)(v^2 + 1)^3 + 72v^{10} - 454v^8 - 2064v^6 - 3060v^4 - 1944v^2 - 486)]/[162v(v^2 + 1)^4]$, $r = 0$, $s = [(18bv^2(v^2 + 1)^2 + 27c(v^2 + 1)^3 + 2(10v^6 + 33v^4 + 54v^2 + 27))(v^4 - 18v^2 - 27)]/[486(v^2 + 1)^4]$;

(c_{10}) $a = [108(ct - 6f)(t^2 + 4)^2 - t^7 + 144t^5 + 432t^3 + 3456t]/[18(t^2 + 4)^2 t^2]$, $d = [9((t^2 + 36)f - 6ct)(t^2 + 4)^2 - t^7 - 100t^5 - 432t^3 - 1728t]/[9(t^2 + 4)^2 t^2]$, $g = [72(b + c)(t^2 + 4)^2 + (t^2 + 12)(t^2 - 12)^2]/[72(t^2 + 4)^2]$, $k = -a$, $l = [16(5t^4 + 24t^2 + 144)t^3 - 9(16bt^3 - 15ft^4 - 72ft^2 - 432f)(t^2 + 4)^2]/[243(t^2 + 4)^4]$, $m = -c - 1$, $n = [(36ct(5t^2 + 36)(t^2 + 4) - 36bt(5t^4 + 24t^2 + 144) - 9f(7t^4 + 40t^2 + 432)(t^2 + 12))(t^2 + 4)^2 + 8t(33t^8 + 592t^6 + 3936t^4 + 11520t^2 +$

$20736)]/[324t(t^2+4)^4]$, $p=-f$, $q=[9f(t^4+40t^2+48)(t^4-72t^2-432)(t^2+4)^2-72b(t^2+12)^2(t^2+4)^2t^3-18ct(7t^4-144t^2-432)(t^2+4)^3-8t(17t^{10}-422t^8-4992t^6-28224t^4-103680t^2-124416)]/[648t^2(t^2+4)^4]$, $r=0$, $s=[(18bt^3(t^2+4)^2+27ct(t^2+4)^3-54f(t^2+12)(t^2+4)^2+4t(5t^4+24t^2+144)(t^2+6))(t^4-72t^2-432)]/[1944t(t^2+4)^4]$;

(c_{11}) $a=[(8ft-3t^4-2t^2w^2-4t^2w+w^4+4w^3+4w^2)(3t^2+3w^2+6w+4)]/[8t(t^2+w^2+2w)]$, $c=[(4f-3t^3)(w+1)-t(3w^3+9w^2+7w+4)]/(2t)$, $d=[3t^6+3t^4(3w^2+6w+2)-8ft^3+t^2w(9w^3+36w^2+46w+20)-8tf(w^2+2w+2)+w^2(w+2)^2(3w^2+6w+4)]/[4t(t^2+w^2+2w)]$, $g=[-3t^4(w+1)+2t^2(4b-3w^3-9w^2-2w-2)+16tf(w+1)-3w^2(w^3+5w^2+8w+4)]/(8t^2)$, $k=-a$, $l=[(4b-3t^2w-3t^2-3w^3-9w^2-8w-4)(t^2+w^2+2w)t+4f(3t^2w+3t^2+3w^3+9w^2+8w+4)]/16$, $m=-c-1$, $n=[(9t^4w^2+18t^4w+6t^4+18t^2w^4+72t^2w^3+96t^2w^2+42t^2w+9w^6+54w^5+126w^4+138w^3+64w^2+8w)(t^2+w^2+2w)t-4f((2(9w^3+27w^2+22w+2)(w+1)+(9w^2+18w+7)t^2)t^2+(9w^4+36w^3+55w^2+34w+8)(w+2)w)-4b(3t^2w+3t^2+3w^3+9w^2+8w+4)(t^2+w^2+2w)t]/[16t(t^2+w^2+2w)]$, $p=-f$, $q=[4b(3t^2+3w^2+6w+4)(t^2+w^2+2w)(w+2)tw+4f((9w^3+27w^2+14w+6)t^4+(9w^2+18w+14)(w+2)^2(w+1)w^2+2(9w^5+45w^4+77w^3+56w^2+16w+4)t^2)-(9t^4w^3+27t^4w^2+12t^4w+9t^4+18t^2w^5+90t^2w^4+150t^2w^3+114t^2w^2+48t^2w+12t^2+9w^7+63w^6+174w^5+249w^4+200w^3+84w^2+16w)(t^2+w^2+2w)t]/[16t^2(t^2+w^2+2w)]$, $r=0$, $s=[((3t^2w^2+3t^2w-3t^2+3w^4+9w^3+5w^2+4w-4bw)(t^2+w^2+2w)t-4(3t^2w^2+3t^2w-2t^2+3w^4+9w^3+6w^2)f)(t^2w+3t^2+w^3+5w^2+8w+4)w]/[16(t^2+w^2+2w)t^3]$;

(c_{12}) $a=[(8ft+3t^4+2t^2w^2+4t^2w-w^4-4w^3-4w^2)(3t^2+3w^2+6w+4)]/[8t(t^2+w^2+2w)]$, $c=-[(4f+3t^3)(w+1)+t(3w^3+9w^2+7w+4)]/(2t)$, $d=[-3t^6-3t^4(3w^2+6w+2)-8ft^3-t^2w(9w^3+36w^2+46w+20)-8tf(w^2+2w+2)-w^2(w+2)^2(3w^2+6w+4)]/[4t(t^2+w^2+2w)]$, $g=[-3t^4(w+1)+2t^2(4b-3w^3-9w^2-2w-2)-16tf(w+1)-3w^2(w^3+5w^2+8w+4)]/(8t^2)$, $k=-a$, $l=[4((3w^2+3w+2)(w+2)+3(w+1)t^2)f-t(4b-3t^2w-3t^2-3w^3-9w^2-8w-4)(t^2+w^2+2w)]/16$, $m=-c-1$, $n=[(9t^4w^2+18t^4w+6t^4+18t^2w^4+72t^2w^3+96t^2w^2+42t^2w+9w^6+54w^5+126w^4+138w^3+64w^2+8w)(t^2+w^2+2w)t+4f((2(9w^3+27w^2+22w+2)(w+1)+(9w^2+18w+7)t^2)t^2+(9w^4+36w^3+55w^2+34w+8)(w+2)w)-4bt(3t^2w+3t^2+3w^3+9w^2+8w+4)(t^2+w^2+2w]/[16t(t^2+w^2+2w)]$, $p=-f$, $q=[-4b(3t^2+3w^2+6w+4)(t^2+w^2+2w)(w+2)tw+4f((9w^3+27w^2+14w+6)t^4+(9w^2+18w+14)(w+2)^2(w+1)w^2+2(9w^5+45w^4+77w^3+56w^2+16w+4)t^2)+t(9t^4w^3+27t^4w^2+12t^4w+9t^4+18t^2w^5+90t^2w^4+150t^2w^3+114t^2w^2+48t^2w+12t^2+9w^7+63w^6+174w^5+249w^4+200w^3+84w^2+16w)(t^2+w^2+2w)]/[16t^2(t^2+w^2+2w)]$, $r=0$, $s=[((3t^2w^2+3t^2w-3t^2+3w^4+9w^3+5w^2+4w-4bw)(t^2+w^2+2w)t+$

$4f(3t^2w^2+3t^2w-2t^2+3w^4+9w^3+6w^2))(t^2w+3t^2+w^3+5w^2+8w+4)w]/[16t^3(t^2+w^2+2w)]$;

(c_{13}) $a=[(8ft+3t^4+2t^2w^2-4t^2w-w^4+4w^3-4w^2)(3t^2+3w^2-6w+4)]/[8(t^2+w^2-2w)t]$, $c=[(4f+3t^3)(w-1)+(3w^3-9w^2+7w-4)t]/(2t)$, $d=[-8ft(t^2+w^2-2w+2)-(3t^4+6t^2w^2-12t^2w+6t^2+3w^4-12w^3+16w^2-8w)(t^2+w^2-2w)]/[4(t^2+w^2-2w)t]$, $g=[8bt^2+16ft(w-1)+3((w-2)^2w^2+t^4)(w-1)+2(3w^3-9w^2+2w-2)t^2]/(8t^2)$, $k=-a$, $l=[4bt(-t^2-w^2+2w)-4f((3w^2-3w+2)(w-2)+3(w-1)t^2)-t(3t^2w-3t^2+3w^3-9w^2+8w-4)(t^2+w^2-2w)]/16$, $m=-c-1$, $n=[(9t^4w^2-18t^4w+6t^4+18t^2w^4-72t^2w^3+96t^2w^2-42t^2w+9w^6-54w^5+126w^4-138w^3+64w^2-8w)(t^2+w^2-2w)t+4f((2(9w^3-27w^2+22w-2)(w-1)+(9w^2-18w+7)t^2)t^2+(9w^4-36w^3+55w^2-34w+8)(w-2)w)+4b(3t^2w-3t^2+3w^3-9w^2+8w-4)(t^2+w^2-2w)t]/[16t(t^2+w^2-2w)]$, $p=-f$, $q=[4b(3t^2+3w^2-6w+4)(t^2+w^2-2w)^2(2-w)tw-4f(9t^4w^3-27t^4w^2+14t^4w-6t^4+18t^2w^5-90t^2w^4+154t^2w^3-112t^2w^2+32t^2w-8t^2+9w^7-63w^6+176w^5-250w^4+184w^3-56w^2)(t^2+w^2-2w)-(9t^4w^3-27t^4w^2+12t^4w-9t^4+18t^2w^5-90t^2w^4+150t^2w^3-114t^2w^2+48t^2w-12t^2+9w^7-63w^6+174w^5-249w^4+200w^3-84w^2+16w)(t^2+w^2-2w)^2t]/[16t^2(t^2+w^2-2w)]$, $r=0$, $s=[((3t^2w^2-3t^2w-3t^2+3w^4-9w^3+5w^2-4w+4bw)(t^2+w^2-2w)t+4f(3t^2w^2-3t^2w-2t^2+3w^4-9w^3+6w^2))(t^2w-3t^2+w^3-5w^2+8w-4)w]/[16(t^2+w^2-2w)t^3]$;

(c_{14}) $a=[(8ft-3t^4-2t^2w^2+4t^2w+w^4-4w^3+4w^2)(3t^2+3w^2-6w+4)]/[8t(t^2+w^2-2w)]$, $c=[(3w^3-9w^2+7w-4+3(w-1)t^2)t-4f(w-1)]/(2t)$, $d=[(3t^4+6t^2w^2-12t^2w+6t^2+3w^4-12w^3+16w^2-8w)(t^2+w^2-2w)-8(t^2+w^2-2w+2)ft)/[4t(t^2+w^2-2w)]$, $g=[3(w-1)(w-2)^2w^2+8bt^2-(16f-3t^3)(w-1)t+2(3w^3-9w^2+2w-2)t^2]/(8t^2)$, $k=-a$, $l=[4((w-2)w+t^2)bt-4((3w^2-3w+2)(w-2)+3(w-1)t^2)f+(3t^2w-3t^2+3w^3-9w^2+8w-4)(t^2+w^2-2w)t]/16$, $m=-c-1$, $n=[4b(3t^2w-3t^2+3w^3-9w^2+8w-4)(t^2+w^2-2w)t-4f((2(9w^3-27w^2+22w-2)(w-1)+(9w^2-18w+7)t^2)t^2+(9w^4-36w^3+55w^2-34w+8)(w-2)w)+(9t^4w^2-18t^4w+6t^4+18t^2w^4-72t^2w^3+96t^2w^2-42t^2w+9w^6-54w^5+126w^4-138w^3+64w^2-8w)(t^2+w^2-2w)t]/[16t(t^2+w^2-2w)]$, $p=-f$, $q=[4b(3t^2+3w^2-6w+4)(t^2+w^2-2w)(w-2)tw-4f((9w^3-27w^2+14w-6)t^4+(9w^2-18w+14)(w-1)(w-2)^2w^2+2(9w^5-45w^4+77w^3-56w^2+16w-4)t^2)+(9t^4w^3-27t^4w^2+12t^4w-9t^4+18t^2w^5-90t^2w^4+150t^2w^3-114t^2w^2+48t^2w-12t^2+9w^7-63w^6+174w^5-249w^4+200w^3-84w^2+16w)(t^2+w^2-2w)t]/[16t^2(t^2+w^2-2w)]$, $r=0$, $s=[((3t^2w^2-3t^2w-3t^2+3w^4-9w^3+5w^2-4w+4bw)(t^2+w^2-2w)t-4(3t^2w^2-3t^2w-2t^2+3w^4-9w^3+6w^2)f)(t^2w-3t^2+w^3-5w^2+8w-4)w]/[16t^3(t^2+w^2-2w)]$;

(c_{15}) $d = f - a$, $g = (2b + 2c - 3)/2$, $k = -a$, $l = (3f)/2$, $m = -c - 1$, $n = [3(c+1)]/2$, $p = -f$, $q = (3a - 2f)/2$, $r = 0$, $s = (1 - 2b - 2c)/2$;

(c_{16}) $a = k = r = 0$, $d = f$, $l = (3f)/2$, $m = -c-1$, $n = [3(c+1)]/2$, $p = -f$, $q = [f(2g-2b-2c+1)]/2$, $s = (3-4b^2-10bc+4bg-6c^2+6cg+3c)/6$;

(c_{17}) $d = 2(a + 2f)$, $g = (3ac + 3a + 2bf + 5cf + 3f)/(2f)$, $k = -a$, $l = 2f$, $m = -c-1$, $n = (8af+3c+8f^2+3)/2$, $p = -f$, $q = (5c+3+2b)(a+f)$, $r = 0$, $s = [((3c + 1 + 2b)f + (2b + 3c)a)(c + 1)]/(2f)$;

(c_{18}) $c = (u^3-3au^2-27f-9u)/(9u)$, $g = (5u^3-6au^2+18bu-54f-45u)/(18u)$, $k = -a$, $l = 2f$, $m = (3au^2+27f-u^3)/(9u)$, $n = (4u^3-12au^2+21fu^2-81f)/(18u)$, $p = -f$, $q = (7u^3 - 21au^2 + 81a + 36bu + 9fu^2 - 189f - 63u)/54$, $r = 0$, $s = [(u^3 - 3au^2 + 6bu - 18f - 9u)(u^2 - 9)]/(54u)$, $u = d - f + a$;

(c_{19}) $d = f - a$, $l = -f/2$, $g = (2c + 3 + 2b)/2$, $k = -a$, $m = -c - 1$, $n = -(c + 2 + 2b)/2$, $p = -f$, $q = a/2$, $r = s = 0$;

(c_{20}) $a = k = r = 0$, $d = f$, $l = -f/2$, $m = -c - 1$, $n = (-2b - c - 2)/2$, $p = -f$, $q = l(2b + 2c - 2g + 3)$, $s = -(2b + 3c + 4)(2b + 2c - 2g + 3)/6$;

(c_{21}) $c = (3av^2 + 27f - v^3 - 18v)/(9v)$, $l = -f$, $g = (6av^2 + 18bv + 54f - 5v^3 - 9v)/(18v)$, $k = -a$, $m = -c-1$, $n = (2v^3 - 6av^2 - 18bv + 21fv^2 - 27f)/(18v)$, $p = -f$, $q = (21av^2+27a+36bv+9lv^2-135l-7v^3-27v)/54$, $r = 0$, $s = [v(v^3 - 3av^2 + 9a - 6bv + 18l + 3v)]/54$, $v = a + d - f$;

(c_{22}) $d = 2(a + 2f)$, $l = -f$, $g = (6f + 3ac + 6a + 2bf + 5cf)/(2f)$, $k = -a$, $m = -c - 1$, $n = (8f^2 + 8af - 2b - c - 2)/2$, $p = -f$, $q = 2ab + 5ac + 9a + 2bf + 5cf + 6f$, $r = 0$, $s = [(f + cf + 2a + ac)(3c + 4 + 2b)]/(2f)$;

(c_{23}) $c = -(h^2 + 8)/(h^2 + 4)$, $d = -((h^2 + 4)a + 6h))/(h^2 + 4)$, $f = l = p = r = 0$, $g = [(2b - 5)(h^2 + 4) + 16]/[2(h^2 + 4)]$, $k = -a$, $m = 4/(h^2 + 4)$, $n = (ah^3 + 4ah - 16 - 4(b - 1)(h^2 + 4))/(h^2 + 4)^2$, $q = [3ah^4 + 16ah^2 + 16a - 32h - 2h(4b - 7)(h^2 + 4)]/[2(h^2 + 4)^2]$, $s = [(3h^3 - 2ah^2 - 8a - 2bh^3 - 8bh + 4h)h]/[2(h^2 + 4)^2]$;

(c_{24}) $c = (3au^3 - 9v^2 - 9uv - u^4)/(9uv)$, $f = p = r = 0$, $g = (24au^3 - 27a^2u^2 + 18buv + 9v^2 - 18uv - 5u^4)/(18uv)$, $k = -a$, $m = -c - 1$, $n = (9au^4 - 15au^3v - 18buv^2 - 9v^3 + 9uv^2 + 5u^4v - 3u^5)/(18u^2v)$, $q = (63a^2u^3 + 108abuv - 42au^4 - 108auv + 54av^2 - 36bu^2v + 7u^5 + 45u^2v - 18uv^2)/(54uv)$, $s = [(9a^2u^3 + 18abuv + 9av^2 - 27auv - 6au^4 - 6bu^2v - 3uv^2 + 6u^2v + u^5)(u - 3a)]/(54v^2)$, $u = 2a - d$, $v = 2a - d + 6l$;

(c_{25}) $c = (h^4l-h^3+8h^2l-8h+16l)/[h(h^2+4)]$, $d = (-ah^2-4a-6h)/(h^2+4)$, $f = p = r = 0$, $k = -a$, $g = (2bh^3 + 8bh + 2h^4l - 5h^3 + 16h^2l - 4h +$

$32l)/[2h(h^2+4)]$, $m=-c-1$, $n=(2ah^4+8ah^2-8bh^3-32bh+3h^6l+20h^4l+8h^3+16h^2l-64l)/[2h(h^2+4)^2]$, $q=(3ah^4+16ah^2+16a-8bh^3-32bh-10h^4l+14h^3-80h^2l+24h-160l)/[2(h^2+4)^2]$, $s=[h(3h^3-2ah^2-8a-2bh^3-8bh-2h^4l-16h^2l+4h-32l)]/[2(h^2+4)^2]$;

(c_{26}) $c=(3au^3-u^4-18uv-9v^2)/(9uv)$, $f=p=r=0$, $g=(9v^2-27a^2u^2+24au^3+18buv-5u^4-36uv)/(18uv)$, $k=-a$, $m=-c-1$, $n=(u^5-3au^4-15au^3v-18bu^2v-18buv^2+5u^4v-9uv^2-9v^3)/(18u^2v)$, $q=(63a^2u^3+108abuv-42au^4-54auv+54av^2-36bu^2v+7u^5+45u^2v-18uv^2)/(54uv)$, $s=[(9a^2u-6au^2+u^3+9v)(u^4-3au^3-6buv+6uv-3v^2)]/(54uv^2)$;

(c_{27}) $d=-a$, $f=p=r=s=0$, $g=(2b-c-3)/2$, $k=-a$, $l=[-(c+2)a]/2$, $m=-c-1$, $n=[(2b-c-2)(c+1)]/2$, $q=a/2$;

(c_{28}) $d=-a$, $f=p=r=0$, $g=(2b-c-6)/2$, $k=-a$, $l=[-a(c+1)]/2$, $m=-c-1$, $n=(2bc+2b-c^2-3c-2)/2$, $q=(3a)/2$, $s=(c-2b+4)/2$.

Proof. Suppose the set of conditions (7) is realized for system (6) and consider the cubic curve (9). By Definition 2.1, the curve (9) is an invariant cubic for (6) if there exist numbers $c_{20}, c_{11}, c_{02}, c_{10}, c_{01} \in \mathbb{R}$ such that

$$P(x,y)\frac{\partial \Phi}{\partial x}+Q(x,y)\frac{\partial \Phi}{\partial y} \equiv \Phi(x,y)(c_{10}x+c_{01}y+c_{20}x^2+c_{11}xy+c_{02}y^2). \quad (10)$$

Identifying the coefficients of the monomials x^iy^j in (10), we reduce this identity to a system of fifteen equations

$$\{U_{ij}=0,\ \ i+j=3,\,4,\,5\} \quad (11)$$

for the unknowns $a_{30}, a_{21}, a_{12}, a_{03}, c_{20}, c_{11}, c_{02}, c_{10}, c_{01}$ and the coefficient of system (6). When $i+j=3$, we find that $c_{10}=2a-a_{21}$, $c_{01}=a_{12}-2b$, $d=(2f-2a-3a_{03}+3a_{21})/2$, $g=(2b+2c+3a_{30}-3a_{12})/2$.

We divide the proof into two cases: $a_{03}\neq 0$ and $a_{03}=0$.

1. $a_{03}\neq 0$. We express c_{02}, c_{11}, c_{20} and s from the equations $\{U_{05}=0, U_{14}=0, U_{23}=0, U_{32}=0\}$ of (11) and obtain:

$c_{02}=-3l$, $c_{20}=[(f-l)a_{12}^2+(n-c-1)a_{03}a_{12}-3qa_{03}^2+2(l-f)a_{03}a_{21}]/a_{03}^2$, $c_{11}=[(l-f)a_{12}-3na_{03}]/a_{03}$, $s=[(q-a)a_{03}^2a_{12}+2(n-c-1)a_{03}^2a_{21}+3(l-f)a_{03}^2a_{30}+(c-n-1)a_{03}a_{12}^2+3(f-l)a_{03}a_{12}a_{21}+(l-f)a_{12}^3]/(3a_{03}^3)$.

Then we calculate the resultant of the polynomials U_{50} and U_{41} with respect to n. We have that $Res(U_{50},U_{41},n)=f_1f_2$, where

$f_1=(a-q)a_{03}+(l-f)a_{21}$, $f_2=27a_{03}^2a_{30}^2+(4a_{21}^3-18a_{12}a_{21}a_{30})a_{03}+4a_{12}^3a_{30}-a_{12}^2a_{21}^2$.

1.1. Assume that $f_1 = 0$. In this case $q = [aa_{03} + (l - f)a_{21}]/a_{03}$ and the equations $U_{50} = 0$, $U_{41} = 0$ of (11) look

$$\begin{aligned} U_{50} &\equiv (3a_{03}a_{12}a_{30} + 2a_{03}a_{21}^2 - a_{12}^2 a_{21})F = 0, \\ U_{41} &\equiv (9a_{03}^2 a_{30} - 7a_{03}a_{12}a_{21} + 2a_{12}^3)F = 0, \end{aligned} \tag{12}$$

where $F = (c - n + 1)a_{03} + (l - f)a_{12}$.

1.1.1. Suppose that $F = 0$, then $n = (ca_{03} + a_{03} - fa_{12} + la_{12})/a_{03}$ and the equations $\{U_{04} = 0, U_{13} = 0\}$ of (11) yield

$l = (a_{12} + b)a_{03}b - fa_{12}$, $a = (2a_{12}^2 - 9a_{03}^2 + 7a_{03}a_{21} + 6fa_{03} - 2ca_{12} - 4fa_{21} - 2c - 2)/(2a_{03})$.

If $a_{03} = (2f)/3$ and $a_{12} = c + 1$, then we obtain the set of conditions (c_1) for the existence of the invariant cubic

$$3(x^2 + y^2) + (c - 2b + 2g + 3)x^3 + 6ax^2y + 3(c + 1)xy^2 + 2fy^3 = 0.$$

If $a_{03} = (2f)/3$ and $a_{12} \neq c + 1$, then the system of equations (11) has not real solutions.

Let $a_{03} \neq (2f)/3$ and express a_{30} from the equation $U_{22} = 0$. Then we calculate the resultant of the polynomials U_{40} and U_{31} with respect to c. We obtain that $Res(U_{40}, U_{31}, c) = 96a_{03}g_1g_2g_3^4g_4g_5g_6$, where

$g_1 = a_{21} - a_{03}$, $g_2 = 3a_{03}a_{21} - (a_{12} + 1)^2$, $g_3 = 3a_{03}^2 - 2a_{03}a_{21} + (a_{12} + 1)^2$, $g_4 = 3a_{03} - 2f \neq 0$, $g_5 = (3a_{03} - a_{21})^2 + 4(a_{12} + 1)^2 \neq 0$, $g_6 = (4a_{03}a_{21} - 3a_{03}^2 - (a_{12} + 1)^2)^2 + 4a_{03}^2(a_{12} + 1)^2 \neq 0$.

Assume that $g_1 = 0$, then $a_{21} = a_{03}$. If $a_{12} = c + 1$, then the cubic curve is reducible. If $a_{12} \neq c + 1$, then $c = [2(3a_{03}a_{12} - fa_{12} - f)]/(3a_{03})$. In this case we get the set of conditions (c_2) for the existence of the cubic

$$9(x^2 + y^2)(4x - 4 - 3fy + ay) + \sqrt{3}(a - 3f)(x^2 + 9y^2)x = 0,$$

and the set of conditions (c_3) for the existence of the cubic

$$9(x^2 + y^2)(4x - 4 - 3fy + ay) - \sqrt{3}(a - 3f)(x^2 + 9y^2)x = 0.$$

Assume that $g_1 \neq 0$ and let $g_2 = 0$. Then $a_{21} = (a_{12}^2 + 2a_{12} + 1)/(3a_{03})$ and $U_{31} \equiv h_1h_2 = 0$, where $h_1 = 6a_{12}a_{03} - 2fa_{12} - 3ca_{03} - 2f$, $h_2 = a_{12} - c - 1$.

The equation $h_1 = 0$ yields $c = [2(3a_{03}a_{12} - fa_{12} - f)]/(3a_{03})$. Denote $u = d - 2a + 8f$. Then $a_{03} = u/12$, $a = (16a_{12}^2 + 32a_{12} + 24fu - 3u^2 + 16)/(8u)$ and $d = u + 2a - 8f$. In this case we obtain the set of conditions (c_4) for the existence of the invariant cubic

$$\begin{aligned} &12u^2(x^2 + y^2) + 4(16a_{12}^3 + 48a_{12}^2 + 48a_{12} - 3u^2 + 16)x^3 + \\ &+48u(a_{12} + 1)^2x^2y + 12u^2a_{12}xy^2 + u^3y^3 = 0. \end{aligned}$$

If $h_1 \neq 0$ and $h_2 = 0$, the system of equations (11) has no solutions.

Assume that $g_1g_2 \neq 0$ and let $g_3 = 0$. In this case the system of algebraic equations (11) is not consistent.

1.1.2. Assume that $F \neq 0$, then

$$U_{50} \equiv 3a_{03}a_{12}a_{30} + 2a_{03}a_{21}^2 - a_{12}^2a_{21} = 0,$$

$$U_{41} \equiv 9a_{03}^2 a_{30} - 7a_{03}a_{12}a_{21} + 2a_{12}^3 = 0.$$

The equation $U_{41} = 0$ yields $a_{30} = [a_{12}(7a_{03}a_{21} - 2a_{12}^2)]/(9a_{03}^2)$. Then

$$U_{50} \equiv (3a_{03}a_{21} - a_{12}^2)(a_{03}a_{21} + a_{12}^2) = 0.$$

The case $a_{21} = a_{12}^2/(3a_{03})$ is contained in 1.2 ($j_2 = 0$) and the case $a_{21} = (-a_{12}^2)/(a_{03})$ is contained in 1.3 ($v_2 = 0$).

1.2. Assume that $f_1 \neq 0$ and let $f_2 = 0$. If $a_{21} = a_{12}^2/(3a_{03})$, then $f_2 = 0$ yields $a_{30} = a_{12}^3/(27a_{03}^2)$. In this case we express l, n, q from the equations $U_{04} = 0, U_{13} = 0, U_{22} = 0$ of (11) and calculate the resultant of the polynomials U_{40}, U_{31} with respect to c. We obtain that

$$Res(U_{40}, U_{31}, c) = -6a_{03}(9a_{03}^2 + a_{12}^2)j_1 j_2,$$

where $j_1 = 27a_{03}^2 - 18fa_{03} - a_{12}^2 + 6aa_{03}$, $j_2 = 729a_{03}^4 + 162a_{03}^2a_{12}^2 + 324a_{03}^2a_{12} + 108a_{03}^2 + 9a_{12}^4 + 4a_{12}^3$.

Suppose that $j_1 = 0$, then $a = (a_{12}^2 - 27a_{03}^2 + 18fa_{03})/(6a_{03})$. In this case we have

$$\begin{aligned} U_{31} &\equiv (63a_{03}^2a_{12} + 18a_{03}^2 - 9a_{12}^3 - 4a_{12}^2)H, \\ U_{40} &\equiv a_{03}(27a_{03}^2 - 13a_{12}^2 - 6a_{12})H, \end{aligned} \tag{13}$$

where $H = 6a_{03}a_{12} - 3ca_{03} - 3a_{03} - 2fa_{12}$.

If $H = 0$, then we obtain the set of conditions (c_5) for the existence of the invariant cubic

$$12u^2(x^2 + y^2) + (4a_{12}x + uy)^3 = 0.$$

When $H \neq 0$ the system of equations $\{U_{31} = 0, U_{40} = 0\}$ is consistent if and only if $a_{12} = (-1)/2$ and $108a_{03}^2 - 1 = 0$. In this case we get the set of conditions (c_6) for the existence of the invariant cubic

$$54a_{03}(x - 2)(x^2 + y^2) - y(9x^2 + y^2) = 0.$$

Suppose that $j_1 \neq 0$ and let $j_2 = 0$. In this case we denote $a_{12} = (u^2 - 1)/2$. Then $j_2 \equiv e_1e_2 = 0$, where

$$e_1 = (3u + 1)(u - 1)^3 + 108a_{03}^2,\ e_2 = (3u - 1)(u + 1)^3 + 108a_{03}^2.$$

The equation $e_1 = 0$ has the parametrization

$$a_{03} = (-16t^3)/[27(t^2 + 4)^2],\ u = (12 - t^2)/[3(t^2 + 4)]$$

and $U_{31} \equiv i_1i_2 = 0$, where $i_1 = t^2 - 12$, $i_2 = 108ct(t^2 + 4)^2 + 18(at^2 + 36f)(t^2 + 4)^2 + t(t^4 + 168t^2 + 144)(t^2 + 12)$.

If $i_1 = 0$, then we obtain the set of conditions (c_7) for the existence of the invariant cubic

$$3t(x - 2)(x^2 + y^2) + 2y(9x^2 + y^2) = 0.$$

If $i_1 \neq 0$ and $i_2 = 0$, then we get the set of conditions (c_8) for the existence of the invariant cubic

$$108(t^2 + 4)^2(x^2 + y^2) - (t^2x + 12x + 4ty)^3 = 0.$$

Assume that $e_1 \neq 0$ and let $e_2 = 0$. The equation $e_2 = 0$ has the parametrization $a_{03} = (-16t^3)/[27(t^2 + 4)^2]$, $u = (t^2 - 12)/[3(t^2 + 4)]$, where

$t^2 \neq 12$. In this case we express c from the equation $U_{31} = 0$ and also obtain the set of conditions (c_8).

1.3. Assume that $f_1 \neq 0$, $a_{21} \neq a_{12}^2/(3a_{03})$ and let $f_2 = 0$. Denote $a_{21} = (a_{12}^2 - h^2)/(3a_{03})$. In this case $f_2 \equiv u_1 u_2 = 0$, where

$u_1 = 27a_{03}^2 a_{30} - a_{12}^3 + 3a_{12}h^2 - 2h^3$, $u_2 = 27a_{03}^2 a_{30} - a_{12}^3 + 3a_{12}h^2 + 2h^3$.

1.3.1. If $u_1 = 0$, then $a_{30} = (a_{12}^3 - 3a_{12}h^2 + 2h^3)/(27a_{03}^2)$ and the equations $U_{50} = 0$ and $U_{41} = 0$ imply

$q = [(n-1-c)(2a_{12}+h)a_{03} + (a_{12}^2 + a_{12}h + h^2)(f-l) + 3aa_{03}^2]/(3a_{03}^2)$.

Suppose that $a_{12} \neq (h-3)/4$. In this case we express l, n, a from the equations $U_{04} = 0$, $U_{13} = 0$, $U_{22} = 0$ of (11) and calculate the resultant of the polynomials U_{40}, U_{31} with respect to c. We obtain that

$$Res(U_{40}, U_{31}, c) = 6a_{03}v_1 v_2^4 v_3 v_4 v_5 v_6,$$

where $v_1 = 3a_{03} - 2f$, $v_2 = 729a_{03}^4 + 54a_{03}^2(3a_{12}^2 + 6a_{12} + 2h^3 + 3h^2 + 2) + (9a_{12}^2 + 18a_{12}h + 4a_{12} + 12h^3 - 3h^2 + 8h)(a_{12} - h)^2$, $v_3 = 9a_{03}^2 + (a_{12} + 2h)^2 \neq 0$, $v_4 = 9a_{03}^2 + (a_{12} - h)^2 \neq 0$, $v_5 = 4a_{12} - h + 3 \neq 0$.

If $v_1 = 0$, then $a_{12} - c - 1 \neq 0$. In this case the system of equations $\{U_{40} = 0, U_{31} = 0\}$ has solutions different from that obtained above only if

$$a_{12} = -(h^2+1)/2 \text{ and } H \equiv 48f^2 + (3h-1)(h+1)^3 = 0.$$

The equation $H = 0$ admits the following parametrization

$$f = (-4v^3)/(9(v^2+1)^2),\ h = (v^2-3)/(3(v^2+1)).$$

We obtain the set of conditions (c_9) for the existence of the invariant cubic

$$27(v^2+1)^2(x^2+y^2) + (v^4x - 8v^3y - 18v^2x - 27x)(xv+y)^2 = 0.$$

Assume that $v_1 \neq 0$ and let $v_2 = 0$. If $a_{12} = -(h^2+1)/2$, then $v_2 = (3h-1)(h+1)^3 + 108a_{03}^2$. The equation $v_2 = 0$ has the following parametrization

$$a_{03} = (-16t^3)/[27(t^2+4)^2], h = (t^2-12)/[3(t^2+4)].$$

We get the set of conditions (c_{10}) for the existence of the invariant cubic

$$108(t^2+4)^2(x^2+y^2) + (t^4x - 16t^3y - 72t^2x - 432x)(xt+2y)^2 = 0.$$

Suppose that $a_{12} \neq -(h^2+1)/2$ and denote $a_{12} = (u^2 - h^2 - 1)/2$. In this case the equation $v_2 = 0$ looks as $v_2 \equiv s_1 s_2 = 0$, where

$$s_1 = 108a_{03}^2 + (3h^2 - 6hu + 2h + 3u^2 + 2u - 1)(h+u+1)^2,$$
$$s_2 = 108a_{03}^2 + (3h^2 + 6hu + 2h + 3u^2 - 2u - 1)(h-u+1)^2.$$

Let $s_1 = 0$ and denote $h = (-3t^2 - 3w^2 - 8w - 4)/4$, $u = (-3t^2 - 3w^2 - 4w)/4$. Then $s_1 \equiv p_1 p_2 = 0$, where $p_1 = 4a_{03} + t^3 + tw^2 + 2tw$, $p_2 = 4a_{03} - t^3 - tw^2 - 2tw$.

If $p_1 = 0$, then the equations $U_{40} = 0$ and $U_{31} = 0$ of (11) imply $c = (4fw + 4f - 3t^3w - 3t^3 - 3tw^3 - 9tw^2 - 7tw - 4t)/(2t)$. In this case we have the set of conditions (c_{11}) for the existence of the invariant cubic

$$4t^2(x^2+y^2) + (t^3y - t^2wx - 3t^2x + tw^2y +$$
$$+2twy - w^3x - 5w^2x - 8wx - 4x)(yt - xw)^2 = 0.$$

If $p_1 \neq 0$ and $p_2 = 0$, then the equations $U_{40} = 0$ and $U_{31} = 0$ yield $c = -[(4f + 3t^3)(w + 1) + t(3w^3 + 9w^2 + 7w + 4)]/(2t)$. In this case we have the set of conditions (c_{12}) for the existence of the invariant cubic

$$4t^2(x^2 + y^2) - (t^3y + t^2wx + 3t^2x + tw^2y+ \\ +2twy + w^3x + 5w^2x + 8wx + 4x)(yt + xw)^2 = 0.$$

Assume that $s_1 \neq 0$ and let $s_2 = 0$. Denote $h = (-3t^2 - 3w^2 + 8w - 4)/4$, $u = (3t^2 + 3w^2 - 4w)/4$. Then $s_2 \equiv q_1q_2 = 0$, where $q_1 = 4a_{03} + t^3 + tw^2 - 2tw$, $q_2 = 4a_{03} - t^3 - tw^2 + 2tw$.

If $q_1 = 0$, then the equations $U_{40} = 0$ and $U_{31} = 0$ of (11) imply $c = [(4f + 3t^3)(w - 1) + (3w^3 - 9w^2 + 7w - 4)t]/(2t)$. In this case we have the set of conditions (c_{13}) for the existence of the invariant cubic

$$4t^2(x^2 + y^2) - (t^3y - t^2wx + 3t^2x + tw^2y- \\ -2twy - w^3x + 5w^2x - 8wx + 4x)(yt - xw)^2 = 0.$$

If $q_1 \neq 0$ and $q_2 = 0$, then the equations $U_{40} = 0$ and $U_{31} = 0$ yield $c = [(3t^3 - 4f)(w - 1) + (3w^3 - 9w^2 + 7w - 4)t]/(2t)$. In this case we have the set of conditions (c_{14}) for the existence of the invariant cubic

$$4t^2(x^2 + y^2) + (t^3y + t^2wx - 3t^2x + tw^2y- \\ -2twy + w^3x - 5w^2x + 8wx - 4x)(yt + xw)^2 = 0.$$

The case $a_{12} = (h-3)/4$ is contained in $v_2 = 0$ $((64h^2 - 53h + 11)(h+1)^3 + 6912a_{03}^4 + 32(32h - 13)(h + 1)^2a_{03}^2 = 0)$.

1.3.2. Assume that $u_1 \neq 0$ and let $u_2 = 0$. Then $a_{30} = (a_{12}^3 - 3a_{12}h^2 + 2h^3)/(27a_{03}^2)$ and the equations $U_{50} = 0$ and $U_{41} = 0$ imply

$$q = [(n - 1 - c)(2a_{12} - h)a_{03} + (a_{12}^2 - a_{12}h + h^2)(f - l) + 3aa_{03}^2)/(3a_{03}^2).$$

Suppose that $a_{12} \neq (-h - 3)/4$. In this case we express l, n, a from the equations $U_{04} = 0$, $U_{13} = 0$, $U_{22} = 0$ of (11) and calculate the resultant of the polynomials U_{40}, U_{31} with respect to c. We obtain that

$$Res(U_{40}, U_{31}, c) = 6a_{03}w_1w_2^4w_3w_4w_5w_6,$$

where $w_1 = 3a_{03} - 2f$, $w_2 = 729a_{03}^4 + 54a_{03}^2(3a_{12}^2 + 6a_{12} - 2h^3 + 3h^2 + 2) + (9a_{12}^2 - 18a_{12}h + 4a_{12} - 12h^3 - 3h^2 - 8h)(a_{12} + h)^2$, $w_3 = 9a_{03}^2 + (a_{12} - 2h)^2 \neq 0$, $w_4 = 9a_{03}^2 + (a_{12} + h)^2 \neq 0$, $w_5 = 4a_{12} + h + 3 \neq 0$.

If $w_1 = 0$, then $a_{12} - c - 1 \neq 0$. In this case the system of equations $\{U_{40} = 0, U_{31} = 0\}$ has solutions different from that obtained above if

$$a_{12} = -(h^2 + 1)/2 \text{ and } G \equiv 48f^2 + (3h + 1)(h - 1)^3 = 0.$$

The equation $G = 0$ admits the following parametrization

$$f = (-4v^3)/(9(v^2 + 1)^2),\ h = (v^2 - 3)/(3(v^2 + 1)).$$

In this case we have the set of conditions (c_9) obtained above.

Assume that $w_1 \neq 0$ and let $w_2 = 0$. If $a_{12} = -(h^2 + 1)/2$, then $w_2 = (3h + 1)(h - 1)^3 + 108a_{03}^2$. The equation $w_2 = 0$ has the following parametrization $a_{03} = (-16t^3)/[27(t^2 + 4)^2], h = (12 - t^2)/[3(t^2 + 4)]$. In this case we get the condition (c_{10}) which was determined above.

Suppose that $a_{12} \neq -(h^2+1)/2$ and denote $a_{12} = (u^2-h^2-1)/2$. In this case the equation $w_2 = 0$ looks as $w_2 \equiv r_1 r_2 = 0$, where

$$r_1 = 108a_{03}^2 + (3h^2+6hu-2h+3u^2+2u-1)(h-u-1)^2,$$
$$r_2 = 108a_{03}^2 + (3h^2-6hu-2h+3u^2-2u-1)(h+u-1)^2.$$

Let $r_1 = 0$ and denote $h = (3t^2+3w^2+8w+4)/4$, $u = (-3t^2-3w^2-4w)/4$. Then $r_1 \equiv k_1 k_2 = 0$, where $k_1 = 4a_{03}+t^3+tw^2+2tw$, $k_2 = 4a_{03}-t^3-tw^2-2tw$.

If $k_1 = 0$, then the equations $U_{40} = 0$ and $U_{31} = 0$ of (11) imply $c = (-4fw-4f-3t^3w-3t^3-3tw^3-9tw^2-7tw-4t)/(2t)$. In this case we have the set of conditions (c_{12}) obtained above.

If $k_1 \neq 0$ and $k_2 = 0$, then the equations $U_{40} = 0$ and $U_{31} = 0$ yield $c = [(4f-3t^3)(w+1)-t(3w^3+9w^2+7w+4)]/(2t)$. In this case we have the set of conditions (c_{11}) determined above.

Assume that $r_1 \neq 0$ and let $r_2 = 0$. Denote $h = (3t^2+3w^2+8w+4)/4$, $u = (3t^2+3w^2+4w)/4$. Then $r_2 \equiv l_1 l_2 = 0$, where $l_1 = 4a_{03}+t^3+tw^2+2tw$, $l_2 = 4a_{03}-t^3-tw^2-2tw$.

If $l_1 = 0$, then the equations $U_{40} = 0$ and $U_{31} = 0$ of (11) imply $c = (-4fw-4f-3t^3w-3t^3-3tw^3-9tw^2-7tw-4t)/(2t)$. In this case we have the set of conditions (c_{12}) obtained above.

If $l_1 \neq 0$ and $l_2 = 0$, then the equations $U_{40} = 0$ and $U_{31} = 0$ yield $c = [(4f-3t^3)(w+1)-t(3w^3+9w^2+7w+4)]/(2t)$. In this case we have the set of conditions (c_{11}) determined above.

The case $a_{12} = (-h-3)/4$ is contained in $w_2 = 0$ $(6912a_{03}^4 - ((64h^2+53h+11)(h-1)+32(32h+13)a_{03}^2)(h-1)^2 = 0)$.

2. $a_{03} = 0$. In this case we express $c_{02}, c_{11}, c_{20}, q, n$ from the equations $\{U_{ij} = 0,\ \ i+j = 4\}$ of (11). Then the equation $U_{14} = 0$ looks

$$U_{14} \equiv f a_{12}(a_{12}+1) = 0.$$

2.1. Assume that $a_{12} = 0$. If $a_{21} = 0$, then $l = (3f)/2$. When $a_{30} = -1$ we obtain the set of conditions (c_{15}) for the existence of the invariant cubic

$$x^2+y^2-x^3 = 0.$$

If $a_{30} \neq -1$ and $a = 0$, then we obtain the sets of conditions (c_{16}) for the existence of the invariant cubic

$$3(x^2+y^2)+2(g-b-c)x^3 = 0.$$

If $a_{21} \neq 0$, then $l = 2f$. We express s from the equation $U_{50} = 0$ and calculate the resultant of the polynomials U_{41}, U_{32} with respect to c. We obtain that $Res(U_{41}, U_{32}, c) = -2a_{21}(a_{21}^2+a_{30}^2)m_1 m_2$, where $m_1 = a_{21}-2f-2a$, $m_2 = a_{21}^2-4a_{30}-4$.

If $m_1 = 0$, then we get the set of conditions (c_{17}) for the existence of the invariant cubic

$$f(x^2+y^2)+(a+f)(cx+2fy+x)x^2 = 0.$$

If $m_1 \neq 0$ and $m_2 = 0$, then we obtain the set of conditions (c_{18}) for the existence of the invariant cubic

$$9(x^2+y^2)+x^2(u^2x+6uy-9x)=0.$$

2.2. Assume that $a_{12}=-1$ and $a_{21}=0$. If $a_{30}=0$, then we obtain the set of conditions (c_{19}) for the existence of the invariant cubic

$$x^2+y^2-xy^2=0.$$

If $a_{30}\neq 0$ and $a=0$, then we obtain the set of conditions (c_{20}) for the existence of the invariant cubic

$$3(x^2+y^2)-x(2bx^2+2cx^2-2gx^2+3x^2+3y^2)=0.$$

2.3. Assume that $a_{12}=-1$ and let $a_{21}\neq 0$. The equation $U_{13}=0$ yields $f=-l$. We express n and q from the equations $U_{22}=0$ and $U_{40}=0$. In this case we have that $U_{50}\equiv n_1n_2=0$, where

$$n_1=a_{21}^2+4a_{30},\ n_2=a_{21}(2c+3-a_{30})+2(a+l)(a_{30}+1).$$

When $n_1=0$ we obtain the set of conditions (c_{21}) for the existence of the invariant cubic

$$9(x^2+y^2)-x(vx-3y)^2=0.$$

Suppose that $n_1\neq 0$ and let $n_2=0$. If $l\neq 0$, then we get the set of conditions (c_{22}) for the existence of the invariant cubic

$$f(x^2+y^2)+(ac+2a+cf+f)x^3+2f(a+f)x^2y-fxy^2=0.$$

If $l=0$, then we get the set of conditions (c_1) $(c=-2,\ f=0)$.

2.4. Assume that $a_{12}(a_{12}+1)\neq 0$ and let $f=0$.

2.4.1. If $a_{21}\neq 0$, then express c, s from the equations $U_{23}=0$, $U_{32}=0$ and calculate the resultant of the polynomials U_{50} and U_{41} with respect to a. We obtain that $Res(U_{50},U_{41},a)=4la_{12}(a_{12}+1)^3z_1z_2z_3$, where $z_1=4a_{12}a_{30}-a_{21}^2$, $z_2=4a_{12}a_{30}+4a_{12}-a_{21}^2+4a_{30}+4$, $z_3=(a_{12}-a_{30})^2+a_{21}^2\neq 0$.

Let $l=0$. If $a_{21}=2a$, then we obtain the set of conditions (c_1) $(f=0)$.

If $a_{21}\neq 2a$ and $a_{30}=-a_{12}-1$, then the equation $U_{50}\equiv 4a_{12}^2+4a_{12}+a_{21}^2=0$ admits the parametrization $a_{12}=(-4)/(h^2+4)$, $a_{21}=(-4h)/(h^2+4)$. In this case we get the set of conditions (c_{23}) for the existence of the invariant cubic $(x^2+y^2)-x(hx+2y)^2=0$.

Assume that $l\neq 0$ and let $z_1=0$. In this case $a_{30}=a_{21}^2/(4a_{12})$ and $U_{41}\equiv l_1l_2=0$, where $l_1=2aa_{12}+2a-a_{12}a_{21}-a_{21}+4l$, $l_2=4a_{12}^2+4a_{12}+a_{21}^2$.

If $l_1=0$, then we have the set of conditions (c_{24}) for the existence of the invariant cubic

$$9uv(x^2+y^2)-x(3aux-u^2x-3vy)^2=0.$$

If $l_1\neq 0$ and $l_2=0$, then the equation $l_2=0$ admits the parametrization $a_{12}=(-4)/(h^2+4)$, $a_{21}=(-4h)/(h^2+4)$. In this case we get the set of conditions (c_{25}) for the existence of the invariant cubic

$$(h^2+4)(x^2+y^2)-x(hx+2y)^2=0.$$

Assume that $lz_1\neq 0$ and let $z_2=0$. In this case $a_{30}=(a_{21}^2-4a_{12}-4)/[4(a_{12}+1)]$ and $U_{41}\equiv b_1b_2=0$, where $b_1=2aa_{12}-a_{12}a_{21}+4l$, $b_2=4a_{12}^2+4a_{12}+a_{21}^2\neq 0$. If $b_1=0$, then we have the set of conditions (c_{26})

for the existence of the invariant cubic $9uv(x^2+y^2)+u(6au^2-9a^2u-u^3-9v)x^3+6uv(3a-u)x^2y-9v(u+v)xy^2=0$.

2.4.2. If $a_{21}=0$, then $a_{12}=c+1$ and $U_{50}\equiv aa_{30}(a_{30}+1)=0$. When $a_{30}=0$ we have the set of conditions (c_{27}) and the invariant cubic curve

$$x^2+y^2+(c+1)xy^2=0.$$

When $a_{30}=-1$ we get the set of conditions (c_{28}) and the invariant cubic

$$x^2+y^2+x(y^2+cy^2-x^2)=0.$$

If $a_{30}(a_{30}+1)\neq 0$ and $a=0$, the cubic system has two parallel invariant straight lines. ∎

4. CUBIC SYSTEMS AND INTEGRATING FACTORS

Let the conditions of Theorem 3.1 be fulfilled. Then the cubic system (6) has one invariant straight line and one irreducible cubic curve. In this section we determine the center conditions for cubic system (6) by constructing an integrating factor of the form

$$\mu=\frac{1}{(1-x)^{\alpha_1}\Phi^{\alpha_2}},\tag{14}$$

where $\Phi=x^2+y^2+a_{30}x^3+a_{21}x^2y+a_{12}xy^2+a_{03}y^3$ and α_1,α_2 are real exponents.

According to [8] the function (14) is an integrating factor for system (6) if and only if the following identity holds

$$P(x,y)\frac{\partial\mu}{\partial x}+Q(x,y)\frac{\partial\mu}{\partial y}+\mu\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)=0.\tag{15}$$

The identity (15) will be used to find the integrating factors for cubic system (6) with two invariant algebraic curves.

Theorem 4.1. *The cubic system (6) has an integrating factor of the form (14) if and only if one of the following twenty sets of conditions holds:*

(i) $d=2a$, $k=-a$, $l=[f(2b-c-1)]/3$, $m=-(c+1)$, $n=(2bc+2b-c^2-3c-2)/2$, $p=-f$, $q=a(2b-c-3)$, $r=0$, $s=[(2b-c-2g-3)(c-2b+4)]/6$;

(ii) $b=[\sqrt{3}(a+f)+4]/4$, $c=[\sqrt{3}(5f-3a)-12]/6$, $d=f-a$, $g=(b-4)/3$, $k=-a$, $l=f$, $m=-c-1$, $n=-m$, $p=-f$, $q=a$, $r=s=0$;

(iii) $b=[4-\sqrt{3}(a+f)]/4$, $c=[\sqrt{3}(3a-5f)-12]/6$, $d=f-a$, $g=(b-4)/3$, $k=-a$, $l=f$, $m=-c-1$, $n=-m$, $p=-f$, $q=a$, $r=s=0$;

(iv) $a=[(2b+3c)(2b+3c+8)+24fu-3u^2+16]/(8u)$, $d=2a-8f+u$, $g=[(2b+3c)^3+12(2b+3c)^2+(2b+3c)(48-u^2+8fu)+32fu-12u^2+64]/(8u^2)$, $k=-a$, $l=f$, $m=-c-1$, $n=-m$, $p=-f$, $q=a$, $r=s=0$;

(v) $a = [(2b+3c+3)^2 + 24fu - 3u^2]/(8u)$, $d = [(2b+3c+3)^2 - 8fu + u^2]/(4u)$, $g = [(2b+3c+3)^3 + (2b+3c+3)(8fu-u^2) - 8u^2]/(8u^2)$, $k = -a$, $l = r = 0$, $m = -c-1$, $n = [(2b+3c+3)(u-12f)]/(4u)$, $p = -f$, $q = -d$, $s = [(2b+3c+3)(u^2-8fu-(2b+3c+3)^2)]/(8u^2)$, $4f(2b+3c+3) - u(2b+c+1) = 0$;

(vi) $a = 3f$, $c = (-2b-5)/3$, $d = [2f(2b+5)]/(1-2b)$, $g = (b-5)/3$, $k = -a$, $l = r = 0$, $m = (2(b+1))/3$, $n = -b$, $p = -f$, $q = -d$, $s = (2-b)/3$, $(2b-1)^2 - 108f^2 = 0$;

(vii) $b = 1/2$, $c = (-3)/2$, $d = (a+15f)/3$, $g = -1$, $k = -a$, $l = f/2$, $m = 1/2$, $n = (3-20a^2-60af)/3$, $p = -f$, $q = (2a-15f)/6$, $r = 0$, $s = (4a^2+12af+3)/12$, $(2a+6f)^2 - 3 = 0$;

(viii) $b = 1/2$, $c = (-3)/2$, $d = (11a+21f)/3$, $g = -1$, $k = -a$, $l = f/2$, $m = 1/2$, $n = (27-84af-140a^2)/2$, $p = -f$, $q = (-2a-21f)/18$, $r = 0$, $s = (20a^2+12af+9)/36$, $(10a+6f)^2 - 27 = 0$;

(ix) $c = -1$, $d = -2a$, $f = -a$, $g = (2b-5)/2$, $k = -a$, $l = (-3a)/2$, $m = n = r = 0$, $p = a$, $q = (5a)/2$, $s = (3-2b)/2$;

(x) $c = 1-2b$, $d = 2(a+2f)$, $g = [(3a+4f)(1-b)]/(f)$, $k = -a$, $l = 2f$, $m = 2(b-1)$, $n = 4af - 3b + 4f^2 + 3$, $p = -f$, $q = 8(1-b)(a+f)$, $r = 0$, $s = [(4ab-3a+4bf-4f)(b-1)]/(f)$;

(xi) $b = (2a^2+ad-d^2+9)/9$, $c = -b$, $f = l = p = r = 0$, $g = (a^2+2ad+d^2-9)/6$, $k = -a$, $m = -c-1$, $n = -2m$, $q = [(d-2a)(a+d+3)(a+d-3)]/18$, $s = [(a+d+3)(a+d-3)(d^2-2a^2-ad-9)]/162$;

(xii) $b = 1/4$, $c = -2$, $d = -2a$, $f = -a$, $g = (-1)/4$, $k = -a$, $l = a/2$, $m = 1$, $n = (-1)/4$, $p = a$, $q = a/2$, $r = s = 0$;

(xiii) $b = (v^3 - 3av^2 + 27a - 9v)/(18v)$, $c = -2b-2$, $d = [2(2v-3a)]/3$, $f = (v-3a)/3$, $g = (3av^2 - 27a - 4v^3)/(18v)$, $k = -a$, $m = -c-1$, $n = [4v(v-3a)]/9$, $p = (3a-v)/3$, $q = (12av^2 - 27a - 4v^3)/27$, $r = 0$, $s = [v(v^3 - 3av^2 + 27a)]/81$, $v = 3(a-l)$;

(xiv) $c = -2(b+1)$, $d = 2(a+2f)$, $g = [-(2(2b+1)f+3ab)]/f$, $k = -a$, $l = -f$, $m = 2b+1$, $n = 4f(a+f)$, $p = -f$, $q = -((8b+1)a+4(2b+1)f)$, $r = 0$, $s = [(2bf+f+2ab)(2b+1)]/f$;

(xv) $b = (2u^4+9uv-9v^2-6au^3)/(18uv)$, $c = -(u^4+9uv+9v^2-3au^3)/(9uv)$, $f = p = r = 0$, $g = (6au^2 - 9a^2u - u^3 - 3v)/(6v)$, $k = -a$, $m = -c-1$, $n = [(3a-u)(u-v)u]/(6v)$, $q = (9a^2u^2 - 6au^3 - 18av + u^4 + 9uv)/(18v)$, $s = [(9a^2u^2 - 6au^3 - 54av + u^4 + 9uv)(u-3a)u]/(162v^2)$, $u = 2a-d$, $v = 2a-d+6l$;

(xvi) $a = [-2(h^4l + h^3 + 8h^2l + 16l)]/[h^2(h^2 + 4)]$, $b = (-2h^4l + h^3 - 16h^2l - 32l)/[2h(h^2+4)]$, $c = (h^4l - h^3 + 8h^2l - 8h + 16l)/[h(h^2+4)]$, $d = [2(h^4l - 2h^3 + 8h^2l + 16l)]/[h^2(h^2+4)]$, $f = p = r = 0$, $g = [-2(h^2+1)]/(h^2+4)$, $k = -a$, $m = -c-1$, $n = (3hl)/2$, $q = [-2(2h^4l - h^3 + 10h^2l + 8l)]/[(h^2+4)h^2]$, $s = (h^3 + 2h^2l + 8l)/[(h^2+4)h]$;

(xvii) $a = (h^4l + 8h^2l - 4h + 16l)/[2(h^2+4)]$, $b = (h^3 - h^4l - 8h^2l + 2h - 16l)/[(h^2+4)h]$, $c = (h^4l - h^3 + 8h^2l - 8h + 16l)/[(h^2+4)h]$, $d = -(h^4l + 8h^2l + 8h + 16l)/[2(h^2+4)]$, $f = p = r = 0$, $g = (-3h^2)/[2(h^2+4)]$, $k = -a$, $m = -c-1$, $n = [2(h^4l + 5h^2l - h + 4l)]/[(h^2+4)h]$, $q = (3h^2l)/4$, $s = [(h - h^2l - 4l)h]/[2(h^2+4)]$;

(xviii) $b = (2u^4 + 9uv - 9v^2 - 6au^3)/(18uv)$, $c = -(u^4 + 18uv + 9v^2 - 3au^3)/(9uv)$, $f = p = r = 0$, $g = (6au^2 - 9a^2u - u^3 - 9v)/(6v)$, $k = -a$, $m = -c-1$, $n = (3au^3 - 9au^2v - u^4 + 3u^3v - 9uv - 9v^2)/(18uv)$, $q = (9a^2u^2 - 6au^3 + u^4 + 9uv)/(18v)$, $s = [(9a^2u - 6au^2 + u^3 + 9v)(u^3 + 9v - 3au^2)]/(162v^2)$, $u = 2a - d$, $v = 6l - 2a + d$;

(xix) $c = 2(b-1)$, $d = -a$, $f = n = p = r = s = 0$, $g = (-1)/2$, $k = -a$, $l = -ab$, $m = 1 - 2b$, $q = a/2$;

(xx) $c = 2b - 3$, $d = -a$, $f = p = r = 0$, $g = (-3)/2$, $k = -a$, $l = a(1-b)$, $m = 2(1-b)$, $n = b - 1$, $q = (3a)/2$, $s = 1/2$.

Proof. Suppose at least one set of the conditions (c_1)–(c_{28}) from Theorem 3.1 is satisfied. Then the cubic system (6) has the invariant straight line $1 - x = 0$ and the invariant cubic $\Phi = 0$ of the form (9). The function (14) is an integrating factor for system (6) if and only if the identity (15) holds. Identifying the coefficients of the monomials x^iy^j in (15), we obtain a system of five equations

$$\{F_{ij} = 0, \ \ i + j = 1, 2\} \tag{16}$$

for the unknowns α_1, α_2 and the coefficients of system (6).

In Case (c_1) the equations of (16) give $\alpha_1 = (c + 1 - 2b)\alpha_2$. We have the set of conditions (i) for the existence of the integrating factor (14), where $\Phi = 3(x^2 + y^2) + (c - 2b + 2g + 3)x^3 + 6ax^2y + 3(c+1)xy^2 + 2fy^3$ and $\alpha_1 = c + 1 - 2b$, $\alpha_2 = 1$.

In Case (c_2) the equations $F_{10} = 0$, $F_{01} = 0$ of (16) yield $\alpha_2 = 4/3$, $\alpha_1 = [\sqrt{3}(a + f) - 4b + 4]/6$ and the equations $F_{ij} = 0, i + j = 2$ imply $b = (\sqrt{3}(a + f) + 4)/4$. We obtain the set of conditions (ii) and the integrating factor (14), where $\Phi = 9(x^2 + y^2)(4x - 4 - 3fy + ay) + \sqrt{3}(a - 3f)(x^2 + 9y^2)x$ and $\alpha_1 = 0$, $\alpha_2 = 4/3$.

In Case (c_3) the equations $F_{10} = 0$, $F_{01} = 0$ of (16) yield $\alpha_2 = 4/3$, $\alpha_1 = [-\sqrt{3}(a+f) - 4b + 4]/6$ and the equations $F_{ij} = 0$, $i+j = 2$ imply $b = (-\sqrt{3}(a+$

$f)+4)/4$. We get the set of conditions (iii) for the existence of the integrating factor (14), where $\Phi = 9(x^2+y^2)(4x-4-3fy+ay)-\sqrt{3}(a-3f)(x^2+9y^2)x$ and $\alpha_1 = 0$, $\alpha_2 = 4/3$.

In Case (c_4), from the equations $F_{10} = 0$, $F_{01} = 0$ of (16), we obtain $\alpha_2 = 4/3$ and $\alpha_1 = [2(12fa_{12} - ua_{12} - bu + 12f)]/(3u)$. Then the equations $F_{ij} = 0$, $i+j = 2$ imply $12fa_{12} - ua_{12} - bu + 12f = 0$. Since $c = [2(a_{12}u - 4fa_{12} - 4f)]/u$, we find that $a_{12} = (2b+3c)/4$. In this case we get the set of conditions (iv) for the existence of the integrating factor (14), where $\Phi = 12u^2(x^2+y^2) + ((2b+3c+4)^3 - 12u^2)x^3 + 3u(2b+3c+4)^2x^2y + 3u^2(2b+3c)xy^2 + u^3y^3$ and $\alpha_1 = 0$, $\alpha_2 = 4/3$.

In Case (c_5) the equations $F_{10} = 0$, $F_{01} = 0$ of (16) yield $\alpha_2 = 4/3$, $\alpha_1 = (24fa_{12} - 2ua_{12} - 2bu + 3u)/(3u)$ and the equations $F_{ij} = 0$, $i+j = 2$ imply $12fa_{12} - a_{12}u - bu = 0$. Since $c = (2ua_{12} - 8fa_{12} - u)/u$, we find that $a_{12} = (2b+3c+3)/4$. In this case we obtain the set of conditions (v) for the existence of the integrating factor (14), where $\Phi = 12u^2(x^2+y^2) + ((2b+3c+3)x+uy)^3$ and $\alpha_1 = 1$, $\alpha_2 = 4/3$.

In Case (c_6) the equations $F_{10} = 0$, $F_{01} = 0$ of (16) yield $\alpha_1 = (-2b-3c-2)/2$, $\alpha_2 = 4/3$ and the equations $F_{ij} = 0$, $i+j = 2$ imply $c = (-2b-5)/3$, $a_{03} = (-2b-3c)/2$. Since $108a_{03}^2 - 1 = 0$, we find that the set of conditions (vi) for the existence of the integrating factor (14), where $\Phi = (2b-1)(x^2+y^2)(x-2) + 2(9x^2+y^2)fy$ and $\alpha_1 = 1$, $\alpha_2 = 4/3$.

In Case (c_7) the system of equations (16) has real solutions if and only if $b = 1/2$, $c = (-3)/2$, $\alpha_1 = (5-3\alpha_2)/2$ and $(3\alpha_2 - 5)(3\alpha_2 - 7) = 0$.

If $\alpha_2 = 5/3$, then $\alpha_1 = 0$. We find the set of conditions (vii) for the existence of the integrating factor (14), where

$$\Phi = 6(a+3f)(x^2+y^2)(x-2) - (9x^2+y^2)y.$$

If $\alpha_2 = 7/3$, then $\alpha_1 = -1$. We get the set of conditions (viii) for the existence of the integrating factor (14), where

$$\Phi = 2(5a+3f)(x^2+y^2)(x-2) - (9x^2+y^2)y.$$

In Case (c_8) we express α_1 and α_2 from the equations $F_{10} = 0$, $F_{01} = 0$ of (16). Then we find b from $F_{02} = 0$ and a from $F_{20} = 0$. In this case $F_{11} \neq 0$ and we cannot construct an integrating factor (14) for system (6).

In Case (c_9) the equations of (16) yield $\alpha_1 = [-(5v^4 + 6v^2 + 9 + 18(v^2+1)^2b)\alpha_2]/[9(v^2+1)^2]$, $c = [2(-7v^4 - 12v^2 - 9)]/[9(v^2+1)^2]$. This case is contained in (i).

In Case (c_{10}) the equations of (16) yield $\alpha_1 = 0$, $\alpha_2 = 5/3$, $c = (-3)/2$, $b = 1/2$, $t^2 = 12$. This case is contained in (vii).

In Case (c_{11}) we express α_1 and α_2 from the equations $F_{10} = 0$, $F_{01} = 0$ of (16). Then the equations $F_{ij} = 0$, $i+j = 2$ imply $b = 1/2$, $w = -1$, $3t^2 - 1 = 0$. Since $a = (-6ft-1)/(2t)$, we find that the set of conditions (vii) for the existence of the integrating factor (14).

In Cases (c_{12}), (c_{13}) and (c_{14}) we have the set of conditions (vii) for the existence of the integrating factor (14).

In Case (c_{15}) the equations $F_{10}=0$, $F_{01}=0$ of (16) yield $\alpha_1=(-1)/2$, $\alpha_2=2$ and the equations $F_{ij}=0$, $i+j=2$ imply $c=-1$, $f=-a$. We obtain the set of conditions (ix) for the existence of the integrating factor (14), where $\Phi=x^2+y^2-x^3$.

In Case (c_{16}) we have $F_{11}\equiv 2f\neq 0$. We cannot construct an integrating factor (14) for system (6).

In Case (c_{17}), from the equations of (16), we find that $\alpha_1=-1$, $\alpha_2=2$, $c=1-2b$. We obtain the set of conditions (x) and the integrating factor (14), where $\Phi=f(x^2+y^2)+2(a+f)((1-b)x+fy)x^2$.

In Case (c_{18}) the equations $F_{10}=0$, $F_{01}=0$ of (16) yield $\alpha_2=4/3$, $\alpha_1=0$ and the equations $F_{ij}=0$, $i+j=2$ imply $f=0$, $b=(9+3au-u^2)/9$. Since $u=a+d$ we obtain the set of conditions (xi) and the integrating factor (14), where $\Phi=9(x^2+y^2)+x^2((a+d)^2x+6(a+d)y-9x)$.

In Case (c_{19}) the equations of (16) give $\alpha_1=(-1)/2$, $\alpha_2=2$, $f=-a$, $b=1/4$, $c=-2$ and we obtain the set of conditions (xii) for the existence of the integrating factor (14), where $\Phi=x^2+y^2-xy^2$.

In Case (c_{20}) we have $F_{10}\equiv 2f\neq 0$. In this case we cannot construct an integrating factor (14) for system (6).

In Case (c_{21}) the equations of (16) imply $\alpha_1=0$, $\alpha_2=2$, $b=(v^3-3av^2+27a-9v)/(18v)$, $v=3(a-l)$. We obtain the set of conditions (xiii) for the existence of the integrating factor (14), where $\Phi=9(x^2+y^2)-x(3(a-l)x-3y)^2$.

In Case (c_{22}) the equations of (16) yield $\alpha_1=0$, $\alpha_2=2$, $c=-2(b+1)$. We get the set of conditions (xiv) for the existence of the integrating factor (14), where $\Phi=f(x^2+y^2)-(2ab+2bf+f)x^3+2f(a+f)x^2y-fxy^2$.

In Case (c_{23}) we obtain the set of conditions (i) ($a=(-2h)/(h^2+4)$, $c=(-h^2-8)/(h^2+4)$, $f=0$, $g=(2bh^2+8b-5h^2-4)/[2(h^2+4)]$) for the existence of the integrating factor (14).

In Case (c_{24}) the equations of (16) give $\alpha_1=1/2$, $\alpha_2=3/2$, $b=(2u^4+9uv-9v^2-6au^3)/(18uv)$. We get the set of conditions (xv) for the existence of the integrating factor (14), where $\Phi=9uv(x^2+y^2)-x(3aux-u^2x-3vy)^2$.

In Case (c_{25}) we express α_1,α_2 and a from the equations of (16). If $b=(h^3-2h^4l-16h^2l-32l)/[2h(h^2+4)]$, then we have the set of conditions (xvi) and the integrating factor (14), where $\Phi=(h^2+4)(x^2+y^2)-x(hx+2y)^2=0$ and $\alpha_1=1/2,\alpha_2=3/2$.

If $b=(h^3-h^4l-8h^2l+2h-16l)/[h(h^2+4)]$, then we obtain the set of conditions (xvii) for the existence of the integrating factor (14), where $\Phi=(h^2+4)(x^2+y^2)-x(hx+2y)^2=0$ and $\alpha_1=0,\alpha_2=3/2$.

In Case (c_{26}) the equations of (16) yield $\alpha_1=0$, $\alpha_2=3/2$, $b=(2u^4+9uv-9v^2-6au^3)/(18uv)$. We determine the set of conditions (xviii) and the integrating factor (14), where $\Phi = 9uv(x^2+y^2)+u(6au^2-9a^2u-u^3-9v)x^3+6uv(3a-u)x^2y-9v(u+v)xy^2$.

In Case (c_{27}) the equations of (16) give $\alpha_1 = 1/2$, $\alpha_2 = 3/2$, $c = 2(b-1)$. We find the set of conditions (xix) for the existence of the integrating factor (14), where $\Phi = x^2+y^2+(2b-1)xy^2$.

In Case (c_{28}) the equations of (16) yield $\alpha_1 = 0$, $\alpha_2 = 3/2$, $c = 2b-3$ and we obtain the set of conditions (xx) for the existence of the integrating factor (14), where $\Phi = x^2+y^2+x(2by^2-2y^2-x^2)$. ∎

Theorem 4.2. *The origin is a center for cubic differential system* (6) *with the invariant straight line* $1-x=0$ *and one irreducible invariant cubic* $x^2+y^2+a_{30}x^3+a_{21}x^2y+a_{12}xy^2+a_{03}y^3=0$ *if one of the conditions* (i)–(xx) *holds.*

The proof of Theorem 4.2 follows directly from Theorem 4.1 and local integrability of the cubic system (6) in the neighborhood of $O(0,0)$.

References

[1] V. V. Amel'kin, N. A. Lukashevich, A. P. Sadovskii, *Non-linear oscillations in the systems of second order*, Belarusian University Press, Belarus, 1982 (in Russian).

[2] B. Arcet, V. G. Romanovski, *On some reversible cubic systems*, Mathematics **9**(2021), 1446.

[3] C. Christopher, J. Llibre, C. Pantazi, X. Zhang, *Darboux integrability and invariant algebraic curves for planar polynomial systems*, J. Phys. A. **35**, 10(2002), 2457–2476.

[4] J. Chavarriga, J. Giné, *Integrability of cubic systems with degenerate infinity*, Differential Equations Dynam. Systems, **6**(1998), 425-–438.

[5] D. Cozma, *Darboux integrability of a cubic differential system with two parallel invariant straight lines*, Carpathian J. Math., **38**, 1(2022), 129–137.

[6] D. Cozma, *The problem of the center for cubic systems with two parallel invariant straight lines and one invariant conic*, Nonlinear Differ. Equ. and Appl., **16**(2009), 213–234.

[7] D. Cozma, *The problem of the center for cubic systems with two homogeneous invariant straight lines and one invariant conic*, Annals of Differential Equations, **26**, 4(2010), 385-399.

[8] D. Cozma, *Integrability of cubic systems with invariant straight lines and invariant conics*, Ştiinţa, Chişinău, 2013.

[9] D. Cozma, *Darboux integrability and rational reversibility in cubic systems with two invariant straight lines*, Electronic Journal of Differential Equations, **2013** 23(2013), 1–19.

[10] D. Cozma, A. Dascalescu, *Integrability conditions for a class of cubic differential systems with a bundle of two invariant straight lines and one invariant cubic*, Buletinul Academiei de Ştiinţe a Republicii Moldova. Matematica, **86**, 1(2018), 120–138.

[11] D. Cozma, A. Dascalescu, *Center conditions for a cubic system with a bundle of two invariant straight lines and one invariant cubic*, ROMAI Journal, **13**, 2(2017), 39–54.

[12] D. Cozma, A. Matei, *Center conditions for a cubic differential system having an integrating factor*, Bukovinian Mathematical Journal, **8**, 2(2020), 6–13.

[13] D. Cozma, A. Şubă, *Conditions for the existence of four invariant straight lines in a cubic differential system with a singular point of a center or a ficus type*, Bul. Acad. de Şt. a Republicii Moldova. Matematica, **3**(1993), 54–62.

[14] D. Cozma, A. Şubă, *The solution of the problem of center for cubic differential systems with four invariant straight lines*, Sci. Annals of the "Al.I.Cuza" University, Mathematics, **XLIV** (1998), s.I.a, 517–530.

[15] M. Dukarić, *On integrability and cyclicity of cubic systems*, Electr. J. Qual. Theory Differ. Eqns., **55**(2020), 1–19.

[16] M. Han, V. Romanovski, X. Zhang, *Integrability of a family of 2-dim cubic systems with degenerate infinity*, Rom. Journ. Phys., **61**, 1-2(2016), 157–166.

[17] J. Llibre, *On the centers of cubic polynomial differential systems with four invariant straight lines*, Topological Methods in Nonlinear Analysis, **55**, 2(2020), 387–402.

[18] A. M. Lyapunov, *The general problem of stability of motion*. Gostekhizdat, Moscow, 1950 (in Russian).

[19] V. G. Romanovski, D. S. Shafer, *The center and cyclicity problems: a computational algebra approach*, Birkhäuser, Boston, Basel, Berlin, 2009.

[20] A. P. Sadovskii, T. V. Shcheglova, *Solution of the center-focus problem for a nine-parameter cubic system*, Differential Equations, **47**, 2(2011), 208–223.

[21] A. Şubă, D. Cozma, *Solution of the problem of center for cubic differential systems with three invariant straight lines in generic position*, Qualitative Theory of Dynamical Systems, **6**(2005), 45–58.

[22] Xiang Zhang, *Integrability of Dynamical Systems: Algebra and Analysis*, Springer Nature Singapure, Singapure, 2017.

ROMAI J., v.17, no.1(2021), 87–104

WEAK DISCONTINUITY WAVES IN AN ULTRA-RELATIVISTIC HEAT-CONDUCTING FLUID IN A GENERAL FRAME

Sebastiano Giambò[1], Veronica La Rosa[1], Giuseppa Muscianisi[2]
[1] *Dep. of Mathematics, University of Messina, Messina, Italy*
[2] *CINECA - SCAI Department, Casalecchio di Reno, Italy*
sgiambo@unime.it, veronica-larosa@istruzione.it, g.muscianisi@cineca.it

Abstract A second-order theory for relativistic heat-conducting fluids is derived in a general frame, independently of the conventions used by Eckart [3] and Landau-Lifshitz [4].

Based on the hydrodynamic for such a relativistic heat–conducting fluid, the propagation of weak discontinuity in the ultra–relativistic limit is studied.

The general features of weak discontinuity waves are presented and two kinds of waves are identified: the hydrodynamic and the heat waves. Both these two kind of waves can be specialized in Landau–Lifshitz [15] and Eckart [16] schemes respectively.

Moreover, a differential equation, named growth equation, is obtained to describe the decay and the growth of discontinuity, for the hydrodynamical wave propagating along the rays. The solution is in an integral form and special case of diverging waves are also discussed.

Keywords: Hyperbolic relativistic model, characteristic velocities, relativity, fluid dynamics, irreversible thermodynamics, weak discontinuity.
2020 MSC: 35L60, 83A05, 53E10, 53E20.

1. INTRODUCTION

The study of space–time evolution and non–equilibrium properties of matter produced in high energy heavy ion collisions, such as those at the Relativistic Heavy Ion Collider (RHIC) and Large Hadron Collider (LHC) using relativistic dissipative fluid dynamics, is relevant to understand the observables. High energy heavy ion collisions offer the opportunity to study the properties of hot and dense matter. To do so, its space-time evolution has to be followed, which is affected both by the state's equation and by dissipative, non–equilibrium processes. In contexts like RHIC and LHC [12, 26, 27], some interesting applications require to develop a robust model for dissipative processes in relativistic hydrodynamics [28, 29, 30, 31].

In order to unify in a single and coherent scheme all irreversible phenomena occurring in a simple fluid or in mixtures, standard non–equilibrium thermodynamics works with two basic ideas [1, 2]. The first one is the local equilibrium hypothesis, whose mathematical expression is given by the equilibrium Gibbs law in its local form. It implies that, out of equilibrium, the basic state functions such as the entropy, depend locally on the same of thermodynamic variables as in equilibrium. The second idea is that, in the presence of dissipative processes, there is a local entropy source strength τ (entropy variation per unit volume and unit time), which, by the second law of thermodynamics, is always non–negative.

The first theories of relativistic dissipative fluid dynamics are due to Eckart [3] and Landau–Lifshitz [4]. The difference in formal appearance stems from different choices for the definition of the hydrodynamical 4–velocity. In Eckart's formulation [3], the 4–velocity is directly related to the particle flux, while, in Landau–Lifshitz's approach [4], it is directly related to the energy flux. Both these conventional theories of dissipative fluid dynamics are based on the assumption that the entropy 4–current contains terms up to linear order in dissipative quantities and hence they are referred to as *first order theories* of dissipative fluids. By this assumption, the resulting equation for heat–flux is linear in the thermodynamic forces, and the resulting equations of motion are parabolic in structure and does not satisfy causality principle [5].

In order to solve this feature, extended theories of dissipative fluid were introduced. These causal theories are based on the assumption that the entropy 4–current should include quadratic terms in the dissipative fluxes and hence they are referred to as *second order theories*. Two path can be followed: the first is named the extended irreversible thermodynamics, developed by Jou et al. [6], Müller and Ruggeri [7], that allows the inclusion of the dissipative quantities in the expression of the entropy density and the entropy flux due to a generalized Gibbs relation,; the second, following Israel and Stewart [8, 9, 10], Muronga [11, 12], but also Giambò et al. [15, 16], introduces additional dynamical fields through the assumption that the entropy 4–current includes quadratic terms in the heat flux.

These models have been taken up and developed later by Garcia–Colin and Sandoval–Villalbalzo [17], Mondragon–Suarez et al. [18], Lopez–Monsalvo [19]. There is also an important approach to the heat conduction, in which a multi-fluids system is considered whose species are represented by a particle number density current and an entropy flux, in general not aligned with the particle flux [20, 21]. This model has been recovered and developed recently by Lopez-Monsalvo and Andersson [22] and Andersson and Comer [23]. The resulting equations for the dissipative fluxes are hyperbolic and a causal propagation of signals is admitted [8, 9, 10, 11, 12, 24, 25].

In the second order theories the space of thermodynamic quantities is extended to include the dissipative quantities which are treated as field variables as well.

In light of some recent results due to Silva et al. [32], Heinz et al. [33], Ván and Biró [34], Muronga [12], Maartens [30], the authors aim to develop a general relativistic second-order causal theory for heat-conducting, viscous and particle–creating relativistic fluid valid both in Eckart and Landau-Lifshitz frames.

The present paper is organized as follows. After a brief review (Section 2) of basic concepts of equilibrium thermodynamics for relativistic ideal fluids, i.e. every dissipative effects are neglected and the entropy 4-current is conserved, in Section 3 a second-order theory for relativistic heat-conducting fluids is formulated. By using the fundamental Gibbs equation at equilibrium, i.e. no extra terms are added in this fundamental thermodynamic relations although dissipative effects are considered, the expression for the non-equilibrium entropy 4-current is derived, depending on the heat flux and the equilibrium variables as well. By imposing the second law of thermodynamics, hyperbolic transport equations for dissipative fluxes are obtained. In Section 4, weak discontinuity wave propagation compatible with the hyperbolic system of governing equations for an ultra–relativistic heat–conducting fluid is studied. Among others, two propagation modes are found: the hydrodynamic and the heat waves. In Section 5, the discontinuities associated to hydrodynamic wave are derived, as well as the transport equation for the discontinuity amplitude, still in the ultra-relativistic case. Finally, the behaviour of discontinuity amplitude at the wave front is investigated in detail. Comments and conclusions are reported in Section 6.

Throughout the article, a coordinate system x^α, being $x^0 = t$ the time and x^i the spatial coordinates in the flat-space time of special relativity is introduced. The units adopted are such that the velocity of light is unitary. The sign convention used follows the time-like convention with the signature $(+,-,-,-)$, and u^α is a time-like vector, $u^\alpha u_\alpha > 0$. The metric tensor is taken to be $g^{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1)$, the Minkowski tensor.

Greek indices range over 0, 1, 2, 3, and the Latin ones over 1, 2, 3.

The notation $\partial_\alpha = \partial/\partial x^\alpha$ represents the partial derivative with respect to x^α.

2. BASIC OF EQUILIBRIUM FLUID DYNAMICS

When we study the equilibrium thermodynamics three important variables must be taken into account: the energy density ρ, the particle number density r and the specific entropy S. The energy density and the particle number density are related by

$$\rho = r(1+\varepsilon)\,, \tag{1}$$

where ε is the specific internal energy.
These basic quantities will be referred to as primary thermodynamic variables, from which to deduce all other state variables, such as the pressure p.

From the equation of state for the entropy density $s = rS$, $s = s(\rho, r)$, and Euler relation

$$\mu = \frac{\rho + p}{r} - TS \,, \tag{2}$$

defining chemical potential μ, the fundamental Gibbs equation can be written as:

$$Tds = d\rho - \mu dr \,, \tag{3}$$

where T denotes the temperature. Eq. (2) defines the last unknown thermodynamic function p.

In relativistic fluid dynamics it is useful to rewrite the thermodynamic quantities in terms of covariant objects, namely the 4–vector particle number current R^α_{eq}, the energy-momentum tensor $T^{\alpha\beta}_{eq}$ and the entropy 4–current S^α_{eq}, defined by [4, 35]

$$R^\alpha_{eq} = ru^\alpha \,, \tag{4}$$

$$T^{\alpha\beta}_{eq} = \rho u^\alpha u^\beta - p\gamma^{\alpha\beta} \,, \tag{5}$$

$$S^\alpha_{eq} = rSu^\alpha \,, \tag{6}$$

where u^α is the unitary hydrodynamical 4–velocity and $\gamma^{\alpha\beta} = g^{\alpha\beta} - u^\alpha u^\beta$ is the spatial projection tensor orthogonal to u^α. The subscript eq denotes quantities evaluated at equilibrium.

Relating the pressure to the energy density and the particle number density by a state equation, the equilibrium state of an ideal relativistic fluid can be described by the five independent variables, ρ, r, u^α.

The thermodynamic relation (3) can be written in a form involving the covariant variables R^α_{eq}, $T^{\alpha\beta}_{eq}$ and S^α_{eq}:

$$dS^\alpha_{eq} = -\frac{\mu}{T} dR^\alpha_{eq} + \frac{1}{T} u_\beta dT^{\alpha\beta}_{eq} \,. \tag{7}$$

Starting from (2) and (3), simple computations yield the following relations

$$S^\alpha_{eq} = \frac{p}{T} u^\alpha - \frac{\mu}{T} R^\alpha_{eq} + \frac{1}{T} u_\beta T^{\alpha\beta}_{eq} \,, \tag{8}$$

$$d\left(\frac{p}{T} u^\alpha\right) = R^\alpha_{eq} d\left(\frac{\mu}{T}\right) - T^{\alpha\beta}_{eq} d\left(\frac{u_\beta}{T}\right) \,. \tag{9}$$

Thus, the conservation laws for particle number 4–current, energy- momentum tensor and entropy 4–current, describing the motion of an ideal relativistic fluid,

$$\partial_\alpha R^\alpha_{eq} = 0 \,, \tag{10}$$

$$\partial_\alpha T^{\alpha\beta}_{eq} = 0\,, \tag{11}$$

$$\partial_\alpha S^{\alpha}_{eq} = 0\,, \tag{12}$$

can be written in the following covariant form

$$\begin{cases} u^\alpha \partial_\alpha r + r \partial_\alpha u^\alpha = 0\,, \\ u^\alpha \partial_\alpha \rho + (\rho + p)\partial_\alpha u^\alpha = 0\,, \\ (\rho + p) u^\beta \partial_\beta u^\alpha - \gamma^{\alpha\beta}\partial_\beta p = 0\,, \\ u^\alpha \partial_\alpha (rS) + rS\partial_\alpha u^\alpha = 0\,, \end{cases} \tag{13}$$

where equation $(13)_2$ is the projection of (11) along u^α, while $(13)_3$ is the spatial projection of (11).

3. NON–EQUILIBRIUM STATES

In order to take into account dissipative processes, as heat conduction, additional terms in the expressions of primary variables, R^α_{eq}, $T^{\alpha\beta}_{eq}$ and S^α_{eq} have to be considered.

The next step is to find evolution equations for these extra variables. Whereas the evolution equations for the equilibrium variables are given by the usual conservation laws, general criteria do not exist concerning the evolution equations of the dissipative fluxes, with the exception of the restriction imposed on them by the second law of thermodynamics.

Moreover, the presence of a heat transfer usually involves a problem regarding the definition of the hydrodynamical 4–velocity u^α. Following Landau-Lifshitz approach [4], u^α is defined as the 4–velocity of energy transport, i.e. u^α is the unique unit time–like eigenvector of $T^{\alpha\beta}$; in Eckart's formulation, u^α is identified by the 4–velocity of particle transport (particle frame) [3]. Formally, the particle frame is the unique time–like vector parallel to R^α_{eq}. Here, a unified covariant description of a relativistic heat–conducting fluid is developed in a general frame independently of the conventions used by Eckart and Landau–Lifshitz. The theories of Eckart and Landau–Lifshitz will be seen to be special cases of this general formulation. Thus, in presence of irreversible processes, small terms ΔR^α, $\Delta T^{\alpha\beta}$ and ΔS^α are added in (4), (5) and (6) respectively, namely

$$R^\alpha = r u^\alpha + \Delta R^\alpha\,, \qquad \text{with} \quad u_\alpha \Delta R^\alpha = 0\,, \tag{14}$$

$$T^{\alpha\beta} = T^{\alpha\beta}_{eq} + \Delta T^{\alpha\beta}\,, \tag{15}$$

$$S^\alpha = rSu^\alpha + \Delta S^\alpha \tag{16}$$

the corresponding conservation laws for particle current and for energy-momentum tensor must be satisfied

$$\partial_\alpha R^\alpha = 0\,, \tag{17}$$

$$\partial_\alpha T^{\alpha\beta} = 0\,, \tag{18}$$

while the second law of thermodynamics requires the entropy source to be non negative

$$\partial_\alpha S^\alpha \geq 0\,. \tag{19}$$

The condition $u_\alpha \Delta R^\alpha = 0$ means that ΔR^α is a space–like vector. The deviations ΔR^α, $\Delta T^{\alpha\beta}$ and ΔS^α from local equilibrium contain the information about particle drift, heat flux and entropy 4–current at non–equilibrium states.

Now, the expressions of such deviations have to be deduced and the evolution equation for particle drift and heat flux must be determined, according to the second law of thermodynamics.

The conservation of R^α, (17), yields

$$u^\alpha \partial_\alpha r + r\partial_\alpha u^\alpha + \partial_\alpha(\Delta R^\alpha) = 0\,, \tag{20}$$

whereas, from equation (18) it follows that

$$\partial_\alpha T^{\alpha\beta}_{eq} + \partial_\alpha(\Delta T^{\alpha\beta}) = 0\,. \tag{21}$$

Using $(13)_1$, $(13)_2$, (20) and (21), and remembering the definition of the chemical potential μ, from Gibbs equation (3) the following relation on the covariant derivative along the world lines is obtained

$$T\partial_\alpha(rSu^\alpha) = \mu\partial_\alpha(\Delta R^\alpha) - u_\beta\partial_\alpha(\Delta T^{\alpha\beta})\,. \tag{22}$$

In order to derive the additional equations for ΔR^α and $\Delta T^{\alpha\beta}$ from a phenomenological treatment, the expression of the non–equilibrium (also referred as extended) entropy 4–current is needed [7]. The most general non–equilibrium entropy 4–current, $S^\alpha = S^\alpha(R^\alpha, T^{\alpha\beta})$, for a relativistic fluid in presence of particle drift and heat conduction only, i.e. other dissipative phenomena are neglected, has the form [8, 9, 10]

$$S^\alpha = \frac{p}{T}u^\alpha - \frac{\mu}{T}R^\alpha + \frac{1}{T}u_\beta T^{\alpha\beta} + Q^\alpha(\Delta R^\alpha, \Delta T^{\alpha\beta})\,, \tag{23}$$

where Q^α is a function of the deviations ΔR^α and $\Delta T^{\alpha\beta}$.

For small deviations, in order to obtain causal and hyperbolic equations, it is sufficient to keep only quadratic terms in the Taylor expansions of Q^α.

Thus, for the fluids under investigation, by virtue of (2), (4) and (5), together with the relations [35]

$$\Delta R^\alpha = \nu^\alpha\,, \qquad \nu^\alpha u_\alpha = 0\,, \qquad \nu^\alpha \nu_\alpha < 0\,, \tag{24}$$

$$\Delta T^{\alpha\beta} = q^\alpha u^\beta + q^\beta u^\alpha \,, \qquad q^\alpha u_\alpha = 0 \,, \qquad q_\alpha q^\alpha < 0 \,, \tag{25}$$

the most general algebraic form for the 4-current of extended entropy, at most second order in the dissipative fluxes ΔR^α and $\Delta T^{\alpha\beta}$, is [8, 9, 10]

$$S^\alpha = rSu^\alpha - \frac{\mu}{T}\nu^\alpha + \frac{1}{T}q^\alpha + \{\nu_\mu (\alpha_{11}\nu^\mu + \alpha_{12}q^\mu) + q_\mu (\alpha_{21}\nu^\mu + \alpha_{22}q^\mu)\} u^\alpha \,, \tag{26}$$

where q^α and ν^α are, respectively, the heat flow and the particle drift, and the definition of R^α and $T^{\alpha\beta}$ have been considered. Recalling that the phenomenological coefficients α_{ij} are supposed to obey Onsanger's reciprocity relations, it follows that $\alpha_{12} = \alpha_{21}$. The deviations ΔR^α and $\Delta T^{\alpha\beta}$ describe the effects due to the presence of particle drift and heat flux in non–equilibrium states. These 4–vector are space–like and orthogonal to the 4–velocity u_α.

From (26) it follows that the effective entropy density measured by co–moving observer is

$$u_\alpha S^\alpha = rS + \{\nu_\mu (\alpha_{11}\nu^\mu + \alpha_{12}q^\mu) + q_\mu (\alpha_{21}\nu^\mu + \alpha_{22}q^\mu)\} \,. \tag{27}$$

Since the entropy density has a maximum at equilibrium, the conditions $\nu^\alpha\nu_\alpha < 0$ and $q^\alpha q_\alpha$ imply that the quadratic form is negative semi definite.

The divergence of extended current (26), together with equation (22), leads to the following expression for the generalized entropy production

$$\begin{aligned}
\partial_\alpha S^\alpha = &-\nu^\alpha \left\{\partial_\alpha \left(\tfrac{\mu}{T}\right) - 2\alpha_{11}u^\lambda\partial_\lambda\nu_\alpha - 2\alpha_{12}u^\lambda\partial_\lambda q_\alpha - \alpha_{12}q_\alpha\partial_\lambda u^\lambda \right.\\
&\left. -q_\alpha\partial_\lambda(\alpha_{12}u^\lambda) - \nu_\alpha\partial_\lambda(\alpha_{11}u^\lambda)\right\}\\
&-q^\alpha \left\{-\partial_\alpha \left(\tfrac{1}{T}\right) - \tfrac{1}{T}u^\lambda\partial_\lambda u_\alpha - \nu_\alpha\partial_\lambda(\alpha_{12}u^\lambda) - 2\alpha_{12}u^\lambda\partial_\lambda\nu_\alpha \right.\\
&\left. -\alpha_{12}\nu_\alpha\partial_\lambda u^\lambda - q_\alpha\partial_\lambda(\alpha_{22}u^\lambda) - 2\alpha_{22}u^\lambda\partial_\lambda q_\alpha\right\} \geq 0 \,.
\end{aligned} \tag{28}$$

Since this production has to be non-negative and the particle drift and the heat flux are space–like vectors, the following phenomenological equations for ν_α and q_α must hold

$$\begin{aligned}
\nu_\alpha = &-\chi_1\gamma_\alpha^\beta \left\{-\mu\partial_\beta T + T\partial_\beta\mu - 2\alpha_{11}T^2u^\lambda\partial_\lambda\nu_\beta - 2\alpha_{12}T^2u^\lambda\partial_\lambda q_\beta \right.\\
&\left. -\alpha_{12}T^2q_\beta\partial_\lambda u^\lambda - q_\beta T^2\partial_\lambda(\alpha_{12}u^\lambda) - \nu_\beta T^2\partial_\lambda(\alpha_{11}u^\lambda)\right\} \,,
\end{aligned} \tag{29}$$

$$\begin{aligned}
q_\alpha = &-\chi_2\gamma_\alpha^\beta \left\{\partial_\beta T - Tu^\lambda\partial_\lambda u_\beta - \nu_\beta T^2\partial_\lambda(\alpha_{12}u^\lambda) - 2\alpha_{12}T^2u^\lambda\partial_\lambda\nu_\beta \right.\\
&\left. -q_\beta T^2\partial_\lambda(\alpha_{22}u^\lambda) - \alpha_{12}T^2\nu_\beta\partial_\lambda u^\lambda - 2\alpha_{22}T^2u^\lambda\partial_\lambda q_\beta\right\} \,,
\end{aligned} \tag{30}$$

where the phenomenological coefficients χ_1 and χ_2 (≥ 0) are, respectively, the diffusion coefficient and heat conductivity coefficient of the fluid.

Thus, the set of hyperbolic equations for a relativistic heat–conducting fluid is

$$\begin{cases} u^\alpha \partial_\alpha r + r\partial_\alpha u^\alpha + \partial_\alpha \nu^\alpha = 0\,, \\ rfu^\beta \partial_\beta u^\alpha - \gamma^{\alpha\beta}\partial_\beta p + u^\beta \partial_\beta q^\alpha + q^\beta \partial_\beta u^\alpha + q^\alpha \partial_\beta u^\beta + u^\alpha u^\beta q^\lambda \partial_\beta u_\lambda = 0\,, \\ rTu^\alpha \partial_\alpha S + \partial_\alpha q^\alpha - f\partial_\alpha \nu^\alpha - u^\alpha q_\beta \partial_\alpha u^\beta = 0\,, \\ \nu_\alpha + \chi_1 \gamma^\beta_\alpha \Big\{ -\mu\partial_\beta T + T\partial_\beta \mu - 2\alpha_{11}T^2 u^\lambda \partial_\lambda \nu_\beta - 2\alpha_{12}T^2 u^\lambda \partial_\lambda q_\beta \\ \qquad -\alpha_{12}T^2 q_\beta \partial_\lambda u^\lambda - q_\beta T^2 \partial_\lambda(\alpha_{12}u^\lambda) - \nu_\beta T^2 \partial_\lambda(\alpha_{11}u^\lambda) \Big\} = 0\,, \\ q_\alpha + \chi_2 \gamma^\beta_\alpha \Big\{ \partial_\beta T - Tu^\lambda \partial_\lambda u_\beta - \nu_\beta T^2 \partial_\lambda(\alpha_{12}u^\lambda) - 2\alpha_{12}T^2 u^\lambda \partial_\lambda \nu_\beta \\ \qquad -q_\beta T^2 \partial_\lambda(\alpha_{22}u^\lambda) - \alpha_{12}T^2 \nu_\beta \partial_\lambda u^\lambda - 2\alpha_{22}T^2 u^\lambda \partial_\lambda q_\beta \Big\} = 0\,, \end{cases} \tag{31}$$

where the 11 independent field variables are r, u^α, T, ν^α and q^α, and f is the index of the fluid, defined by the relation $rf = p + \rho$. The pressure p, the entropy S and the energy density ρ are all functions of the state variables r and T. Also the coefficients α_{11}, α_{12} and α_{22} can be considered as functions of the particle number density and temperature.

4. WEAK DISCONTINUITIES PROPAGATION

A very interesting aspects of dissipative hydrodynamics is the study of dispersion relation for the properties of a perturbed plane-waves [26] or the study of characteristic surfaces [36]. In the present paper it is studied a weak discontinuity wave propagation compatible with the hyperbolic system of governing equations (31), [42, 43, 44, 45].

In a domain Ω of space-time V_4, let Σ be a moving singular hyper–surface, defined by

$$\varphi(x^\alpha) = 0\ , \qquad x^\alpha = x^\alpha(w^A)\ , \tag{32}$$

where w^A, $A = 0, 1, 2$, are the coordinates on Σ; it is supposed to be a weak-discontinuity surface on which the variables r, u^α, T, ν^α and q^α are continuous, but their normal derivatives may exhibit jump discontinuities. The jump on Σ in a flow quantity F is denoted by $[F] = F_1 - F_0$, where F_0 and F_1 are the limits of F from each side of Σ.

Thus, the jumps on Σ in the normal derivatives of the particle number r, the unitary hydrodynamical 4–velocity u^α, the temperature T, the particle drift ν^α and the heat flux q^α can be denoted as follows

$$\begin{aligned} &[\partial_\alpha r]\, N^\alpha = -\xi\,, \qquad [\partial_\beta u^\alpha]\, N^\beta = -\omega^\alpha\,, \\ &[\partial_\alpha T]\, N^\alpha = -\vartheta\,, \qquad [\partial_\beta \nu^\alpha]\, N^\beta = -\eta^\alpha\,, \qquad [\partial_\beta q^\alpha]\, N^\beta = -\pi^\alpha\,. \end{aligned} \tag{33}$$

N^α is the space–like unit vector orthogonal to Σ, $N^\alpha N_\alpha = -1$.

With respect to a rest frame determined by some preferred time–like unit vector u^α, the velocity, λ, of propagation in the direction of some orthogonal unit space–like vector n^α will be given, for a suitably normalized N^α, by

$$\lambda = \frac{L}{\ell}\,, \qquad n^\alpha = \frac{1}{\ell}\left(N^\alpha - L u^\alpha\right)\,, \tag{34}$$

where

$$L = u^\alpha N_\alpha\,, \qquad \ell^2 = 1 + L^2\,. \tag{35}$$

If the pressure is a function of the energy density, i.e. $p = p(\rho) = p\,[\rho(r,T)]$, the Gibbs relation yields

$$\left(\frac{\partial p}{\partial r}\right)_T = \frac{dp}{d\rho}\left(\frac{\partial \rho}{\partial r}\right)_T = p'(f + TrS'_r)\,, \tag{36}$$

$$\left(\frac{\partial p}{\partial T}\right)_r = \frac{dp}{d\rho}\left(\frac{\partial \rho}{\partial T}\right)_r = p'TrS'_T \tag{37}$$

For an ultra–relativistic fluid, the energy density ρ can be approximated to

$$\rho = r(1+\varepsilon) = r + r\varepsilon \simeq r\varepsilon = C_V r T\ , \tag{38}$$

where C_V is the specific heat at constant volume. The pressure depends only on ρ, $p = \rho(\gamma - 1) = RrT$, where

$$\gamma = \frac{C_P}{C_V}\,, \qquad R = C_P - C_V\,, \tag{39}$$

R is the universal constant of gas and C_P is the specific heat at constant pressure. Then, we obtain the following relations for the index of the fluid and for the chemical potential holds

$$f = T(C_V + R)\,, \qquad \mu = f - TS\,. \tag{40}$$

Moreover, the entropy, $S = S(r,T)$, is such that

$$\left(\frac{\partial S}{\partial r}\right)_T = -\frac{R}{r}\,, \qquad \left(\frac{\partial S}{\partial T}\right)_r = \frac{C_V}{T}\,. \tag{41}$$

Making use of Hadamard's method for characteristic hyper-surfaces of possible discontinuities [37, 38, 39], taking account of the first order compatibility conditions derived by Thomas [40, 41], and assuming that the state ahead of Σ is uniform and in thermodynamic equilibrium, i.e. $\nu^\alpha = 0$ and $q^\alpha = 0$ along the unperturbed flow direction from (31) the following system in the unknown discontinuities ξ, ω^α, ϑ, η^α and π^α, is obtained

$$\begin{cases} L\xi + r\omega^\alpha N_\alpha + \eta^\alpha N_\alpha = 0\,, \\ rfL\omega^\beta - \gamma^{\alpha\beta} N_\alpha (RT\xi + rR\vartheta) + L\pi^\beta = 0\,, \\ RTL\xi - rC_V L\vartheta - \pi_N + f\eta_N = 0\,, \\ \gamma^\beta_\alpha N_\beta \left(-\dfrac{R}{r} T^2 \xi + C_V T\vartheta \right) + 2\alpha_{11} T^2 L\eta_\alpha + 2\alpha_{12} T^2 L\pi_\alpha = 0\,, \\ -\gamma^\beta_\alpha N_\beta \vartheta + TL\omega_\alpha + 2\alpha_{12} T^2 L\eta_\alpha + 2\alpha_{22} T^2 L\pi_\alpha = 0\,. \end{cases} \tag{42}$$

As first, from system (42), can be obtained the solution $L = 0$, which represents a wave moving with the fluid. The corresponding discontinuities are given by

$$\begin{aligned} &\omega^\alpha N_\alpha = -\frac{1}{r}\eta^\alpha N_\alpha\,, \\ &\pi^\alpha N_\alpha = f\eta^\alpha N_\alpha\,, \\ &\xi = 0\,, \\ &\vartheta = 0\,. \end{aligned} \tag{43}$$

Since the coefficients characterizing the discontinuities exhibit seven degrees of freedom, the system (42) admits seven independent eigenvectors corresponding to $L = 0$ in the space of field variables.

In what follows, it is suppose $L \neq 0$. The equations $(42)_2$, $(42)_4$ and $(42)_5$, multiplying by N^α, give the reduced system

$$\begin{cases} L\xi + r\omega_N + \eta_N = 0\,, \\ RT\ell^2\xi + rR\ell^2\vartheta + rfL\omega_N + L\pi_N = 0\,, \\ RTL\xi - rC_V L\vartheta - \pi_N + f\eta_N = 0\,, \\ \dfrac{R}{r}T^2\ell^2\xi - C_V T\ell^2\vartheta + 2\alpha_{11}T^2L\eta_N + 2\alpha_{12}T^2L\pi_N = 0\,, \\ \ell^2\vartheta + TL\omega_N + 2\alpha_{12}T^2L\eta_N + 2\alpha_{22}T^2L\pi_N = 0\,, \end{cases} \tag{44}$$

in the unknown discontinuities ξ, ω_N, ϑ, η_N and π_N, being ω_N, η_N and π_N the components of the vectors ω^α, η^α and π^α normal to Σ.

System (44) admits non trivial solutions if, and only if, the determinant of the coefficient matrix vanishes. Thus, the characteristic equation for the velocity of propagation $\lambda = L/\ell$ is

$$\begin{aligned} \left(C_V L^2 - R\ell^2\right)\big\{\left(4\alpha_{11}\alpha_{22}r^2fT^4 - 2\alpha_{11}rT^3 - 4\alpha_{12}^2r^2fT^4\right)L^2 \\ - \left(2\alpha_{12}rfT^3 + 2\alpha_{11}rT^3 + 2\alpha_{22}rf^2T^3 - fT^3\right)\ell^2\big\} = 0\,. \end{aligned} \tag{45}$$

Thus, two well–behaved propagation modes exist, which are interpretable as

- a *hydrodynamic wave*, with velocity $\lambda_1^2 = \dfrac{dp}{d\rho} = \dfrac{R}{C_V}$;
- a *heat wave*, with velocity $\lambda_2^2 = \dfrac{4\alpha_{12}rfT + 2\alpha_{11}rT + 2\alpha_{22}rTf^2 - f}{4(\alpha_{11}\alpha_{22} - \alpha_{12}^2)r^2T^2f - 2\alpha_{11}rT}$.

5. DISCONTINUITY TRANSPORT EQUATION

In this section the discontinuities associated to the hydrodynamic wave for an ultra-relativistic fluid, and the transport equation describing the evolution, along the rays, of the amplitude of the discontinuities are determined. Finally, the main features of the solution of this equation are investigated.

From system (44), the discontinuities associated to the hydrodynamic wave are

$$\begin{aligned} &\omega^\alpha = n^\alpha\psi\,, \\ &\xi = \frac{r\ell}{L}\psi\,, \\ &\vartheta = \frac{TL}{\ell}\psi\,, \\ &\eta_\alpha = 0\,, \qquad \eta_N = 0\,, \\ &\pi_\alpha = 0\,, \qquad \pi_N = 0\,, \end{aligned} \tag{46}$$

where $\psi = -\omega_N/\ell$ is the amplitude of discontinuity associated to this wave.

In order to deduce the transport equation for ψ, the discontinuities in the second partial derivatives of the field variables must be considered and are denoted by

$$\begin{aligned} &\left[\frac{\partial^2 r}{\partial x^\alpha \partial x^\beta}\right] N^\alpha N^\beta = \bar{\xi}\,, \qquad \left[\frac{\partial^2 u^\alpha}{\partial x^\beta \partial x^\gamma}\right] N^\beta N^\gamma = \bar{\omega}^\alpha\,, \\ &\left[\frac{\partial^2 T}{\partial x^\alpha \partial x^\beta}\right] N^\alpha N^\beta = \bar{\theta}\,, \qquad \left[\frac{\partial^2 \nu^\alpha}{\partial x^\beta \partial x^\gamma}\right] N^\beta N^\gamma = \bar{\eta}^\alpha\,, \\ &\left[\frac{\partial^2 q^\alpha}{\partial x^\beta \partial x^\gamma}\right] N^\beta N^\gamma = \bar{\pi}^\alpha\,. \end{aligned} \tag{47}$$

Differentiating the equations (31) with respect to x^γ, computing the jumps and then multiplying by N^γ, using (47) and the first and second order compatibility conditions [40, 41], the following system for an ultra-relativistic fluid is obtained

$$\begin{cases} L\bar{\xi} + r\bar{\omega}_N + \bar{\eta}_N = A\,, \\ RT\ell^2\bar{\xi} + rR\ell^2\bar{\vartheta} + rfL\bar{\omega}_N + L\bar{\pi}_N = B\,, \\ RTL\bar{\xi} - rC_V L\bar{\theta} + f\bar{\eta}_N - \bar{\pi}_N = C\,, \\ \dfrac{R}{r}T^2\ell^2\bar{\xi} - C_V T\ell^2\bar{\vartheta} + 2\alpha_{11}T^2 L e\bar{t}a_N + 2\alpha_{12}T^2 L\bar{\pi}_N = D\,, \\ \ell^2\bar{\vartheta} + TL\bar{\omega}_N + 2\alpha_{12}T^2 L\bar{\eta}_N + 2\alpha_{22}T^2 L\bar{\pi}_N = E\,, \end{cases} \tag{48}$$

where

$$\begin{aligned} A &= -2\xi\omega_N - \ell\frac{d\xi}{d\sigma} - ra^{AB}\omega^\alpha_{,A}x_{\alpha,B} - a^{AB}\eta^\alpha_{,A}x_{\alpha,B}\,, \\ B &= -rf\ell\frac{d\omega_N}{d\sigma} - RTL\ell\frac{d\xi}{d\sigma} - rRL\ell\frac{d\vartheta}{d\sigma} - \ell\frac{d\pi_N}{d\sigma} - 2R\ell^2\xi\vartheta - L^2\pi^\lambda\omega_\lambda \\ &\quad -3\pi_N\omega_N - rf\omega_N^2 - (f+2RT)L\xi\omega_N - r(C_V+3R)L\vartheta\omega_N\,, \end{aligned} \tag{49}$$

$$
\begin{aligned}
C &= -RT\ell\frac{d\xi}{d\sigma} + rC_V\ell\frac{d\vartheta}{d\sigma} + a^{AB}\pi^\alpha_{,A}x_{\alpha,B} - fa^{AB}\eta^\alpha_{,A}x_{\alpha,B} + rC_V\vartheta\omega_N \\
&\quad -(R-C_V)L\xi\vartheta - RT\xi\omega_N - (R+C_V)\vartheta\eta_N - L\pi_\beta\omega^\beta\,,
\end{aligned}
$$

$$
\begin{aligned}
D &= -\left\{L\ell\frac{d\vartheta}{d\sigma} + L\ell\frac{d\xi}{d\sigma} + 2\alpha_{11}T^2\ell\frac{d\eta_N}{d\sigma} + 2\alpha_{12}T^2\ell\frac{d\pi_N}{d\sigma} + 2\frac{R}{r}T\ell^2\xi\vartheta \right.\\
&\quad -C_V\ell^2\vartheta^2 - \frac{R}{r^2}T^2\ell^2\xi^2 + 3\frac{\partial\alpha_{11}}{\partial r}T^2L\xi\eta_N + 3\alpha_{11}T^2\omega_N\eta_N \\
&\quad +4\alpha_{11}TL\vartheta\eta_N + 3\frac{\partial\alpha_{12}}{\partial r}T^2L\xi\pi_N + 3\frac{\partial\alpha_{12}}{\partial T}T^2L\vartheta\pi_N \\
&\quad +4\alpha_{12}T^2\omega_N\pi_N + 4\alpha_{12}TL\vartheta\pi_N + 3\frac{\partial\alpha_{11}}{\partial T}T^2L\vartheta\eta_N - 2C_VTL\vartheta\omega_N \\
&\quad \left. +2\frac{R}{r}T^2L\xi\omega_N + 2\alpha_{11}T^2L^2\omega^\beta\eta_\beta + 2\alpha_{12}T^2L^2\omega^\beta\pi_\beta + \frac{\eta_N}{\chi_1}\right\},
\end{aligned}
$$

$$
\begin{aligned}
E &= -\left\{L\ell\frac{d\vartheta}{d\sigma} + T\ell\frac{d\omega_N}{d\sigma} + 2\alpha_{12}T^2\ell\frac{d\eta_N}{d\sigma} + 2\alpha_{22}T^2\ell\frac{d\pi_N}{d\sigma} + 3L\vartheta\omega_N \right.\\
&\quad +T\omega_N^2 + 3\frac{\partial\alpha_{12}}{\partial r}T^2L\xi\eta_N + 3\frac{\partial\alpha_{12}}{\partial T}T^2L\vartheta\eta_N + 4\alpha_{12}T^2\omega_N\eta_N \\
&\quad +4\alpha_{12}T^2\omega_N\eta_N + 4\alpha_{12}TL\vartheta\eta_N + 3\frac{\partial\alpha_{22}}{\partial r}T^2L\xi\pi_N + 3\frac{\partial\alpha_{22}}{\partial T}T^2L\vartheta\pi_N \\
&\quad 3\alpha_{22}T^2\omega_N\pi_N + 4\alpha_{22}TL\vartheta\pi_N - TL^2\omega^\beta\omega_\beta - 2\alpha_{12}T^2L^2\omega^\beta\eta_\beta \\
&\quad \left. -2\alpha_{22}T^2L^2\omega^\beta\pi_\beta + \frac{\pi_N}{\chi_2}\right\}.
\end{aligned}
$$

In (48)-(49), a^{AB} are the components of the first fundamental covariant tensor of Σ [40, 41], and $x_{\alpha,B} = \partial x^\alpha/\partial w^B$. Moreover, the expression of the derivative along the ray is used, which for a given function F is

$$\frac{dF}{d\sigma} = \frac{1}{\ell}a^{AB}u^\alpha x_{\alpha,B}F_{,A}\,,$$

where σ is the ray parameter. The derivatives along the rays of the jumps of u^α ν_α and q_α are given by

$$\frac{d\omega_N}{d\sigma} = \frac{d\omega^\alpha}{d\sigma}N_\alpha\,, \qquad \frac{d\eta_N}{d\sigma} = \frac{d\eta_\alpha}{d\sigma}N^\alpha\,, \qquad \frac{d\pi_N}{d\sigma} = \frac{d\pi_\alpha}{d\sigma}N^\alpha\,.$$

Solving equations $(48)_{1,2,3}$ by substitution, and remembering the expression for the velocity of propagation λ_1, the following equation is derived,

$$fLA - B - LC = 0\,, \tag{50}$$

where A, B, C are given by (49).

Following Thomas [40, 41], the following relation holds

$$a^{AB}\omega^\alpha_{,A}x_{\alpha,B} = -L\frac{d\psi}{d\sigma} + \frac{2}{\ell}\Omega\psi\,, \tag{51}$$

where Ω is the mean curvature of the hyper-surface Σ and, using (46),

$$\frac{d\xi}{d\sigma} = r\frac{\ell}{L}\frac{d\psi}{d\sigma}\,, \quad \frac{d\theta}{d\sigma} = T\frac{L}{\ell}\frac{d\psi}{d\sigma}\,, \quad \frac{d\omega_N}{d\sigma} = -\ell\frac{d\psi}{d\sigma}\,. \tag{52}$$

Introducing (46), (49), (51), (52) into equation (50), the following transport equation for the amplitude ψ can be derived

$$\frac{d\psi}{d\sigma} + \frac{L}{\ell}\Omega\psi - \psi^2 = 0\,. \tag{53}$$

In order to integrate this equation, it is convenient to rewrite it in terms of the proper time, τ, defined by $\ell d\sigma = d\tau$. Thus the following differential equation is obtained

$$\frac{d\psi}{d\tau} + \frac{L}{\ell^2}\Omega\psi - \frac{1}{\ell}\psi^2 = 0\,, \tag{54}$$

where $1/\ell^2 = 2-\gamma$ and $1/\ell = \sqrt{2-\gamma}$. Setting $a_0 = -L/\ell^2 = -(2-\gamma)L$ and $P_0 = -\sqrt{2-\gamma}$, (54) can be rewritten as

$$\frac{d\psi}{d\tau} - a_0\Omega\psi + P_0\psi^2 = 0\,. \tag{55}$$

The coefficient a_0 represents the constant speed of propagation of the wave front multiplied for the constant value ℓ^{-1}.

For a family of parallel surfaces propagating with constant velocity the mean curvature $\Omega(\tau)$ at any point of the wave front Σ is [46]

$$\Omega(\tau) = \frac{\Omega_0 - K_0 a_0 \tau}{1 - 2\Omega_0 a_0 \tau + K_0 a_0 \tau^2}\,, \tag{56}$$

where $\Omega_0 = (k_{01} + k_{02})/2$ and $K_0 = k_{01}k_{02}$ are, respectively, the mean and the Gaussian curvatures of Σ at initial time, with k_{01} and k_{02} being the initial principal curvatures. Then, the integration of (55) yields

$$\psi = \frac{\psi_0\left(1 - 2a_0\Omega_0\tau + K_0 a_0^2\tau^2\right)^{-1/2}}{1 + P_0\psi_0\int_0^\tau\left(1 - 2a_0\Omega_0\hat{\tau} + K_0 a_0^2\hat{\tau}^2\right)^{-1/2} d\hat{\tau}}\,, \tag{57}$$

where ψ_0 is the value of ψ on the wave front at $\tau = 0$. Since the transport equation is non linear, a critical time may exists, at which the weak discontinuity wave Σ evolves into a shock wave, in the sense that the amplitude ψ of the discontinuity of the first order derivatives blows up as τ tends to the critical time.

Eq. (57) can be specialized for plane, cylindrical and spherical wave.

- Plane wave: $k_{01} = k_{02} = 0$. The amplitude ψ is

$$\psi = \frac{\psi_0}{1 + P_0\psi_0\tau}\,, \tag{58}$$

and, since $P_0 < 0$, if $\psi_0 > 0$ (i.e. an expansive wave), the denominator of (58) vanishes at a finite time $\tau_c = (|P_0|\psi_0)^{-1}$. Thus, $\psi \to +\infty$ as $\tau \to \tau_c$, so the weak discontinuity evolves into a shock wave. Conversely, if $\psi_0 < 0$ (i.e. a compressive wave), the denominator of (58) is positive for any τ, so the discontinuity amplitude decays, $\psi \to 0$ as $\tau \to +\infty$.

- Cylindrical waves: $k_{01} = -R_0^{-1}$, $k_{02} = 0$, where R_0 is the cylinder radius at initial time. The amplitude ψ is

$$\psi = \frac{\psi_0 \left(1 + \dfrac{a_0}{R_0}\tau\right)^{-1/2}}{1 + P_0\psi_0 \displaystyle\int_0^{\tau} \left(1 + \dfrac{a_0}{R_0}\hat{\tau}\right)^{-1/2} d\hat{\tau}} . \tag{59}$$

Again, since $P_0 < 0$, if $\psi_0 > 0$, a critical time, τ_c, exists given by

$$\int_0^{\tau_c} \left(1 + \frac{a_0}{R_0}\hat{\tau}\right)^{-1/2} d\hat{\tau} = \frac{1}{|P_0|\psi_0} , \tag{60}$$

and the wave degenerates into a shock at time τ_c. Conversely, if $\psi_0 < 0$, the denominator of ((59)) remains positive for any time and the amplitude damps out in time.

- Spherical waves: $k_{01} = k_{02} = -R_0^{-1}$, where R_0 is the sphere radius of the outward travelling discontinuity surface at time $\tau = 0$. The amplitude ψ is

$$\psi = \frac{\psi_0 \left(\dfrac{R_0}{R_0 + a_0\tau}\right)}{1 + P_0\psi_0 \displaystyle\int_0^{\tau} \left(\dfrac{R_0}{R_0 + a_0\hat{\tau}}\right) d\hat{\tau}} . \tag{61}$$

Analogously to the previous cases, if $\psi_0 > 0$, a critical time, τ_c, appears given by

$$\int_0^{\tau_c} \left(\frac{R_0}{R_0 + a_0\hat{\tau}}\hat{\tau}\right) d\hat{\tau} = \frac{1}{|P_0|\psi_0} . \tag{62}$$

On the contrary, if $\psi_0 < 0$, the amplitude decays in time.

6. CONCLUSION

A depth discussion of the relativistic dynamics of fluids includes a number of dissipative processes [28]. The effects of internal dissipation in fluids-viscosity and thermal conductivity are well modelled by a generalization of the Sonic theory, called Navier Stokes equations, result in rather pathological theories [3, 4]. These theories are non-causal and without a well-posed initial value formulation (see for example [25]).

Less straightforward approaches have succeded in producing a class of causal dissipative fluid theories, e.g. Israel and Stewart [10], Jou, Casa-Vasquez and Lebon [6], Hiscock and Lindblom [24, 25], Carter [20, 21], Müller and Ruggeri [7], Muronga [11, 12], Maartens [30]. Almost all formulation used Eckart scheme [3] or Landau-Lifshitz approach [4] for problem of heat conduction.

In this paper, in light of some result due to Silva et al [32], Heinz et al [33], Van and Biró [34], Muronga [11, 12], Maartens [30], a generic causal theory of heat–conduction fluid has been derived, valid both in Eckart and Landau–Lifshitz frames.

In two previous papers, [15] and [16], the same authors developed a second-order theory for relativistic fluid with thermal conduction and examined the propagation of weak discontinuities in the special case of ultra-relativistic fluid in Landau-Lifshitz and Eckart scheme respectively. In this paper, the previous results have been generalized in order to consider the hypothesis (found in Carter [21] and recently picked up by Anderson and Comer [23]) of state equilibrium in which particle current and entropy flux are not aligned. The results obtained in the present paper can be specialized in those discussed in [15] and [16].

Acknowledgement. Work supported by G.N.F.M of I.N.d.A.M., by research grants of the University of Messina.

References

[1] S. R. deGroot and P. Mazur, *Non-equilibrium thermodynamics*, Dover, New York 1984.

[2] H. J. Kreuzer, *Nonequilibrium thermodynamics and its statistical foundations*, Oxford University Press, Oxford, 1981.

[3] C. Eckart, *The thermodynamics of irreversible process III: relativistic theory of the simple fluid*, Phys. Rev., **58**, (1940), 919-924.

[4] L. D. Landau, E. M. Lifshitz, *Fluid Mechanics*, Pergamon Press, New York, 1959.

[5] W. A. Hiscock, L. Lindblom, *Generic instabilities in first-order dissipative relativistic fluid theories*, Phys. Rev. D, **31**, (1985), 725-733.

[6] D. Jou, J. Casas-Vázquez, G. Lebon, *Extended irreversible thermodynamics*, Rep. Prog. in Phys., **51**, (1988), 1105-1179.

[7] I. Müller, T. Ruggeri, *Extended Thermodynamics*, Springer-Verlag, Berlin, 1993.

[8] W. Israel, *Nonstationary irreversible thermodynamics: a causal relativistic theory*, Ann. Phys. (N.Y.), **100**, (1976), 310-331.

[9] J. M. Stewart, *On transient relativistic thermodynamics and kinetic theory*, Proc. R. Soc. Lond., Ser. A, **357**, (1977), 59-75.

[10] W. Israel, J. M. Stewart, *Transient relativistic thermodynamics and kinetic theory*, Ann. Phys. (N.Y.), **118**, (1979), 341-372.

[11] A. Muronga, *Causal theories of dissipative relativistic fluid dynamics for nuclear collisions*, Phys. Rev. C, **69**, (2004), 034903.

[12] A. Muronga, *Second order dissipative fluid dynamics for ultrarelativistic nuclear collisions*, Phys. Rev. Lett., **88**, (2002), 062302.

[13] U. W. Heinz, H. Song, *Causal relativistic hydrodynamics for viscous fluids*, J. Phys. G, **35**, (2008), 104126.

[14] B. Bets, D. Henkel, D. H. Rishe, *Complete second-order dissipative fluid dynamics*, J. Phys. G: Nucl. Part. Phys., **36(6)**, (2009), 064029.

[15] S. Giambò, V. La Rosa, G. Muscianisi, *Nonlinear waves in an ultra-relativistic heat-conducting fluid I (Landau-Lifshtz formulation)*, Rep. Math. Phys., **67**, (2011), 375-394.

[16] S. Giambò, V. La Rosa, G. Muscianisi, *Nonlinear waves in an ultrarelativistic heat-conducting fluid II (Eckart formulation)*, AAPP, **90(1)**, (2012), A2 (19pp).

[17] L. S. Garcia-Colin, A. Sandoval-Villalbazo, *Relativistic non-equilibrium thermodynamics revisited*, Journal of Non Equilibrium Thermodynamics, **31**, (2006), 11.

[18] J. H. Mondragon-Suarez, A. Sandovall-Villabozo, A. L. Garcia- Perciante, *Structure formation in the presence of relativistic heat conduction corrections to the jeans wave number with a stable first order the gradient formalism*, Gen. Relativ. Gravit., **45**, (2013), 1805-1814.

[19] C. S. Lopez-Monsalvo, *A two-fluid model for heat conduction*, AIP Conf. Proc., **178**, (2014), 1577.

[20] B. Carter, *Regular and anomalous heat conduction: the canonical diffusion equation in relativistic thermodynamics*, M. Cahen, R. Debever and J. Geheniau editors, Journées Relativistes, Université Libre de Bruxelles, Bruxelles, 1976.

[21] B. Carter, *Covariant theory of conductivity in ideal fluid or solid media in Relativistic Fluid Dynamics*, A. Anile and Y.choquet-Bruhat eds. Springer-Verlag, Berlin, 1989.

[22] C. S. Lopez-Monsalvo, N. Andersson, *Thermal dynamics in general relativity*, Proc. R. Soc. Lond. A, (2010).

[23] N. Andersson, G. L. Comer, *Variational multi-fluid dynamics and causal heat conductivity*, Proc. R. Soc. Lond. A, **466**, (2010), 1373-1371.

[24] W. A. Hiscock, L. Lindblom, *Stability and causality in dissipative relativistic fluids*, Ann. Phys. (N.Y.), **151**, (1983), 466-496.

[25] W. A. Hiscock, L. Lindblom, *Linear plane waves in dissipative relativistic fluids*, Phys. Rev. D, **35**, (1987), 3723-3732.

[26] G. S. Denicol, T. Kodama, T. Koide, P. Mota, *Shock propagation and stability in causal dissipative hydrodynamics*, Phys. Rev. C, **78**, (2008), 034901.

[27] G. S. Denicol, T. Kodama, T. Koide, P. Mota, *Causal theory of relativistic dissipative hydrodynamics*, Brazilian Journal of Physics, **37**, (2007), 1047-1054 .

[28] P. Danielewicz, M. Gyulassy, *Dissipative phenomena in quark-gluon plasmas*, Phys. Rev. D, **31**, (1985), 53-62.

[29] M. O. Calvao, J. M. Salim, *Extended thermodynamics of friedmann–robertson–walker models in the Landau–Lifshits frame*, Class. Quantum Grav., **9**, (1992), 127-135.

[30] R. Maartens, *Dissipative cosmology*, Class. Quantum Grav., **12**, (1995), 1455-1465.

[31] L. Lindblom, *The relaxation effect in dissipative relativistic fluid theories*, Ann. of Phys., **247**, (1996), 1-18.

[32] R. Silva, J. A. S. Lima, M. O. Calvao, *Temperature evolution law of imperfect relativistic fluids*, Gen. Rel. Grav., **34**, (2002), 865-875.

[33] U. Heinz, H. Song, A. K. Chandhuri, *Dissipative hydrodynamics for viscous relativistic fluids*, Phys. Rev. C, **73**, (2006), 034904.

[34] P. Ván, T. S. Biró, *Relativistic hydrodynamics - causality and stability*, Eur. Phys. J, **155**, (2008), 201-212.

[35] W. G. Dixon, *Special Relativity, The foundations of Macroscopic Physycs*, Cambridge University Press, Cambridge, 1978.

[36] T. S. Olson, W. A. Hiscock, *Stability, causality and hyperbolicity in carter's "regular" theory of relativistic heat-conducting fluids*, Phys. Rev. D, **41**, (1990), 3687-3695.

[37] G. A. Nariboli, *On some aspects of wave propagation*, J. Math. Phys. Sci., **2(3)**, (1968), 2249.

[38] W. Israel, *Singular hypersurfaces and thin shells in general relativity*, Il nuovo Cimento B, **44**, (1966), 1-14.

[39] G. A. Maugin, *Conditions de compatibilité pour une hypersurface singulière en mécanique relativistes des milieux continus*, Ann. Inst. H.Poincaré, **24**, (1976), 213-241.

[40] T. Y. Thomas, *Extended compatibility conditions for the study of surfaces of discontinuity in continuum mechanics*, J. Math. Mech., **6**, (1957), 311-322 .

[41] T. Y. Thomas, *Hypersurfaces in Einstein-Riemann space and their compatibility conditions*. J. Math. Analys. Applns., **7**, (1963), 225.

[42] G. Boillat, *La Propagation des Ondes*, Gauthier-Villars, Paris, 1965.

[43] A. M. Anile, *Relativistic Fluids and Magneto-fluids*, Cambridge University Press, Cambridge, 1989.

[44] J. Gariel, M. Cissoko, *Wavefront velocities in a cosmic fluid with bulck viscosity and matter creation*, Call. Quantum Grav., **13**, (1996), 2977-2985.

[45] W .Israel, *Singular hypersurfaces and thin shelles in General Relativity*, Nuovo Cimento B, **44**, (1996), 1-14.

[46] E. P. Lane, *Metric Differential Geometry of Curves and Surfaces*, University of Chicago Press, Chicago, 1940.

ROMAI J., v.17, no.1(2021), 105–114

AN ANALOGUE OF THE ELGAMAL SCHEME BASED ON THE MARKOVSKI ALGORITHM

Nadeghda Malyutina[1], Victor Shcherbacov[2]

[1] *Moldova State University, Chişinău, Republic of Moldova*

[2] *Vladimir Andrunachievici Institute of Mathematics and Computer Science, Chişinău, Republic of Moldova*

231003.bab.nadezhda@mail.ru, victor.scerbacov@math.md

Abstract We give an analogue of the ElGamal encryption system based on the Markovski algorithm [4; 5].

Keywords: quasigroup, ElGamal's scheme, Markovski algorithm, encryption, decryption.
2020 MSC: 34C14, 34C40.

1. INTRODUCTION

Usually the classical Taher ElGamal encryption system is formulated in the language of number theory using multiplication modulo a prime [1].

ElGamal's scheme is a public key cryptosystem based on the difficulty of computing discrete logarithms in a finite field. The cryptosystem includes an encryption algorithm and a digital signature algorithm. ElGamal Scheme Underlies US Former Electronic Digital Signature Standards (DSA) and Russia (GOST R 34.10-94).

The scheme was proposed by Taher ElGamal in 1985. ElGamal developed one of the variants of the Diffie-Hellman algorithm. He improved the Diffie-Hellman system and obtained two algorithms that were used for encryption and for authentication. Unlike RSA, the ElGamal algorithm was not patented and, therefore, became a cheaper alternative, since it did not require payment of license fees [2].

The sender of messages and their recipient can be individuals, organizations, or technical systems. These may be subscribers of a network, users of a computer system, or abstract "parties" involved in information interaction. But more often participants are identified with people and replaced with the formal designations A and B by Alice and Bob. It is assumed that messages are transmitted through the so-called "open" communication channel, available for listening to some other persons.

In cryptography, it is usually assumed that a person sending messages or receiving them has some opponent E and this opponent can intercept messages transmitted over an open channel. The enemy is considered as a certain person named Eve, who has at her disposal powerful computing equipment and owns cryptanalysis methods. Naturally, Alice and Bob want their messages to be incomprehensible to Eve, and use special ciphers for this.

Before sending a message over an open communication channel from A to B, A encrypts the message, and B, having received the encrypted message, decrypts it, restoring the original text. The important thing is that Alice and Bob can agree on the cipher they use not on an open channel, but on a special "closed" channel, inaccessible for listening to the enemy. It should be borne in mind that usually the organization of such a closed channel and the transmission of messages through it is too expensive compared to an open channel or a closed channel cannot be used at any time. Each attempt to break the cipher is called an attack on the cipher. In cryptography, it is generally accepted that the adversary can know the encryption algorithm used, the nature of the transmitted messages and the intercepted ciphertext, but does not know the secret key.

Developers of modern cryptosystems strive to make attacks on known and selected text invulnerable. Significant successes have been achieved along this path.

2. ELGAMAL'S SCHEME

Suppose there are subscribers $A, B, C, \dots$ who want to transmit encrypted messages to each other without having any secure communication channels. We will consider the code proposed by ElGamal, which solves this problem, using, in contrast to the Shamir code, only one message forwarding. In fact, the Diffie -Hellman scheme is used here to form a common secret key for two subscribers transmitting a message to each other, and then the message is encrypted by multiplying it by this key. For each subsequent message, the secret key is recalculated. A large prime number is selected p and number g, such that different degrees of g are different numbers modulo p. The numbers p and g are transmitted to subscribers in the clear.

Then each subscriber of the group selects his secret number c_i, $1 < c_i < p-1$, and calculates the corresponding open number d_i,

$$d_i = g^{c_i} \pmod{p} \tag{1}$$

Table 1.User keys in the ElGamal system

Subscriber	Private key	Public key
A	c_A	d_A
B	c_B	d_B
C	c_C	d_C

We show now how A sends message m to the subscriber B. We will assume, that the message is presented as a number $m < p$.

Step 1. A forms a random number $k, 1 \geq k \geq p-2$, calculates numbers:

$$r = g^k \mod p \tag{2}$$

$$e = m \cdot {d_B}^k \mod p \tag{3}$$

and passes a couple of numbers (r, e) to the subscriber B.

Step 2. B, getting (r, e), calculates

$$m' = e \cdot r^{p-1-c_B} \mod p \tag{4}$$

Statement 1 (properties of the ElGamal cipher).

(1) The subscriber B received a message, i.e. $m' = m$;

(2) the adversary, knowing p, g, d_B, r and e, cannot calculate m.

Example 2.1. *Consider the transmission of message m = 15 from A to B.*

We take p = 23, g = 5. Let subscriber B choose for himself a secret number $c_B = 13$ *and calculate (1):* $d_B = 5^{13} \pmod{23} = 21$.

Subscriber A randomly selects the number k, for example, $k = 7$, *and calculates from (2), (3):* $r = 5^7 \pmod{23} = 17$, $e = 15 \cdot 21^7 \pmod{23} = 15 \cdot 10 \pmod{23} = 12$.

Now A sends to B an encrypted message in the form of a pair of numbers $(17, 12)$ *and B calculates:* $m' = 12 \cdot 17^{23-1-13} \pmod{23} = 12 \cdot 17^9 \pmod{23} = 12 \cdot 7 \pmod{23} = 15$. *So B was able to decrypt the transmitted message.*

By a similar scheme, all subscribers in the network can send messages. Moreover, any subscriber who knows the public key of subscriber B can send him messages encrypted using the public key d_B. But only subscriber B, and no one else, can decrypt these messages using the secret key c_B known only to him.

The Shamir cipher completely solves the problem of exchanging messages that are closed for reading, in the case when subscribers can use only open communication lines.

However, this message is sent three times from one subscriber to another, which is a drawback. The ElGamal cipher allows you to solve the same problem in one data transfer, but the amount of transmitted ciphertext is twice the size of the message.

It is easy to see that this system can also be formulated in terms of a residue ring modulo p or, equivalently, using the language of the Galois field $GF(p)$.

In addition, we can use the concept of the action of a group of automorphisms of a cyclic group $(Z_p, +)$ on this group. Let $(Z_p, +)$ be a cyclic group of residues of large simple order with respect to addition of residues and element a be the generator of the group $(Z_{(p-1)}, \cdot) \cong Aut(Z_p, +)(gcd(a, p-1) = 1)$.

Alice's keys are the following: Public key p, a and a^m, $m \in \mathbb{N}$. Private key m.

Encryption. To send a message $b \in (Z_{(p-1)}, \cdot)$, Bob is calculating a^r and a^{mr} for random $r \in \mathbb{N}$ (sometimes the number r is called an ephemeral key [3]).

Ciphertext: $(a^r; a^{mr} \cdot b)$.

Decryption. Alice knows m, so if she gets the ciphertext $(a^r; a^{mr} \cdot b)$, she will calculate a^{mr} from a^r and then $a^{(-mr)}$ and then from $a^{mr} \cdot b$ calculate b.

Example 2.2. *Example Alice picks* $p = 107, a = 2, m = 67$, *and calculates* $a^m = 2^{67} \equiv 94 \pmod{107}$.

Her public key $(p, a^m) = (107, 94)$, *and her private key is* $m = 67$. *Bob wants to send a message "B "to Alice. He selects a random integer* $r = 45$ *and encrypts* $B = 66$ *like* $(a^m)^r \cdot B$.

Bob gets: $(2^{45}, 94^{45} \cdot 66) \equiv (28, 5 \cdot 66) \equiv (28, 9) \pmod{107}$.

He sends an encrypted message $(28, 9)$ *to Alice. Alice receives this message and using her private key* $m = 67$ *she decrypts as follows:* $28^{(-67)} \cdot 9 = 28^{(106-67)} \cdot 9 \equiv 28^{39} \cdot 9 \equiv 43 \cdot 9 \equiv 66 \pmod{107}$.

The complexity of this system is based on the complexity of the discrete logarithm problem. ElGamal's encryption system is not secure according to the selected attack ciphertext [3]. ElGamal cryptosystems are usually used in a hybrid cryptosystem, i.e. the message itself is encrypted using a symmetric cryptosystem and ElGamal also uses a symmetric cryptosystem to encrypt the key.

3. AN ANALOGUE OF THE ELGAMAL SCHEME BASED ON THE MARKOVSKI ALGORITHM

We give an analogue of the ElGamal encryption system based on the Markovski algorithm [4; 5].

Let (Q, f) be a binary quasigroup and $T = (\alpha, \beta, \gamma)$ its isotopy.

Alice's keys are as follows: The public key is (Q, f), T, $T^{(m,n,k)} = (\alpha^m, \beta^n, \gamma^k)$, $m, n, k \in \mathbb{N}$, and the Markovski algorithm.

Private key m, n, k.

Encryption. To send a message $b \in (Q, f)$, Bob calculated $T^{(r,s,t)}$, $T^{(mr,ns,kt)}$ for random $r, s, t \in \mathbb{N}$ and $(T^{(mr,ns,kt)}(Q, f))$.

The ciphertext is $(T^{(r,s,t)}, T^{(mr,ns,kt)}(Q, f)b)$.

To obtain $(T^{(mr,ns,kt)}(Q, f)b)$, Bob uses the Markovski algorithm which is known to Alice.

Decryption Alice knows m, n, k, so if she gets the ciphertext $(T^{(r,s,t)}, (T^{(mr,ns,kt)}(Q, f))b)$, she will calculate $(T^{(mr,ns,kt)}(Q, f))^{(-1)}$ using $T^{(r,s,t)}$ and finally she will calculate b.

Example 3.1. *Let (Q, f) be a binary quasigroup defined by the following Cayley table:*

Table 2

f	0	1	2	3	4	5	6
0	5	2	6	4	0	3	1
1	1	6	5	3	4	2	0
2	0	5	4	6	3	1	2
3	4	1	3	0	2	6	5
4	2	4	0	1	6	5	3
5	6	3	1	2	5	0	4
6	3	0	2	5	1	4	6

and $T = (\alpha, \beta, \gamma)$ its isotopy, where: $\alpha = (234)(0516)$ corresponds to a permutation of rows of a quasigroup table Q; $\beta = (0321)(56)$ corresponds to a permutation of the columns of a quasigroup table Q obtained after application α; $\gamma = (1236054)$ substitution applied to the table obtained after application β.

And for γ we have the inverse $\gamma^{(-1)}$ the following kind: $\gamma^{(-1)} = (1450632)$.

Cayley tables of these permutations are of the form:

Table 3.

α	0	1	2	3	4	5	6
0	6	3	1	2	5	0	4
1	3	0	2	5	1	4	6
2	4	1	3	0	2	6	5
3	2	4	0	1	6	5	3
4	0	5	4	6	3	1	2
5	1	6	5	3	4	2	0
6	5	2	6	4	0	3	1

Table 4.

β	0	1	2	3	4	5	6
0	2	6	3	1	5	4	0
1	5	3	0	2	1	6	4
2	0	4	1	3	2	5	6
3	1	2	4	0	6	3	5
4	6	0	5	4	3	2	1
5	3	1	6	5	4	0	2
6	4	5	2	6	0	1	3

Table 5.

γ^{-1}	0	1	2	3	4	5	6
0	1	3	2	4	0	5	6
1	0	2	6	1	4	3	5
2	6	5	4	2	1	0	3
3	4	1	5	6	3	2	0
4	3	6	0	5	2	1	4
5	2	4	3	0	5	6	1
6	5	0	1	3	6	4	2

Then Alice's keys are as follows: The private key: $m = 3, n = 6, k = 5$. *The public key is* $(Q, f), T, T^{(3,6,5)} = (\alpha^3, \beta^6, \gamma^5)$ *and the Markovski algorithm, where:* $\alpha^3 = (0615); \beta^6 = (02)(13); \gamma^5 = (0315624), \gamma^{-5} = (0426513)$.

As a result, we get the following Cayley tables:

Table 6.

α^3	0	1	2	3	4	5	6
0	3	0	2	5	1	4	6
1	6	3	1	2	5	0	4
2	0	5	4	6	3	1	2
3	4	1	3	0	2	6	5
4	2	4	0	1	6	5	3
5	5	2	6	4	0	3	1
6	1	6	5	3	4	2	0

Table 7.

β^6	0	1	2	3	4	5	6
0	2	5	3	0	1	4	6
1	1	2	6	3	5	0	4
2	4	6	0	5	3	1	2
3	3	0	4	1	2	6	5
4	0	1	2	4	6	5	3
5	6	4	5	2	0	3	1
6	5	3	1	6	4	2	0

Table 8.

$(\gamma^5)^{(-1)}$	0	1	2	3	4	5	6
0	6	1	0	4	3	2	5
1	3	6	5	0	1	4	2
2	2	5	4	1	0	3	6
3	0	4	2	3	6	5	1
4	4	3	6	2	5	1	0
5	5	2	1	6	4	0	3
6	1	0	3	5	2	6	4

Encryption. To send a message $b = 630512403$, *Bob computes from the known* $T = (\alpha, \beta, \gamma)$*:* $\alpha = (234)(0516)$*;* $\beta = (0321)(56)$*;* $\gamma = (1236054)$, *calculates isotopy* $T^{(r,s,t)}$ *for random numbers* $r = 5, s = 3, t = 6$, *i.e.* $T^{(5,3,6)}$*:*

$$\alpha^5 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 4 & 2 & 3 & 1 & 0 \end{pmatrix}$$

$$\beta^3 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 0 & 4 & 6 & 5 \end{pmatrix}$$

$$\gamma^6 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 1 & 2 & 5 & 0 & 3 \end{pmatrix}$$

In our example $T^{(5,3,6)}$ *we get:* $\alpha^5 = (0516)(243)$*;* $\beta^3 = (0123)(56)$*;* $\gamma^6 = (0632145)$.

Then he calculates $T^{(mr,ns,kt)}$ *using the public key:*

$T^{(m,n,k)} = (\alpha^m, \beta^n, \gamma^k) = (\alpha^*, \beta^*, \gamma^*)$*:*

$$\alpha^* = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 3 & 4 & 0 & 1 \end{pmatrix}$$

$$\beta^* = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 0 & 1 & 4 & 5 & 6 \end{pmatrix}$$

$$\gamma^* = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 4 & 1 & 0 & 6 & 2 \end{pmatrix}$$

Then he raises these permutations, respectively, to the power $r = 5$, $s = 3, t = 6$ *and gets:*

$$\alpha^{*5} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 3 & 4 & 0 & 1 \end{pmatrix}$$

$$\beta^{*3} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 0 & 1 & 4 & 5 & 6 \end{pmatrix}$$

$$\gamma^{*6} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 6 & 0 & 2 & 1 & 5 \end{pmatrix}$$

$\alpha^{5m} = \alpha^5 = (0615)$*;* $\beta^{3n} = \beta^3 = (02)(13)$*;* $\gamma^{6k} = (0426513)$, $(\gamma^{6k})^{(-1)} = (0315624)$.

As a result of the application of the new isotopy $T^{(5m,3n,6k)}$ *to the quasigroup* (Q, f) *we obtain:*

Table 9.

α^{5m}	0	1	2	3	4	5	6
0	3	0	2	5	1	4	6
1	6	3	1	2	5	0	4
2	0	5	4	6	3	1	2
3	4	1	3	0	2	6	5
4	2	4	0	1	6	5	3
5	5	2	6	4	0	3	1
6	1	6	5	3	4	2	0

Table 10.

β^{3n}	0	1	2	3	4	5	6
0	2	5	3	0	1	4	6
1	1	2	6	3	5	0	4
2	4	6	0	5	3	1	2
3	3	0	4	1	2	6	5
4	0	1	2	4	6	5	3
5	6	4	5	2	0	3	1
6	5	3	1	6	4	2	0

Table 11.

$(\gamma^{6k})^{(-1)}$	0	1	2	3	4	5	6
0	4	6	1	3	5	0	2
1	5	4	2	1	6	3	0
2	0	2	3	6	1	5	4
3	1	3	0	5	4	2	6
4	3	5	4	0	2	6	1
5	2	0	6	4	3	1	5
6	6	1	5	2	0	4	3

To obtain $(T^{(mr,ns,kt)}(Q,f)b$*, Bob uses the Markovski algorithm known to Alice, with the known leader value* $l=3$*, then the ciphertext for* $b = 6\,3\,0\,5\,1\,2\,4\,0\,3$ *will look like:* $v_1 = 3\cdot 6 = 6, v_2 = 6\cdot 3 = 2, v_3 = 2\cdot 0 = 0, v_4 = 0\cdot 5 = 0,$ $v_5 = 0\cdot 1 = 6, v_6 = 6\cdot 2 = 5, v_7 = 5\cdot 4 = 3, v_8 = 3\cdot 0 = 1, v_9 = 1\cdot 3 = 1,$ $b' = 620065311.$

Decryption. Alice knows $m = 3, n = 6, k = 5$*, so if she gets an isotopy* $T^{(r,s,t)}$ *and ciphertext* $(T^{(mr,ns,kt)}(Q,f))b) = 620065311$*, she will calculate the isotopy first* $T^{(mr,ns,kt)}$ *using* $T^{(r,s,t)} = T^{(**,**,**)}$ *:*

$$\alpha^{**} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 4 & 2 & 3 & 1 & 0 \end{pmatrix}$$

$$\beta^{**} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 0 & 4 & 6 & 5 \end{pmatrix}$$

$$\gamma^{**} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 1 & 2 & 5 & 0 & 3 \end{pmatrix}$$

She calculates $T^{(mr,ns,kt)}$*:*

$$\alpha^{**3} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 3 & 4 & 0 & 1 \end{pmatrix}$$

$$\beta^{**6} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 0 & 1 & 4 & 5 & 6 \end{pmatrix}$$

$$\gamma^{**5} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 6 & 0 & 2 & 1 & 5 \end{pmatrix}$$

As a result, she receives the same table as Bob received in the encryption process. For Table $(\gamma^{6k})^{-1}$ *Alice builds a parastrophe* (23) *used in the Markovski* *algorithm for decryption:*

Table 12.

\	0	1	2	3	4	5	6
0	5	2	6	3	0	4	1
1	6	3	2	5	1	0	4
2	0	4	1	2	6	5	3
3	2	0	5	1	4	3	6
4	3	6	4	0	2	1	5
5	1	5	0	4	3	6	2
6	4	1	3	6	5	2	0

and finally, using this table, she calculates b: $u_1 = 3\backslash 6 = 6, u_2 = 6\backslash 2 = 3,$
$u_3 = 2\backslash 0 = 0, u_4 = 0\backslash 0 = 5, u_5 = 0\backslash 6 = 1, u_6 = 6\backslash 5 = 2, u_7 = 5\backslash 3 = 4,$
$u_8 = 3\backslash 1 = 0, u_9 = 1\backslash 1 = 3.$

Therefore, $b = 630512403$.

In this algorithm, isostrophy [6] can also be used instead of isotopy, the modified algorithm instead of the Markovski algorithm and n-ary ($n > 2$) quasigroups [7; 8] instead of binary quasigroups.

A generalization of the Diffie-Hellman scheme of the open key distribution is given in [9].

The generalization is based on the concepts of the left and right powers of the elements of some non-associative groupoids.

For medial quasigroups, this approach is implemented in [10]. The protocol of the elaboration of a common secret key based on Moufang loops is given in [10].

This protocol is a generalization of the results from [11]. Generalizations of the ElGamal scheme based on Moufang loops are given in [10].

In [12], the discrete logarithmic problem with Moufang loops is reduced to the same problem over finite simple fields. Another generalization of the ElGamal scheme based on quasi-automorphisms of quasigroups is presented in [10].

4. CONCLUSION

Today, different points of view on the same mathematical idea lead to different generalizations. We considered in our work an analogue of the ElGamal encryption system based on the Markovski algorithm. This algorithm is under improvement and its other modifications are planned.

References

[1] T. ElGamal, *A public key cryptosystem and a signature scheme based on discrete logarithms*, - IEEE Transactions on Information Theory, **31**, (4)1985, 469-472.

[2] B.Ya. Ryabko, A.N. Fionov, *Cryptographic methods of information protection: a training manual*, - M. Hotline-Telecom, 2005, p. 12-34. ISBN 5-89176-233-1.

[3] Wikipedia, *ElGamal encryption*, 2014. http:// en.wikipedia.org/wiki/ ElGamal-encryption.

[4] N.A. Moldovyan, A.V. Shcherbacov, V.A. Shcherbacov, *On some applications of quasigroups in cryptology*, Workshop on Foundations of Informatics, August 24-29, 2015, Chisinau, Proceedings, pages 331-341.

[5] V.A. Shcherbacov, *On generalisation of Markovski cryptoalgorithm*, Workshop on General Algebra, February 26-March 1, 2015, Technische Universitat at Dresden, Technical Report, Technische Universitat at Dresden, Dresden, 36-37, 2015.

[6] V.A. Shcherbacov, *On the structure of left and right F-, SM- and E-quasigroups*, J. Gen. Lie Theory Appl., **3**, (3)(2009), 197-259, .

[7] V.D. Belousov, *n-Ary Quasigroups*, Stiintsa, Kishinev, 1971 (in Russian).

[8] V.A. Shcherbacov, *Quasigroups in cryptology*, Comput. Sci. J. Moldova, **17**, (2)(2009), 193-228.

[9] S. Yu. Katyshev, V.T. Markov, A.A. Nechaev, *Utilization of nonassociative groupoids for the realization of an open key-distribution procedure*, Diskret. Mat., 26(2014), 45-64 (in Russian).

[10] A.V. Gribov, *Algebraic Non-Associative Structures and Its Applications in Cryptology*, PhD thesis, Moscov State University, 2015 (in Russian).

[11] E. Stickel, *A new method for exchanging secret keys*, Proceedings of the Third International Conference on Information Technology and Applications, volume 2, p. 426-430, 2005.

[12] G. Maze, *Algebraic Methods For Constructing One-Way Trapdoor Functions*, PhD thesis, University of Notre Dame, 2003.

ROMAI J., v.17, no.1(2021), 115–123

LYAPUNOV'S STABILITY OF UNPERTURBED MOTION GOVERNED BY COMPLETE TERNARY CUBIC DIFFERENTIAL SYSTEMS OF THE DARBOUX TYPE

Natalia Neagu[1], Mihail Popa[2]

[1]*Ion Creangă State Pedagogical University, Chişinău, Republic of Moldova*

[2]*Institute of Mathematics and Computer Science, Chişinău, Republic of Moldova*

neagu_natusik@mail.ru, mihailpomd@gmail.com

Abstract The conditions of stability after Lyapunov of the unperturbed motion for the system $s^3(1,2,3)$ in the non-critical case were obtained. The Lyapunov series was constructed for the ternary differential system $s^3(1,2,3)$ of Darboux type in the critical case and was determined the conditions of stability of the unperturbed motion governed by this system.

Keywords: differential systems, stability of unperturbed motion, center-affine comitant and invariant.

2020 MSC: 34C14, 34C40.

1. THE LYAPUNOV FORM OF THE CRITICAL TERNARY DIFFERENTIAL SYSTEM $S^3(1,2,3)$

We examine the ternary differential system with polynomial nonlinearities $s^3(1,2,3)$ of the form

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + a^j_{\alpha\beta}x^\alpha x^\beta + a^j_{\alpha\beta\gamma}x^\alpha x^\beta x^\gamma \quad (j,\alpha,\beta,\gamma = \overline{1,3}), \tag{1}$$

where $a^j_{\alpha\beta}$ and $a^j_{\alpha\beta\gamma}$ are symmetric tensors in the lower indices, by which a total convolution is carried out here.

The system of the first approximation ([1], [2])

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha \quad (j,\alpha = \overline{1,3}). \tag{2}$$

plays an important role in studying the differential system (1).

Taking into account the fact that the unperturbed motion of the system (1) corresponds the zero values of variables $x^j(t)$ $(j = 1,3)$, we have the following definition

Definition of stability by Lyapunov [2]. *If for any small positive value ε, however small, one can find a positive number δ, such that for all perturbations $x^j(t_0)$ satisfying the condition*

$$\sum_{j=1}^{2}(x^j(t_0))^2 \leq \delta, \tag{3}$$

the inequality

$$\sum_{j=1}^{2}(x^j(t))^2 < \varepsilon$$

holds for any $t \geq t_0$, then the unperturbed motion $x^j = 0$ $(j = \overline{1,2})$ is called stable, otherwise it is called unstable.

If the unperturbed motion is stable and the number δ can be found however small such that for any perturbed motions satisfying (3) the condition

$$\lim_{t\to\infty}\sum_{j=1}^{2}(x^j(t))^2 = 0,$$

holds, then the unperturbed motion is called *asymptotically stable.*

The characteristic equation of the system (1) and (2) is

$$\varrho^3 + L_{1,3}\varrho^2 + L_{2,3}\varrho + L_{3,3} = 0, \tag{4}$$

where the coefficients of this equation are center-affine invariants [4] and have the form

$$L_{1,3} = -\theta_1, \quad L_{2,3} = \frac{1}{2}(\theta_2 - \theta_1^2), \quad L_{3,3} = \frac{1}{6}(-\theta_1^3 + 3\theta_1\theta_2 - 2\theta_3), \tag{5}$$

and

$$\theta_1 = a^\alpha_\alpha, \quad \theta_2 = a^\alpha_\beta a^\beta_\alpha, \quad \theta_3 = a^\alpha_\gamma a^\beta_\alpha a^\gamma_\beta. \tag{6}$$

According to I. G. Malkin [3], we have

Definition 1.1. *The system* (1) *is critical if the characteristic equation* (4) *of this system has one zero root, and all other roots of this equation have negative real parts.*

Lemma 1.1. *The system* (1) *is critical if and only if the center-affine invariant conditions*

$$L_{1,3} > 0, \quad L_{2,3} > 0, \quad L_{3,3} = 0 \tag{7}$$

hold, where $L_{i,3}$ $(i = \overline{1,3})$ are from (5).

By means of the Lyapunov theorems on stability of unperturbed motion [1] and the Hurwitz theorem [2] we obtain the following theorems:

Theorem 1.1. *Assume that the center-affine invariants* (5) *of the system* (1) *satisfy the inequalities*

$$L_{1,3} > 0, \quad L_{2,3} > 0, \quad L_{1,3}L_{2,3} - L_{3,3} > 0. \tag{8}$$

Then the unperturbed motion $x^1 = x^2 = x^3 = 0$, *of this system, is asymptotically stable.*

Theorem 1.2. *If at least one of the center-affine invariant expression* (5) *of the system* (1) *is negative, then the unperturbed motion* $x^1 = x^2 = x^3 = 0$ *of this system is unstable.*

2. STABILITY CONDITIONS OF UNPERTURBED MOTION GOVERNED BY CRITICAL THREE-DIMENSIONAL DIFFERENTIAL SYSTEM $S^3(1,2,3)$ OF DARBOUX TYPE

Either

$$\eta = a^{\alpha}_{\beta\gamma} x^{\beta} x^{\gamma} x^{\delta} y^{\mu} \varepsilon_{\alpha\delta\mu} \equiv 0, \tag{9}$$

and

$$\eta_1 = a^{\alpha}_{\beta,\gamma,\delta} x^{\beta} x^{\gamma} x^{\delta} x^{\mu} y^{\nu} \varepsilon_{\alpha\mu\nu} \equiv 0, \tag{10}$$

from [4] (or [5]), where $x = (x^1, x^2, x^3)$ and $y = (y^1, y^2, y^3)$ are cogradient vectors [6].

In the center-affine conditions (9) and (10), by a center-affine transformation, the system (1) can be brought to the critical Lyapunov of Darboux type, of the form

$$\begin{aligned}
\frac{dx}{dt} &= 2x(gx + hy + kz) + 3x(ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz),\\
\frac{dy}{dt} &= px + qy + rz + 2y(gx + hy + kz) + 3y(ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz),\\
\frac{dz}{dt} &= sx + my + nz + 2z(gx + hy + kz) + 3z(ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz),
\end{aligned} \tag{11}$$

where $x = x^1, y = x^2$ şi $z = x^3$, and $a, b, c, d, e, f, g, h, k, m, n, p, q, r, s$ are real arbitrary coefficients.

We analyze the noncritical equations

$$\begin{aligned}
px + qy + rz + 2y(gx + hy + kz) + 3y(ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz) = 0,\\
sx + my + nz + 2z(gx + hy + kz) + 3z(ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz) = 0.
\end{aligned} \tag{12}$$

Because in the system (11), according to the conditions (7), we have $L_{2,3} = nq - mr > 0$, then we can assume, without losing generality, that $nq \neq 0$.

Then from the first relation, from (12), we express y, and from the second relation we express z

$$\begin{aligned} y &= -\frac{p}{q}x - \frac{r}{q}z - \frac{2}{q}y(gx+hy+kz) - \frac{3}{q}y(ax^2+by^2+cz^2+2dxy+2exz+2fyz),\\ z &= -\frac{s}{n}x - \frac{m}{n}y - \frac{2}{n}z(gx+hy+kz) - \frac{3}{n}z(ax^2+by^2+cz^2+2dxy+2exz+2fyz). \end{aligned} \tag{13}$$

We seek y and z as a holomorphic functions of x. Then we can write

$$\begin{aligned} y(x) &= A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + ...,\\ z(x) &= B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + ... \end{aligned} \tag{14}$$

Substituting (14) into (13) we have

$$\begin{gathered} A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + ... = -\frac{p}{q}x - \frac{r}{q}(B_1x + B_2x^2 + B_3x^3 +\\ +B_4x^4 + B_5x^5 + ...) - \frac{2}{q}(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + ...)[gx+\\ +h(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + ...) + k(B_1x + B_2x^2 + B_3x^3 +\\ +B_4x^4 + B_5x^5 + ...)] - \frac{3}{q}(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + ...)[(ax^2+\\ +b(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + ...)^2 + c(B_1x + B_2x^2 + B_3x^3 +\\ +B_4x^4 + B_5x^5 + ...)^2 + 2dx(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + ...)+\\ +2ex(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + ...) + 2f(A_1x + A_2x^2+\\ +A_3x^3 + A_4x^4 + A_5x^5 + ...)(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + ...))], \end{gathered}$$

$$\begin{gathered} B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + ... = -\frac{s}{n}x - \frac{m}{n}(A_1x + A_2x^2 + A_3x^3 +\\ +A_4x^4 + A_5x^5 + ...) - \frac{2}{n}(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + ...)[gx+\\ +h(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + ...) + k(B_1x + B_2x^2 + B_3x^3 +\\ +B_4x^4 + B_5x^5 + ...)] - \frac{3}{n}(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + ...)[(ax^2+\\ +b(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + ...)^2 + c(B_1x + B_2x^2 + B_3x^3 +\\ +B_4x^4 + B_5x^5 + ...)^2 + 2dx(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + ...)+\\ +2ex(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + ...) + 2f(A_1x + A_2x^2+\\ +A_3x^3 + A_4x^4 + A_5x^5 + ...)(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + ...))]. \end{gathered}$$

This implies that

$$A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + ... = -\frac{p + rB_1}{q}x-$$
$$-\frac{2gA_1 + 2hA_1^2 + 2kA_1B_1 + rB_2}{q}x^2 - \frac{1}{q}(3aA_1 + 3bA_1^3 + 3cA_1B_1^2 + 6dA_1^2+$$
$$+6eA_1B_1 + 6fA_1^2B_1 + rB_3 + 2gA_2 + 4hA_1A_2 + +2kA_2B_1 + 2kA_1B_2)x^3-$$
$$-\frac{1}{q}(3aA_2 + 9bA_1^2A_2 + 3cA_2B_1^2 + 6cA_1B_1B_2 + 12dA_1A_2 + 6eA_2B_1+$$
$$+6eA_1B_2 + 12fA_1A_2B_1 + 6fA_1^2B_2 + rB_4 + 2gA_3 + 2hA_2^2 + 4hA_1A_3+$$
$$+2kA_3B_1 + 2kA_2B_2 + 2kA_1B_3)x^4 - \frac{1}{q}(3aA_3 + 9bA_1A_2^2 + 9bA_1^2A_3+$$
$$+3cA_3B_1^2 + 6cA_2B_1B_2 + 3cA_1B_2^2 + 6cA_1B_1B_3 + 6dA_2^2 + 12dA_1A_3+$$
$$+6eA_3B_1 + 6eA_2B_2 + 6eA_1B_3 + 6fA_2^2B_1 + 12fA_1A_3B_1+$$
$$+12fA_1A_2B_2 + 6fA_1^2B_3 + rB_5 + 2gA_4 + 4hA_2A_3 + 4hA_1A_4 + 2kA_4B_1+$$
$$+2kA_3B_2 + 2kA_2B_3 + 2kA_1B_4)x^5 + ...,$$

$$B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + ... = -\frac{mA_1 + s}{n}x-$$
$$-\frac{2gB_1 + 2hA_1B_1 + 2kB_1^2 + mA_2}{n}x^2 - \frac{1}{n}(3aB_1 + 3bA_1^2B_1 + 3cB_1^3 + 6dA_1B_1+$$
$$+6eB_1^2 + 6fA_1B_1^2 + mA_3 + 2gB_2 + 2hA_2B_1 + 2hA_1B_2 + 4kB_1B_2)x^3-$$
$$-\frac{1}{n}(6bA_1A_2B_1 + 3aB_2 + 3bA_1^2B_2 + 9cB_1^2B_2 + 6dA_2B_1 + 6dA_1B_2+$$
$$+12eB_1B_2 + 6fA_2B_1^2 + 12fA_1B_1B_2 + mA_4 + 2gB_3 + 2hA_3B_1 + 2hA_2B_2+$$
$$+2hA_1B_3 + 2kB_2^2 + 4kB_1B_3)x^4 - \frac{1}{n}(3bA_2^2B_1 + 6bA_1A_3B_1 + 6bA_1A_2B_2+$$
$$+3aB_3 + 3bA_1^2B_3 + 9cB_1B_2^2 + 9cB_1^2B_3 + 6dA_3B_1 + 6dA_2B_2+$$
$$+6dA_1B_3 + 6eB_2^2 + 12eB_1B_3 + 6fA_3B_1^2 + 12fA_2B_1B_2 + 6fA_1B_2^2+$$
$$+12fA_1B_1B_3 + mA_5 + 2B_4g + 2hA_4B_1 + 2hA_3B_2 + 2hA_2B_3+$$
$$+2hA_1B_4 + 4kB_2B_3 + 4kB_1B_4)x^5 + ...$$

From this identity we have

$$A_1 = \frac{rs - np}{nq - mr}, \quad B_1 = \frac{mp - qs}{nq - mr};$$
$$A_2 = -\frac{2M(nA_1 - rB_1)}{nq - mr}, \quad B_2 = \frac{2M(mA_1 - qB_1)}{nq - mr},$$

$$A_3 = -\frac{1}{nq-mr}\{3N(nA_1 - rB_1) + 2[A_2(gn + 2hnA_1 + knB_1 - hrB_1)+$$
$$+B_2(knA_1 - gr - hrA_1 - 2krB_1)]\},$$

$$B_3 = -\frac{1}{nq-mr}\{-3N(mA_1 - qB_1) - 2[A_2(gm + 2hmA_1 + kmB_1 - hqB_1)+$$
$$+B_2(kmA_1 - gq - hqA_1 - 2kqB_1)]\},$$

$$A_4 = -\frac{1}{nq-mr}(3anA_2 + 9bnA_1^2A_2 + 3cnA_2B_1^2 + 6cnA_1B_1B_2 + 12dnA_1A_2+$$
$$+12dnA_1A_2 + 6enA_2B_1 + 6enA_1B_2 + 12fnA_1A_2B_1 + 6fnA_1^2B_2 + 2gnA_3+$$
$$+2hnA_2^2 + 4hnA_1A_3 + 2knA_3B_1 + 2knA_2B_2 + 2knA_1B_3 - 6brA_1A_2B_1-$$
$$-3arB_2 - 3brA1^2B2 - 9crB_1^2B_2 - 6drA_2B_1 - 6drA_1B_2 - 12erB_1B_2-$$
$$-6frA_2B_1^2 - 12frA_1B_1B_2 - 2grB_3 - 2hrA_3B_1 - 2hrA_2B_2 - 2hrA_1B_3-$$
$$-2krB_2^2 - 4krB_1B_3),$$

$$B_4 = -\frac{1}{nq-mr}(-3aA_2m - 9A_1^2A_2bm - 3A_2B_1^2cm - 6A_1B_1B_2cm-$$
$$-12A_1A_2dm - 6A_2B_1em - 6A_1B_2em - 12A_1A_2B_1fm - 6A_1^2B_2fm-$$
$$-2A_3gm - 2A_2^2hm - 4A_1A_3hm - 2A_3B_1km - 2A_2B_2km - 2A_1B_3km+$$
$$+6A_1A_2bB_1q + 3aB_2q + 3A_1^2bB_2q + 9B_1^2B_2cq + 6A_2B_1dq+$$
$$+6A_1B_2dq + 12B_1B_2eq + 6A_2B_1^2fq + 12A_1B_1B_2fq + 2B_3gq+$$
$$+2A_3B_1hq + 2A_2B_2hq + 2A_1B_3hq + 2B_2^2kq + 4B_1B_3kq),$$

$$A_5 = -\frac{1}{nq-mr}(3aA_3n + 9A_1A_2^2bn + 9A_1^2A_3bn + 3A_3B_1^2cn+$$
$$+6A_2B_1B_2cn + 3A_1B_2^2cn + 6A_1B_1B_3cn + 6A_2^2dn + 12A_1A_3dn+$$
$$+6A_3B_1en + 6A_2B_2en + 6A_1B_3en + 6A_2^2B_1fn + 12A_1A_3B_1fn+$$
$$+12A_1A_2B_2fn + 6A_1^2B_3fn + 2A_4gn + 4A_2A_3hn + 4A_1A_4hn+$$
$$+2A_4B_1kn + 2A_3B_2kn + 2A_2B_3kn + 2A_1B_4kn - 3A_2^2bB_1r-$$
$$-6A_1A_3bB_1r - 6A_1A_2bB_2r - 3aB_3r - 3A_1^2bB_3r - 9B_1B_2^2cr-$$
$$-9B_1^2B_3cr - 6A_3B_1dr - 6A_2B_2dr - 6A_1B_3dr - 6B_2^2er - 12B_1B_3er-$$
$$-6A_3B_1^2fr - 12A_2B_1B_2fr - 6A_1B_2^2fr - 12A_1B_1B_3fr - 2B_4gr-$$
$$-2A_4B_1hr - 2A_3B_2hr - 2A_2B_3hr - 2A_1B_4hr - 4B_2B_3kr - 4B_1B_4kr),$$

$$B_5 = -\frac{1}{nq-mr}(-3aA_3m - 9A_1A_2^2bm - 9A_1^2A_3bm - 3A_3B_1^2cm-$$
$$-6A_2B_1B_2cm - 3A_1B_2^2cm - 6A_1B_1B_3cm - 6A_2^2dm - 12A_1A_3dm-$$
$$-6A_3B_1em - 6A_2B_2em - 6A_1B_3em - 6A_2^2B_1fm - 12A_1A_3B_1fm-$$
$$-12A_1A_2B_2fm - 6A_1^2B_3fm - 2A_4gm - 4A_2A_3hm - 4A_1A_4hm-$$
$$-2A_4B_1km - 2A_3B_2km - 2A_2B_3km - 2A_1B_4km + 3A_2^2bB_1q + 6A_1A_3bB_1q+$$
$$+6A_1A_2bB_2q + 3aB_3q + 3A_1^2bB_3q + 9B_1B_2^2cq + 9B_1^2B_3cq + 6A_3B_1dq+$$
$$+6A_2B_2dq + 6A_1B_3dq + 6B_2^2eq + 12B_1B_3eq + 6A_3B_1^2fq + 12A_2B_1B_2fq+$$
$$+6A_1B_2^2fq + 12A_1B_1B_3fq + 2B_4gq + 2A_4B_1hq + 2A_3B_2hq+$$
$$+2A_2B_3hq + 2A_1B_4hq + 4B_2B_3kq + 4B_1B_4kq), ... \tag{15}$$

where

$$M = g + hA_1 + kB_1; \;\; N = a + bA_1^2 + cB_1^2 + 2dA_1 + 2eB_1 + 2fA_1B_1. \tag{16}$$

Remark 2.1. *From* (7) *we have*

$$L_{2,3} = nq - mr > 0.$$

Substituting (14) into the right-hand sides of the critical differential equations (11), we get the following identity

$$2x(gx + hy + kz) + 3x(ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz) =$$
$$= C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + ...,$$

or in detailed form

$$2x[gx+h(A_1x+A_2x^2+A_3x^3+A_4x^4+A_5x^5+...)+k(B_1x+B_2x^2+B_3x^3+B_4x^4+$$
$$+B_5x^5 + ...)] + 3x[ax^2 + b(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + ...)^2+$$
$$+c(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + ...)^2 + 2dx(A_1x + A_2x^2+$$
$$+A_3x^3 + A_4x^4 + A_5x^5 + ...) + 2ex(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + ...)+$$
$$+2f(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + ...)(B_1x + B_2x^2 + B_3x^3 + B_4x^4+$$
$$+B_5x^5 + ...)] = C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + ...$$

From here, we obtain

$$C_1 = 0; \;\; C_2 = 2M \;\; C_3 = 3N + 2(hA_2 + B_2k),$$
$$C_4 = 2(3bA_1A_2 + 3cB1B_2 + 3dA_2 + 3eB_2 + 3fA_2B_1 + 3fA_1B_2 + hA_3 + kB_3),$$
$$C_5 = 3bA_2^2 + 6bA_1A_3 + 3cB_2^2 + 6cB_1B_3 + 6dA_3 + 6eB_3 + 6fA_3B_1+$$
$$+6fA_2B_2 + 6fA_1B_3 + 2hA_4 + 2kB_4, ..., \tag{17}$$

where M and N are from (16).

Theorem 2.1. *The stability of the unperturbed motion, described by the critical system* (11) *of Darboux type* $s^3(1,2,3)$ *of the perturbed motion, includes all possible cases in the following four:*

$$\textbf{\textit{I.}}\ \ M \neq 0, \tag{18}$$

then unperturbed motion is **unstable***;*

$$\textbf{\textit{II.}}\ \ M = 0,\ N < 0, \tag{19}$$

then unperturbed motion is **stable***;*

$$\textbf{\textit{III.}}\ \ M = 0,\ N > 0, \tag{20}$$

then unperturbed motion is **unstable***;*

$$\textbf{\textit{IV.}}\ \ M = N = 0, \tag{21}$$

then unperturbed motion is **stable***.*

In the last case, the unperturbed motion belongs to some continuous series of stabilized motion. Moreover, for sufficiently small perturbations, any perturbed motion will asymptotically approach to one of the stabilized motions of the mentioned series. The expressions M and N are from (16)*.*

Proof. According to the Lyapunov theorem [2, §32] and Remark 1, we analyze the coefficients of the series (17). If $C_2 \neq 0$, then $M \neq 0$, so we obtained case I.

If $M = 0$ (which implies $A_2 = B_2 = 0$), then $C_3 = 3N$. Depending on the sign of the expression N, we get cases II and III.

Therefore, if $M = N = 0$, then $A_i = B_i = 0$ $(\forall i)$, so we get case IV of this theorem. Theorem 3 is proved.

Remark 2.2. *The theorem 3 generalizes the stability of unperturbed motion, described by the critical systems of Darboux type of the perturbed motion* $s^3(1,2)$ [7] *and* $s^3(1,3)$ [8]*.*

References

[1] A. M. Liapunov, *Obshchaia zadacha ob ustoichivosti dvijenia,* Sobranie sochinenii, II – Moskva-Leningrad: Izd. Acad. Nauk SSSR, 1956 (in Russian).

[2] D.R. Merkin, *Introduction to the theory of stability.* NY: Springer–Verlag, 1996. 320 p.

[3] I. G. Malkin, *Teoria ustoichivosti dvijenia.* Izd. Nauka, Moskva, 1966, 530 p.

[4] K. S. Sibirsky, *Introduction to the algebraic theory of invariants of differential equations.* Nonlinear Science: Theory and Applications. Manchester: Manchester University Press, 1988.

[5] N. N. Gerştega, *Lie algebras for the three-dimensional differential system and applications.*. Chişinău: PhD thesis, 2006 (in Russian).

[6] G. B. Gurevich, *Fondations of the theory of algebraic invariants*, Moscow: GITTL, 1948. 429 p. (English transl., Nordhoff, 1964).

[7] N. Neagu, *Lie algebras and invariants for differential systems with projections on some mathematical models.* PhD thesis, Chişinău, 2017, 125 p.

[8] N. Neagu, M. N. Popa, *Stability conditions of unperturbed motion governed by critical three-dimensional differential system of Darboux type with cubic nonlinearities.* International Conference "Mathematics & information tehnologies: research and education (MITRE-2021)" (dedicated to the 75th anniversary of the Moldova State University), July 01-03, 2021, Chişinău. Abstracts, p. 62–63.

ROMAI J., v.17, no.1(2021), 125–136

ON SOME NEW THEOREMS FOR SPACES OF SUBHARMONIC FUNCTIONS

Romi F. Shamoyan
Department of Mathematics, Bryansk State Technical University, Bryansk, Russia
rshamoyan@gmail.com

Abstract We introduce new general spaces of subharmonic functions in the unit disk and prove some new parametric representation results for them expanding some previously known assertions.

Keywords: subharmonic functions, Riesz measure, unit disk, Nevanlynna chracteristic.
2020 MSC: Primary 42B15, Secondary 42B30.

1. INTRODUCTION

This paper is devoted to some new results in spaces of subharmonic functions in the unit disk. We introduce new general spaces of subharmonic functions in the unit disk and show some new embedding theorems for them. Embedding theorems and various inequalities for various spaces of subharmonic functions in various domains is an old research area. We refer the reader, for example, to [4] and various references there. See also [1], [3], [4], [5], [8], [10], [12], [13], [15]. Some arguments from [14] are crucial for this paper.

As a result from these embedding theorems we obtain immediately complete parametric representations of these new large spaces of subharmonic functions.

To formulate that result, we first need some definition.

Let $D = \{z \in C;\ |z| \leq 1\}$ be the unit disk and $E = \{z \in C;\ |z| = 1\}$ be the unit circle. Let $SH(D)$ be the space of all subharmonic functions in D.

Let further $u \in SH(D)$, let $u^+ = \max(u, 0)$. Then as usual put

$$T(r,u) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u^+(re^{i\theta})d\theta,\ r \in (0,1)\quad \text{(Nevanlinna characteristics)}$$

Let now $\alpha \geq 0$ and let

$$SH_\alpha(D) = \{u \in SH(D):\ T(r,u) \leq \frac{C_u}{(1-r)^\alpha},\ 0 \leq r < 1\}.$$

For $\alpha = 0$ we have classical Privalov space and known results on parametric representation (see [1], [9] , [10]). For fixed $\xi, z \in D$, $\beta > -1$, $\xi \neq 0$ we denote by $A_\beta(z,\xi)$ the following expression

$$A_\beta(z,\xi) = \left(1 - \frac{z}{\xi}\right) \exp\left\{-\left[\frac{2(\beta+1)}{\pi}\right] \int_D \frac{(1-|t|^2)^\beta}{(1-z\bar{t})^{\beta+2}} \left(\ln\left|1-\frac{t}{\xi}\right|\right) dm_2(t)\right\}.$$

These are so-called Djrbashian factors (see [1], [8], [14]).

Consider further the following spaces of subharmonic functions in the unit disk

$$SH^p_\alpha(D) = \left\{u \in SH(D) : \left(\int_0^1 (1-r)^\alpha \left(\int_{-\pi}^{\pi} u^+(re^{i\theta})d\varphi\right)^p dr\right) < \infty, \right\},$$

$0 < p < +\infty$, $\alpha > -1$.

Let $B^{1,\infty}_\alpha$ be Besov space on a unit circle E

$$B^{1,\infty}_s(E) = \left\{\psi \in L_1(E) \text{ from to } e^{it} : \int_0^1 \frac{\|\triangle^2_t \psi\|_{L_1}}{(t^s)} dt < +\infty\right\},$$

where $\triangle^2_t \psi\left(e^{i\theta}\right) = \psi\left(e^{i(\theta+t)}\right) - 2\psi\left(e^{i\theta}\right) + \psi\left(e^{i(\theta-t)}\right)$, $\theta \in [-\pi,\pi]$, $t \in (0,1)$, $s \in (0,2)$.

Note that the following mapping $t \to e^{it}$ is a homeomorphism between $(-\pi,\pi)$ and E. In [8] Ohlupina showed that the $SH_\alpha(D)$ coincides with the class of functions u, so that

$$u(z) = \int_D \ln|A_\beta(z,\xi)| d\mu(\xi) + Re\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\psi(e^{i\theta})d\theta}{(1-e^{-i\theta}z)^{\beta+1}}\right\},$$

$z \in D$, $\psi \in B^{1,\infty}_{\beta-\alpha+1}$, $\beta > \alpha$, $\alpha > -1$, μ is a nonnegative Borel measure in D. Similar sharp parametric representation theorems were obtained also in mentioned work for SH^p_α spaces of subharmonic functions in the unit disk.

Note these SH_α, SH^p_α, $\alpha > -1$, $p > 0$ spaces and similar type spaces of subharmonic functions were introduced for the first time in [8], where embeddings and various interesting properties were also provided. Note similar spaces and results were given also in $\mathbb{C}^+$ (upper half spaces) of $\mathbb{C}$. We will use $A_\beta(z,\xi)$ factors actively in this paper, and some properties of these factors that were used in [1], (see also [8]).

Throughout the paper, we write C or c (with or without lower indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities), but is independent of the functions or variables being discussed.

Let μ be positive Borel measure in D. Let now $n(r) = \mu(D_r)$, where $D_r = \{z \in \mathbb{C} : |z| < r\}$, $0 < r < 1$.

One of the corner stones of the theory of subharmonic functions is the following result of Riesz on parametric representation of subharmonic functions.

Theorem 1.1. *(see [6], [9]) Let $u \in SH(D)$, $u \not\equiv (-\infty)$. Then there is a unique Borel measure μ so that the following parametric representation is valid for u function*

$$u(z) = \int_{D_r} \ln \left| \frac{r(\xi - z)}{r^2 - \bar{\xi} r} \right| d\mu(\xi) + h(z),$$

where $z \in D_r$ and $h(z)$ is a harmonic function in D_r.

We call μ measure Riesz measure for u function. It is a general problem to find certain concrete conditions on μ so that $u \in \mathcal{X} \subset SH(D)$, where $\mathcal{X}$ is a certain fixed subclass of $SH(D)$. We refer, for example, to [8] for such type results.

Let

$$A = \left\{ f \in SH(D) : \sup_r T(r, u) < +\infty \right\}.$$

The following sharp parametric representation theorem is classical, (see [6], [9], [10], [16]).

Theorem 1.2. *The A class coincides with the space of all subharmonic functions for which*

$$u(z) = \frac{1}{2\pi} \int_D \ln \left| \frac{\xi - z}{1 - \bar{\xi} z} \right| d\mu(\xi) + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1 - r^2) d\varphi(\theta)}{(1 - 2r\cos(\theta - \varphi) + r^2)},$$

where μ is an arbitrary nonnegative Borel measure in the unit disk for which

$$\int_D (1 - |\xi|) d\mu(\xi) < \infty,$$

and φ is an arbitrary function of bounded variation on $[-\pi, \pi]$.

We provide in this paper similar type parametric representation theorems for some new large subharmonic function spaces in the unit disk.

Theorem 1.3. *(see [8]) Let*

$$R_\alpha = \left\{ u \in SH(D) : \; u(z) = \int_D \ln |A_\beta(z, \xi)| d\mu(\xi) + h(z) \right\},$$

where μ is a positive Borel measure in D, $h(z)$ is harmonic function, so that

$$\int_{-\pi}^{\pi} |h(re^{i\theta})| d\theta \le \frac{c}{(1 - r)^\alpha}, \; \beta > \alpha, \; \alpha > -1.$$

Then $u \in SH_\alpha(D)$, in other words the following embedding is valid

$$R_\alpha \subset SH_\alpha, \; \alpha > -1.$$

Let further $M_p^p(f,r) = \int_E |f(r\xi)|^p d\xi,\ 0 < p < \infty,\ 0 < r < 1.$

Theorem 1.4. *(see [8]) Let*

$$R_{p,\alpha} = \left\{ u \in SH(D):\ u(z) = \int_D \ln |A_\beta(z,\xi)| d\mu(\xi) + h(z) \right\},$$

where $\beta > \beta_0$, $\beta_0 = \beta_0(\alpha)$, for large enough β_0, μ is a positive Borel measure in D, so that

$$\int_0^1 (1-r)^{\alpha+p} (n(r))^p dr < \infty,$$

and h is harmonic so that

$$\int_0^1 (1-r)^\alpha M_1^p(h,r) dr < \infty,$$

where $0 < p < \infty$. Then $u \in SH_\alpha^p$, in other words the following embedding is valid

$$R_{p,\alpha} \subset SH_\alpha^p,\ 0 < p < \infty,\ \alpha > -1.$$

First we show similar type embeddings to those we formulated in our theorems above for some new large analytic area Nevanlinna type spaces. Then we show at the end of this paper that our results are sharp using rather transparent arguments.

Various embedding theorems for various spaces of subharmonic functions in various domains can be seen in various papers of various authors. We mention, for example, [1]-[12] and refer the reader for various references which can be seen there in those papers. Our arguments sometimes are sketchy since they are based in elementary estimates.

2. MAIN RESULTS

Our main intention is to extend these embedding theorems 1.3 and 1.4 to large new scales of spaces of subharmonic functions in the unit disk. We in particular consider the following new large spaces of subharmonic functions.

We first introduce new large spaces of subharmonic functions in the unit disk as follows and then formulate our results extending of theorems 1.3, 1.4 to these large scales of functions.

Let further

$$(SA)_{\alpha,\beta}^p(D) = \left\{ f \in SH(D): \int_0^1 \sup_{0<\tau<R} T(f,r)^p (1-r)^\beta (1-R)^\alpha dR < \infty \right\},$$

$\alpha > -1$, $\beta > 0$, $0 < p < \infty$.

$$(SB)^{p,q}_{\alpha,\beta}(D)=\left\{f\in SH(D):\int_0^1\left(\int_0^R T(f,r)^p(1-\tau)^\beta d\tau\right)^{\frac{q}{p}}(1-R)^\alpha dR<\infty\right\},$$

$0<p<\infty$, $\beta>-1$, $\alpha>-1$.

$$\left(S\tilde{B}^p_{\alpha,\beta}\right)(D)=\left\{f\in SH(D):\sup_{0<R<1}\left(\int_0^R (T(f,r))^p(1-r)^\alpha dr\right)(1-R)^\beta<\infty\right\},$$

$\alpha>-1$, $\beta\geq 0$, $0<p<\infty$.

Sometimes, these spaces will be simply denoted as X_1, X_2 and X_3, respectively.

Note $S\tilde{B}^p_{\alpha,\beta}$ classes if $p=\infty$, $\beta=0$ we have classes studied recently in [8] and our theorems can be viewed as direct extensions of theorems of O. Ohlupina.

Note, some sharp results on zero sets and related problems on these type analytic spaces were obtained in recent papers [14] and [15]. In these papers subclasses of our spaces consisting of analytic functions were considered and parametric representations were provided also. We formulate now main results of this paper. As follows, they extend some results provided previously in papers [14] and [8] of Ohlupina. We use actively machinery from [8].

We will assume that u is harmonic in a $U(0)$ where $U(0)$ is a neighborhood of 0 and also $u(0)>-\infty$, though this assumption can be removed probably using regularization procedure for subharmonic functions provided in [8].

Theorem 2.1. *Let*

$$R^p_{\alpha,\beta}=\left\{u\in SH(D):\ u(z)=\int_D \ln|A_{\widetilde{\beta}}(z,\xi)|d\mu(\xi)+h(z),\ z\in D\right\},$$

where $\widetilde{\beta}>\beta_0$, $\beta_0=\beta_0(\alpha,\beta)$, *for large enough* β_0 *and* h *is a harmonic function, so that*

$$\int_0^1\left(\int_0^R\left(\int_E |h(\tau\xi)|d\xi\right)(1-\tau)^\alpha dm_2(\tau\xi)\right)^p(1-R)^\beta dR<\infty$$

and

$$\int_0^1 n(r)^p(1-\rho)^{(\alpha+1)p+\beta+p}d\rho<\infty.$$

Then the following embedding is valid

$$R^p_{\alpha,\beta}\subset SB^p_{\alpha,\beta},\ p\leq 1,\ \alpha,\beta>-1.$$

Theorem 2.2. *Let*

$$\widetilde{R}^p_{\alpha,\beta} = \left\{ u \in SH(D): \ u(z) = \int_D \ln |A_{\tilde{\beta}}(z,\xi)| d\mu(\xi) + h(z), \ z \in D \right\}$$

where $\tilde{\beta} > \beta_0$, $\beta_0 = \beta_0(\alpha, \beta)$, *for large enough* β_0 *and* h *is harmonic function, so that*

$$\int_0^1 \left(\sup_{0<\tau<R} \left(\int_E |h(\tau\xi)| d\xi \right) (1-\tau)^\alpha \right)^p (1-R)^\beta dR < \infty$$

and

$$\int_0^1 (1-R)^{p(\alpha+1)+\beta} (n(R))^p dR < \infty.$$

Then the following embedding is valid

$$\widetilde{R}^p_{\alpha,\beta} \subset SA^p_{\alpha,\beta}, \ p \le 1, \ \alpha > -1.$$

Theorem 2.3. *Let*

$$\widetilde{\widetilde{R}}^p_{\beta,\nu} = \left\{ u \in SH(D): \ u(z) = \int_D \ln |A_{\tilde{\tilde{\beta}}}(z,\xi)| d\mu(\xi) + h(z), \ z \in D \right\}$$

where $\widetilde{\beta} > \beta_0$, $\beta_0 = \beta_0(\beta, \nu)$, *for large enough* β_0 *and* h *is a harmonic function, so that*

$$\sup_{0<R<1} \int_0^R \left(\int_E |h(\tau\xi)| d\xi \right)^p (1-\tau)^\nu d\tau (1-R)^\beta < \infty$$

and

$$n(r) \le c(1-r)^{\dfrac{-(\beta+\nu+p+1)}{p}}, \ r \in (0,1).$$

Then the following embedding is valid

$$\widetilde{\widetilde{R}}^p_{\beta,\nu} \subset S\widetilde{B}^p_{\beta,\nu}, \ \nu > -1, \ \beta \ge 0, \ 0 < p < \infty.$$

Remark 1. *These Theorems 2.1 - 2.3 extends Theorems 1.3 and 1.4 to these new spaces of subharmonic functions.*

Proofs of theorems. We need some properties of $A_\beta(z, \xi)$ first.

Lemma 1. *(see [8])*

1 Let $\xi, z \in D$, $z \ne \xi$, $\xi \ne 0$, $-1 < \beta < +\infty$. *Then*

$$\lim_{|\xi| \to 0} \frac{\ln |A_\beta(z,\xi)|}{\left(2 \ln \frac{1}{|\xi|} \right)} = 2.$$

2 Let $\xi, z \in D$, $\xi \neq 0$, $\beta > -1$. Then

$$\ln |A_\beta(z,\xi)| \leq c \left(\frac{1-|\xi|^2}{|1-\bar{\xi}z|} \right)^{\beta+2}.$$

Lemma 2. *(see [8]) Let μ be positive Borel measure in $\{z : |z| < 1\}$ the unit disk. Then we have*

$$\mu(\Delta_k) \leq \mu(D_k),$$

where Δ_k and D_k are subsets of the unit disk, $D_k = \{z : |z| < 1 - \frac{1}{2^k},\ k = 0, 1, 2, \ldots\}$, $\Delta_k = \left\{\xi : 1 - \frac{1}{2^k} \leq |\xi| \leq 1 - \frac{1}{2^{k+1}},\ k \in \mathbb{Z}_+\right\}$.

First we provide the following observation concerning subharmonic u function in the unit disk and it is Riesz measure μ. Let further $n(r) = \mu(D_r)$. We combine arguments from [8] with some known estimates from [14].

We follow some arguments from [8]. We denote by X (or by $X_j, j = 1, 2, 3$) one of our classes in our theorems. Let $u \in X \cap C^{(2)}(D)$, $u(0) > -\infty$, $\triangle u$ be a Laplacian of u function. Let further

$$n(r) = \int_0^r \int_{-\pi}^{\pi} \triangle u(re^{i\varphi}) d\varphi \ tdt,\ 0 < r < 1.$$

Following arguments of [8] we have

$$I = \int_{-\pi}^{\pi} \int_0^\rho \ln \frac{\rho}{r} \triangle u \left(re^{i\theta}\right) rdrd\varphi \leq \int_{-\pi}^{\pi} u^+ \left(\rho e^{i\varphi}\right) d\varphi,\ \rho \in (0,1).$$

Then (see [8])

$$I \equiv \int_0^\rho \frac{1}{r} \left(\int_0^r \int_{-\pi}^{\pi} \triangle u(te^{i\varphi}) tdtd\varphi \right) dr.$$

Using the fact that $n(r) = \mu(D_r) = \int_0^r \int_{-\pi}^{\pi} \triangle u(te^{i\varphi}) d\varphi tdt$ (see [8]) we have

$$\int_0^\rho \frac{n(r)}{r} dr \leq c \int_{-\pi}^{\pi} u^+(\rho e^{i\varphi}) d\varphi,\ \rho \in (0,1). \tag{A}$$

We now simply note that from (A) directly we have the following inequalities.

$$\int_0^1 \left(\int_0^R (1-\tau)^\alpha \int_0^r \frac{n(u)}{u} dud\tau \right)^p (1-R)^\beta dR \leq C_1 \|f\|_{X_1}$$

$$\int_0^1 \left(\sup_{0<\tau<R} \left(\int_0^r \frac{n(u)}{u} du \right) (1-r)^\nu \right)^p (1-R)^\sigma dR \leq C_2 \|f\|_{X_2}$$

$$\sup_{0<R<1}\left(\int_0^R\left(\int_0^\tau\frac{n(u)}{u}du\right)^p(1-\tau)^\alpha d\tau\right)(1-R)^\beta\leq C_3\|f\|_{X_3}$$

It remains to follow arguments from [14] to get what we need. Namely we have the following estimates for each function space $(X_j)_{j=1,2,3}$.

$$\int_0^1 n(\rho)^p(1-\rho)^{(\alpha+1)p+\beta+p}d\rho<\infty \quad \text{for } X_1 \text{ function space}$$

$$\int_0^1 n(\rho)^p(1-\rho)^{p(\nu+1)+\sigma}dR<\infty \quad \text{for } X_2 \text{ function space}$$

$$n(\rho)\leq\tilde{c}(1-\rho)^{-\frac{1+p+\alpha+\beta}{p}} \quad \text{for } X_3 \text{ function space}$$

For general case, $0<p<\infty$, that is when $u\in C^2(D)$, $u(0)>-\infty$ assumption is not needed we must follow again arguments from [8].

We arrived at the following theorem.

Theorem 2.4. *Let $u\in X_1$ or X_2 or X_3 function space, $\rho\in(0,1)$. Then we have for μ Riesz measure of subharmonic u function*
$\int_0^1(n(\rho))^p(1-\rho)^{(\alpha+1)p+\beta+p}d\rho<\infty$, $0<p<\infty$, $\alpha>-1$, $\beta>-1$ *for* X_1 *space.*
$\int_0^1 n(\rho)^p(1-\rho)^{p(\nu+1)+\sigma}d\rho<\infty$, $0<p<\infty$, $\nu\geq 0$, $\sigma>-1$ *for* X_2 *space.*
$n(\rho)\leq c(1-\rho)^{-\frac{1+p+\alpha+\beta}{p}}$, $\alpha>-1$, $\beta\geq 0$, $0<p<\infty$ *for* X_3 *space.*

Similar results for $SH_\alpha(D)$ and $SH^p_\alpha(D)$ spaces of subharmonic functions were obtained earlier by Ohlupina in [8].

We assume further u is subharmonic in D, $u(0)>-\infty$, and if U_0 is a ball covering zero, then u is harmonic there $A_\beta(z,\xi)$ is defined for all $z,\xi\in D$.

This assumption however can be removed via standard procedure of regularization of subharmonic functions (see [8]).

Let us return now to the proof of our Theorems 2.1 - 2.3 (new embedding theorems for our new general large spaces of subharmonic functions in the unit disk).

Let further $V_\beta(z)=\int_D\ln|A_\beta(z,\xi)|d\mu(\xi)$, $z\in D$, $u(z)=V_\beta(z)+h(z)$ and based on properties of A_β we have (see Lemma 1) for $z\in D$

$$u^+(z)\leq|h(z)|+C\int_D\left(\frac{1-|\xi|^2}{|1-\bar{\xi}z|}\right)^{\beta+2}d\mu(\xi)$$

Following the arguments used in proof of Theorem 1.1, see [8], we arrive at the following estimate

$$\int_{-\pi}^{\pi}u^+(re^{i\varphi})d\varphi\leq\int_{-\pi}^{\pi}|h(re^{i\varphi})|d\varphi+\int_{-\pi}^{\pi}\left(\int_D\left(\frac{1-|\xi|^2}{|1-\bar{\xi}z|}\right)^{\beta+2}d\mu(\xi)\right)d\varphi=I_1+I_2;$$

From here it remains to show that $I_2(r)$, $r \in (0,1)$ function satisfies certain estimates. Namely that the following estimates are valid

$$\int_0^1 \left(\int_0^R I_2(r)(1-r)^\alpha dr \right)^p (1-R)^\beta dR < \infty; \qquad (C_1)$$

$$\sup_R \int_0^R (I_2(r)^p (1-r)^\alpha dr)\,(1-R)^\beta < \infty; \qquad (C_2)$$

$$\int_0^1 \left(\sup_{0<r<R} (I_2(r))(1-r)^\alpha \right)^p (1-R)^\beta dR < \infty; \qquad (C_3)$$

So this arrives at another problem to estimate $I_2(r)$ in each $X_1,\ X_2,\ X_3$, space to show further that $(C_1),\ (C_2),\ (C_3)$ are valid using condition in formulation of our theorems. We have following again same ideas from [8] the following chain of estimates.

The proof of (C_2) is very similar to arguments used in [8]. Indeed we have the following estimates.

Let $\triangle_k = \left\{ \xi :\ 1 - \frac{1}{2^k} \le |\xi| < 1 - \frac{1}{2^{k+1}} \right\}$, $r \in \triangle_k$, then $\frac{1}{2^{k+1}} \le (1-|\xi|) < \frac{1}{2^k}$, $D = \bigcup_{k=0}^{+\infty} \triangle_k$. We have to show for (C_2) that

$$\int_0^R (I_2(r))^p (1-r)^\alpha dr \le \frac{c}{(1-R)^\beta},\ \ R \in (0,1).$$

Note that

$$\int_{-\pi}^{\pi} \left(\int_D \frac{(1-|\xi|^2)^{\beta+2}}{|1-\bar{\xi} z|^{\tilde{\beta}+2}} d\mu(\xi) \right) d\varphi \le (z = re^{i\varphi}) \le$$
$$\le c \sum_{k=0}^{\infty} \int_{\triangle_k} \frac{(1-|\xi|^2)^{\tilde{\beta}+2}}{(1-r|\xi|)^{\tilde{\beta}+1}} d\mu(\xi) \le \frac{c}{(1-r)^{\frac{\alpha+\beta+1}{p}}},\ \ r \in (0,1), \qquad \text{(see [8])}.$$

Since

$$\sum_{k=0}^{\infty} \int_{\triangle_k} \frac{(1-|\xi|^2)^{\tilde{\beta}+2}}{(1-r|\xi|)^{\tilde{\beta}+1}} d\mu(\xi) \le$$
$$\le c \sum_{k=0}^{n} \int_{\triangle_k} \frac{(1-|\xi|^2)^{\tilde{\beta}+2}}{(1-r|\bar{\xi}|)^{\tilde{\beta}+1}} d\mu(\xi) + c_1 \sum_{k=n+1}^{\infty} \int_{\triangle_k} \frac{(1-|\xi|^2)^{\tilde{\beta}+2}}{(1-r|\xi|)^{\tilde{\beta}+1}} d\mu(\xi) =$$
$$= \widetilde{I}_1 + \widetilde{I}_2, \quad |\xi| \in \left[1 - \frac{1}{2^k}; 1 - \frac{1}{2^{k+1}} \right),\ \ k \ge 0$$

It is easy to show

$$\widetilde{I}_1 \le \frac{c}{(1-r)^{\frac{\alpha+\beta+1}{p}}}, \quad \text{(see [8])},$$

and

$$\widetilde{I}_2 \le \frac{c_1}{(1-r)^{\frac{\alpha+\beta+1}{p}}},\ r \in (0,1), \quad \text{(see [8])}.$$

The rest is clear now. We have

$$\int_0^R (I_2(r))^p (1-r)^\alpha dr \le \frac{c}{(1-R)^\beta},\ R \in (0,1).$$

Theorem is proved for X_3 spaces.

Let us show (C_1) and (C_3) now. First (C_1). As it was shown in [8] if $(1-r_k) = \frac{1}{2^k}$, $n(r_k) = n_k$, $r_k - r_{k-1} = \frac{1}{2^k}$, then for $\beta \tilde{>} \beta_0$ we have

$$\begin{aligned} C_1(R) = \int_0^R (1-r)^\alpha I_2(r) dr \le \tilde{\tilde{c}} \int_0^R (1-r)^\alpha \int_0^1 \frac{(1-\rho)^{\tilde{\beta}+1}}{(1-r\rho)^{\tilde{\beta}+1}} n(\rho) d\rho \le \\ \le C_1 \sum_{k=1}^\infty \frac{n_k}{2^{k(\tilde{\beta}+2)}} \left[\int_0^R \frac{(1-r)^\alpha dr}{(1-r_k r)^{\tilde{\beta}+1}} \right] \le \\ \le C \sum_{k=1}^\infty \frac{n_k}{2^{k(\tilde{\beta}+2)}} \frac{1}{(1-r_k R)^{\tilde{\beta}-\alpha}}. \end{aligned}$$

Now for $p \le 1$ (we can easily reformulate condition in our theorem on $n(r)$ in terms of n_k)

$$\int_0^1 C_1(R)^p (1-R)^\beta dr \le c \sum_{k=1}^\infty \frac{(n_k^p)(2^{-k\beta})}{2^{k(\tilde{\beta}+2)p}} \left[2^{-k\alpha p}\right] \left(2^{-k}\right) \left(2^{-k\tilde{\beta}p}\right) \le c.$$

The proof of (C_3) is similar. We use

$$\sup_{0<r<R} \frac{(1-r)^\alpha}{(1-r\rho)^{\tilde{\beta}+1}} \le \frac{1}{(1-R\rho)^{\tilde{\beta}+1-\alpha}},\ R, \rho \in (0,1),$$

(a bit general form of this) and we repeat $0 < p < \infty$ case almost similarly for (C_3).

Note, indeed, that

$$\int_0^1 \sup_{0<r<R} (I_2(r)(1-r)^\alpha)^p (1-R)^\beta dR \leq$$

$$\leq \widetilde{C} \int_0^1 \left(\int_{-\pi}^{\pi} \left(\int_D \frac{(1-|\xi|^2)^{\beta+2} d\mu(\xi)}{|1-\bar{\xi}z|^{\beta+2-\alpha}} \right)^p d\varphi \right) (1-R)^\beta dR,$$

$\alpha > 0,\ \beta > -1$ and it remains to repeat arguments we provided for (C_1) and (C_2) case using the fact that for $\alpha_2 > \alpha_1 + 1,\ \alpha_1 > -1$

$$\int_0^1 (1-R)^{\alpha_1} (1-R\rho)^{-\alpha_2} dR \leq \widetilde{C}(1-\rho)^{-\alpha_2+\alpha_1+1},\ \rho \in (0,1).$$

Theorem is proved.

Remark 2. *For $p > 1$ similar type argument based on Hardy's inequality leads to same conclusion. We however do not consider this case here in details refereing to [8] and leaving this case to interested readers.*

Another case of interest for this theory is that of parametric representation results for

$$D^c = \{z \in C;\ |z| > 1\} (\text{the complement of the unit disk}).$$

This may be the subject of a further study for the authors.

References

[1] K. Avetisyan, *On representations of some classes of subharmonic functions in the unit disk and upper half space* (in Russian), Izv. Nat. Acad. Armenii, Matematika **29**, 1(1994), 3-15. (Published translation: Journal of Contemporary Mathematical Analysis, Armenian Academy of Sciences **29**, 1(1994), 1-12.)

[2] M. Brelo, *The Basis of the Classical Potential Theory*, Mir, Moscow, (1964), (in Russian).

[3] T. McConnell, *Area integrals and subharmonic functions*, Indiana University Mathematics Journal **33**, 2(1984), 289-303.

[4] T. M. Flett, *Inequialities for the p^{th} mean values of harmonic and subharmonic functions with $p \leq 1$*, Proc. London Math. Soc. 3, **20**, 2(1970), 249-275.

[5] T. M. Flett, *Mean values of subharmonic functions on half-spaces*, Journal of the London Mathematical Society, s2-1(1)(1969), 375-383.

[6] W. Hayman, P. Kennedy, *Subharmonic functions*, Moscow, Mir, 1980, 304 pages.

[7] P. Lelong, *Sur la representation d'une fonction plurisussharmonic*, C.R. Acad.Sci Paris, 237(1953), 691-693.

[8] O. V. Ohlupina, *Green type potentials and integral representations of spaces of subharmonic classes of subharmonic functions*, Bryansk St. Univ. Phd Dissertation, 2012, (in Russian)

[9] I. I. Privalov, *Subharmonic functions*, 1951, Moscow, 199 pages, (in Russian)

[10] F. Riesz, *Sur les Fonctions Subharmoniques et Leur Rapport a la Théorie du Potentiel*, Acta Mathematica **48** (3-4)(1926), 329-343.

[11] J. Riihentaus, *A generalized mean value inequality for subharmonic functions*, Expositiones Mathematicae **19** (2)(2001), 187-190.

[12] M. Stoll, *On the Littlewood-Paley inequialities for subharmonic functions on Domains in R^n*, Recent Advances in Harmonic Analysis and Applications, Springer New York (2013), 357-383.

[13] F. Shamoyan, E. Shubabko, *Parametric representations of some clases of holomorphic functions in the disk*, Operator theory: Advanced and Applications, volume 113, (2000).

[14] R. Shamoyan, Haiying Li, *Descriptions of zero sets and parametric representations of certain analytic area Nevanlinna type classes in the unit disk*, Kragujevac Journal of Mathematics 34(2010), 73-89.

[15] R. Shamoyan, Haiying Li, *On the action of differentiation operator in some classes of Nevanlinna-Djrbashian type in the unit disk and polydisk*, Armenian Journal of Mathematics **2**, (4)(2009), 120-134.

[16] M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen Co, Tokio, (1959).

ROMAI J., v.17, no.1(2021), 137–149

SOME APPROACHES AND SOLUTIONS IN DECISIONS MAKING FOR PROCESSING ILL-STRUCTURED DATA AND KNOWLEDGE

Inga Titchiev, Constantin Gaindric

Vladimir Andrunachievici Institute of Mathematics and Computer Science, Chişinău, Republic of Moldova

inga.titchiev@math.md, constantin.gaindric@math.md

Abstract The article presents some up-to-date results of the research obtained by the project team: *Intelligent Information systems for solving ill–structured problems, knowledge and Big Data processing.* The research is carried out in development of intelligent information systems with applications in three social domains: medicine, education and culture. As a rule, in relation to these domains, it is proposed to solve the ill–structured problems which operate with a large volume of data, depend on the decision maker's vision, and need a personalized approach.

Keywords: Data, information, knowledge, Digital technology, Artificial Intelligence, ill–structured problems, knowledge processing.

2020 MSC: 68M01.

1. INTRODUCTION

The existence of society in our days, but even more so in the future, depends on emerging ICT technologies, which integrate with the traditional ones gradually replacing them. The integrated technologies are applied in the real production environment, as well as in the artificially created one, but also in services. They ensure effective interaction, respond to speech, movement and, of course, to the person's traditional commands. 3G technologies already provide fast and easy transfer of data in the form of text, images and voice, which will make private and public services more efficient. Virtual reality will make possible interactive virtual modeling and design of production systems . Advanced speech and motion detection allow you to almost completely control information systems using speech and gestures. The use of distributed data warehouses and extensive information networks (e.g. GRID) already allows the use, if necessary, of sufficiently large amounts of information needed to achieve the desired goals regardless of the user's workplace.

The predominantly unstructured character of the data remains a permanently stringent current problem. We speak about data on the basis of which the computer systems operate, with which the databases are populated but also with which they are traditionally manipulated. Therefore, the processing of ill-structured data and knowledge remains relevant, especially the processing methods. At present, but also until some progress is made, the processing methods depend very much on the examined domain. New IT solutions for information systems, that will be used and specific tools are needed to improve the quality of data, information, and knowledge in daily activity and perspective processes [1]. Looking to the future is not a routine task, which makes it very difficult and important to provide activities with good quality of data, information, and knowledge.

2. BASIC CONCEPTS

As digital technology becomes more and more important in many aspects of everyday life, people should be able to trust it. Ensuring trust is also a prerequisite for the adoption of this technology. This is an opportunity for Europe, given its strong commitment to values and the rule of law and its proven ability to design safe, reliable and complex products and services, from aeronautics to energy, cars, and medical equipment.

The sustainable growth and societal well-being of present and future Europe is increasingly based on the value of data.[20] Artificial Intelligence (AI) [7] is one of the most important applications of the data-based economy. Today, most data is consumer-related and stored and processed in central cloud infrastructures. Instead, much of tomorrow's data, which will be much more abundant, will come from industry, businesses and public sector and will be stored on a variety of systems, especially the computer devices that will operate on the periphery of the network. This creates new opportunities for Europe, which is in a strong position in the digital industry and business-to-business applications, but occupies a relatively weak place in terms of consumer platforms.

The High Level Expert Group proposed "A Definition of AI", [20]: *"Artificial Intelligence (AI) systems are software (and possibly hardware) designed by humans that, if they are given a complex goal, acts in the physical or digital dimension, perceiving the environment by taking data, interpreting structured or ill-structured data collected, reasoning about knowledge or processing information obtained from that data, and deciding on the best action to be taken in order to achieve the goal. AI systems can use symbolic rules or learn a numerical model, and they can also adapt their behavior by analyzing how the environment is affected by their previous actions."*

For some time now, there has been a conceptual and technological discussion about the hierarchy of data, information and knowledge, as well as the quality aspects associated with them.[2] Data are considered as a descriptive element, representing the perception and intensity of an examined phenomenon. Information is more than just a data set; it is the result of a process that interprets and processes data in a specific prescribed format. There is also the term **informational product**, insisting on the idea that information is not only the introduction, but also the procedures used to obtain it, its processing, while the notion of information more frequently used explicitly insists on the exchange of data [3].

The formulation of the notion of knowledge encounters some difficulties that can be overcome by a categorization, for example:

- **Explicit knowledge**: knowledge expressed in words or numbers. This type of knowledge is coded and well defined;
- **Tacit knowledge**: knowledge expressed through introspection, intuition and suspicion. This type of knowledge is very personal and difficult to formalize;
- **Self-transcending knowledge**: the ability to sense the presence of a possible potential to see what does not yet exist, which can be considered as tacit knowledge before its embodiment.

The third type of knowledge, the notion of self-transcendent knowledge was introduced by Sharmer C.O.[4] which sustain that the examination of knowledge management in the next decade will focus on the interaction of three forms of knowledge - explicit, tacit and self-transcendent.

In addition, each of these three types of knowledge can be classified according to whether it can be described as:

- **Declarative knowledge**: facts, know-how, understanding;
- **Explanatory knowledge**: rationalization, knowledge of knowledge;
- **Procedural knowledge**: instructions, know-how, understanding;
- **General/organizational knowledge**: knowledge that is easy to be transmitted on and that is held by a large number of people;
- **Specific/individual knowledge**: knowledge that very few have.

Knowledge is made up of a set of input data: information, experience, relationships and techniques that each individual mentally synthesizes to form a conception of how to approach the problem that needs to be solved or to form an opinion about the decision-making situation or about a person, on whom the solution of the problem in question depends.

Data quality and information quality are not new concepts, but they have been increasingly emphasized in research during the last years. Most of the concerns about data and information quality are due to ICT specialists, information management systems, databases and their management, data security and data warehouse quality. Many researchers, including Pierce, Kahn, and Melkas [1], examine the relationship between data quality, information quality, and knowledge quality. Most researchers agree that improving the quality of data should lead to the improvement in the quality of the information that is generated from this data. It therefore seems reasonable that improving information should, in turn, improve the quality of knowledge.

Convinced that information systems will become the predominant factor in the progress of any field of activity, there are still many questions that need to be answered:

1 Is high data quality the only requirement for improving the quality of information?

2 Does high-quality information automatically "turn" into knowledge and, if so, what kind of knowledge?

3 How does the improvement execute between data, information, and knowledge?

One answer, seems to be natural: one of the possibilities proposed by this project is to structure the data, and matter-of-course the knowledge; and methods of structuring both data and knowledge must be developed according to the field examined, taking into account the specificity of the domain and the nature of the data. Thus, we consider there, opens a way to process knowledge and large volumes of data that will allow the development of intelligent information systems to solve problems of major public interest.

3. SOME LANDMARKS OF THE PROJECT FOR THE DEVELOPMENT OF INTELLIGENT IT SYSTEMS FOR SOLVING ILL– STRUCTURED PROBLEMS AND KNOWLEDGE PROCESSING

In an attempt to answer the above questions, the goal of the project was formulated as follows:

1 Structural design of intelligent information systems, databases, and knowledge bases for applications in medical triage and diagnostics, e-learning and digitization of heterogeneous documents;

2 Adaptation and integration of content and software from the existing developments in the concept of intelligent information systems.

In order to carry out the research, the following objectives were formulated:

1 Design of intelligent IT systems, databases and knowledge bases for medical triage and diagnostic applications;

2 Creating the platform for digitizing heterogeneous documents;

3 Development of the automatic content generation systems for computer-based education (e-learning);

4 Elaborating of the systemic concept of the heterogeneous multi-cloud platform and the methods of creating the execution environment of the imaging information processing applications;

5 Elaborating of the concepts and tools for interpreting and evaluating information.

4. THE MAIN OBTAINED RESULTS

4.1. DESIGN OF INTELLIGENT IT SYSTEMS, DATABASES AND KNOWLEDGE BASES FOR MEDICAL TRIAGE AND DIAGNOSTIC APPLICATIONS

1 The minimum set of parameters required for the registration of victims has been identified, in limited time and comprehensive compliance with the provisions of national and international protocols based on the experience of physicians-experts who determine triage decisions based on vital signs and allow rapid classification of victims.[5]

2 The inference algorithm in the form of a decision table represents the actions of the emergency diagnosis and allows the triage of victims.

3 The SonaRes technology platform, being adapted to EFAST (Figure 1), has been completed with attributes/values, which allow the localization of free fluids. EFAST diagnostic rules have been created.

4 In everyday life, and in particular in the case of the algorithm for emergency diagnosis, we often need to manage para-consistent information, i.e. seemingly contradictory information. One of the proposed methodologies for managing inconsistent information is the use of the concept of para-consistent negation.

4.2. PLATFORM FOR DIGITIZING HETEROGENEOUS DOCUMENTS

4.2.1 Modules to support manual verification of the processed documents.

- Step 1. Improving image quality - by providing online access to the relevant features of ABBYY FineReader (AFR): automatic preprocessing, alignment, splitting, etc [6].;

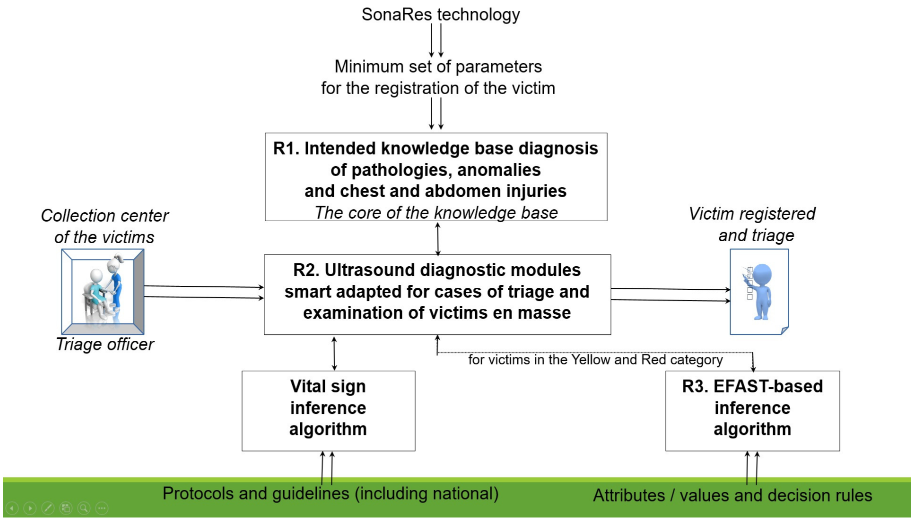


Fig. 1. Adapted SonaRes technology

	Red (I)	Red (II)	Yellow	Green
Glasgow Coma Scale	3-8	9-13	14	15
Airways Permeability	Obstruction/Stridor	Difficult breathing	Normal breathing	Normal breathing
Pulse	>120 or <40	111-120 or 41-45	81-110 or 46-59	60-80
Systolic Blood Pressure	<80	80-89	90-100	>100
Respiratory Rate	>35 or <13	29-35	19-28	14-18
Oxygen saturation	<= 85	86-90	91-95	96-100
Individual mobility	Unable	Unable	With help	Walking

Table 1 Basic attributes and values in triage based on virtual signs.

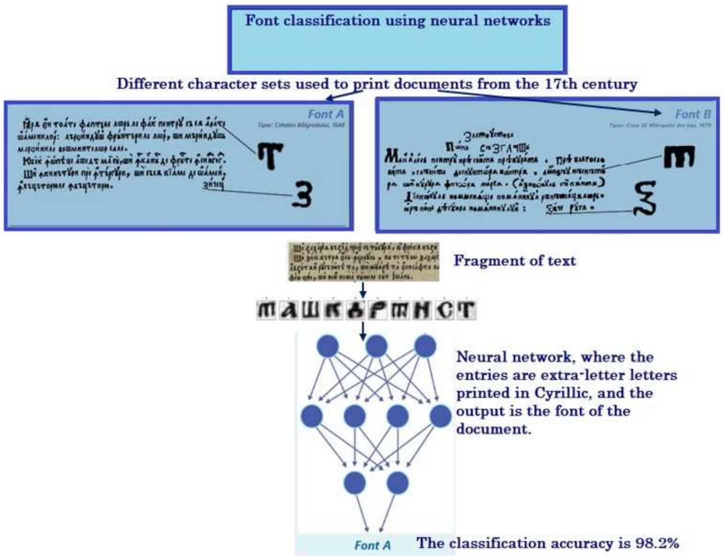


Fig. 2. Neural networks in font classification

- Step 2. Checking/correcting the layout of the recognized document - by displaying the web version of the reconstructed document, in which both the margins and the content of the layout elements can be edited;

- Step 3. For text elements - font correction. Font classification is performed, the platform provides tools for correcting/modifying the font. Fonts were classified using neural networks[9, 10]. In addition, for non-Latin fonts, transliteration may be applied at the user's request. The platform provides support for spelling correction of texts, including old ones.

- Step 4. Writing metadata. Metadata is provided in textual form in the format "name = value."

- Step 5. Checking/correcting scripts - using script-specific writing modules (LaTeX editor for math formulas[8] and other types, Musescore for music notes, etc.)

- Step 6. Preview and save the reconstructed document.

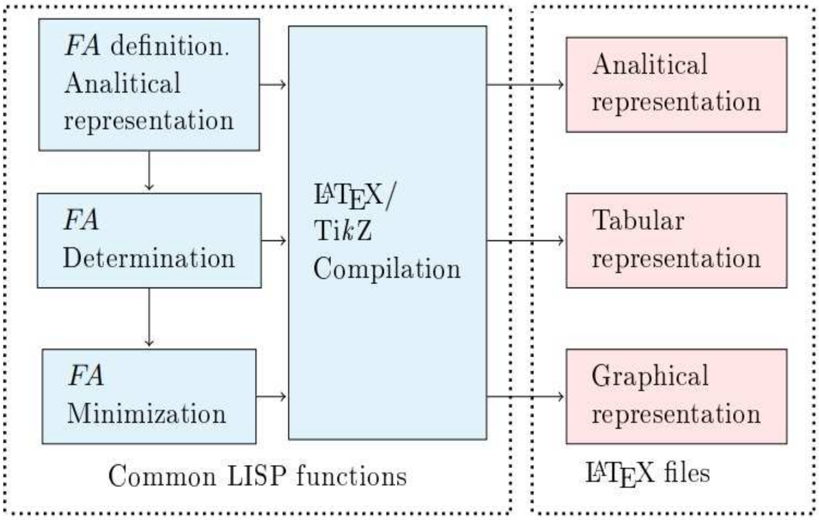


Fig. 3. Automatic generation[18, 19] of graphical representations of finite automata scheme

4.3. AUTOMATIC CONTENT GENERATION SYSTEMS FOR COMPUTER-BASED EDUCATION (E-LEARNING)

1 The possibilities offered by the existing e-learning platforms[14] were researched and analyzed. The evaluation criteria were selected and the comparative analysis of the 8 most popular platforms and learning management systems that support the use of multimedia elements, content creation, and editing, was performed.

2 The system of automatic content generation[15] for computer-based education (e-learning) has been extended by making equivalent transformations on stack memory machines and context-independent grammars.

3 The management of learning progress and collaboration issues that may arise in distance learning through Petri nets have been researched (Figure 4). After modeling the collaborative learning process, analyzing the nets obtained through the coverability trees, the blockages in the system, the learning path and the improvement of the process were estimated. Hierarchical Petri nets (HLPNs) [11] have been applied to build various control sequences in distance learning. Depending on the student's behavior, different ways of learning are proposed. Different training strategies are proposed: linear, choice and arbitrary (combining the first two).

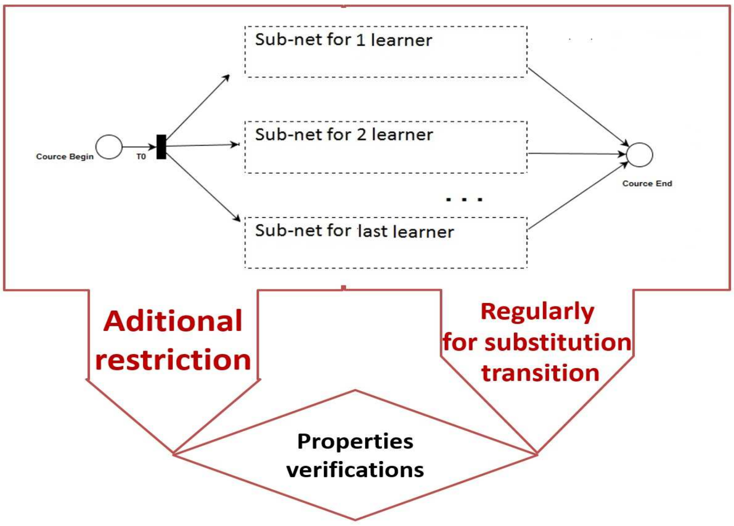


Fig. 4. HLPN in collaborative learning

4 Algorithms for the automatic generation of computer-based education content have been developed [12]. The mechanism for generating the list of questions in an adaptive TestWid test has been developed. The system will generate the list of questions, which were not presented to the previous student. The development steps of the application that generate the content at the request of the user through keywords have been described. Ways to improve the content generation process have been defined.

4.4. SYSTEMIC CONCEPT OF THE HETEROGENEOUS MULTI-CLOUD PLATFORM AND THE METHODS OF CREATING THE EXECUTION ENVIRONMENT OF THE IMAGING INFORMATION PROCESSING APPLICATIONS

1 Methods for storing and archiving the ill-structured data (medical images) based on hierarchical platforms for processing and visualizing the imaging information in distributed computing systems used to implement multi-level memory in archiving have been proposed.

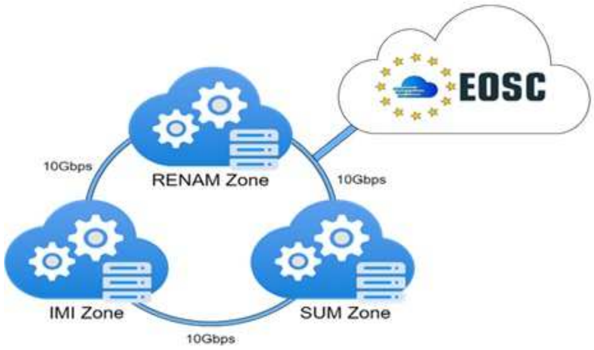


Fig. 5. Distributed Cloud infrastructure

2 Software tools for creating Open Science repositories[17] for storing imaging data have been studied.

3 A solution for the implementation of a distributed Cloud infrastructure (figure 5) with the possibility of its further integration in the European computing infrastructure EGI was chosen.

4 Technologies for the creation of intelligent software agents (e.g. chatbots) have been established for access to the knowledge base for ultrasound diagnosis of pathologies.

4.5. CONCEPTS AND TOOLS FOR INTERPRETING AND EVALUATING INFORMATION

1 Methods to adapt software tools from the existing developments to assess the credibility of online information have been developed[13].

2 Basic features of several tools for verifying the credibility of Web sources were analyzed[16].

3 In order to create a theory of the ontological unity of Information/ Matter/ Energy, the concept of information of natural kinds, the concept of evolutionary information - Plyrophoria [21], the concept of emergent information and embedded in matter information based on the quantum states that make up the matter of systems was formulated and described (Figure 6).

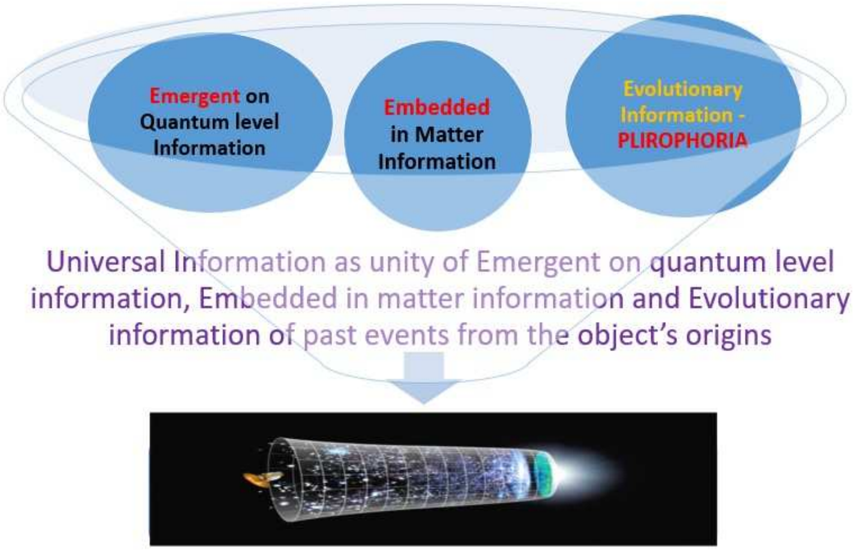


Fig. 6. Information in Wiener's sense

5. CONCLUSIONS

In the Digital Europe and Horizon Europe programs, adopted by the European Commission, great attention is paid to artificial intelligence, which is seen as a tool capable of providing more benefits to citizens and businesses across Europe, including better disease prevention, enhanced cyber security, and more. Several EU countries have adopted strategies to promote Artificial Intelligence, research programs in the field, specialization master programs for applications in artificial intelligence. We consider that these examples would be worth following, otherwise our country risks being left without the capacity to use advanced technologies in all fields of activity: economy, education, health, culture, etc.

The research that comes with a national contribution to a current challenge of enabling online access to European heritage digital resources, will facilitate the generation of digital content of computer-assisted training courses with the application of knowledge bases, reusable language resources. Also it will contribute to the systemic treatment of the poorly structured issues. Modeling of human intelligence is a fundamental concern that influences practical applications developed.

Acknowledgment. 20.80009.5007.22, *Intelligent Information systems for solving ill structured problems, knowledge and Big Data processing* project has supported part of the research for this paper.

References

[1] Pierce, E.; Kahn, B.; Melkas, H., *A comparison of quality issues for data, information, and knowledge*, in: M. Khosrow-Pour (Ed.), Emerging Trends and Challenges in Information Technology Management: Proceedings of the 2006 Information Resources Management Association Conference, 17th IRMA International Conference, Washington, DC, USA, May 2006, pp. 21–24.

[2] Uotila, T.; Melkas, H., *Quality of data, information and knowledge in regional foresight processes*, Futures **39**(2007), pp. 1117-1130.

[3] Wang, R.Y.; Lee, Y.; Pipino, L.; Strong, D., *Manage your information as a product*, Sloan Management Review, **39**(4), (1998), pp. 95–105.

[4] Sharmer, O., *Self-transcending knowledge: organizing around emerging realities*, in: I. Nonaka, D. Teece(Eds.), Managing Industrial Knowledge: Creation, Transfer and Utilization, Sage Publications, London, 2001, pp. 68–90.

[5] Gaindric, C.; Şandru, S.; Puiu, S.; Popcova, O.; Secrieru, I.; Guţuleac, E., *Advanced prehospital triage based on vital signs in mass casualty situations*, Proceedings of the Workshop on Intelligent Information Systems WIIS2021, October 14-15, 2021, Chişinău, Republic of Moldova, pp. 130-134.

[6] Colesnicov, A.; Malahov, L.; Cojocaru, S.; Burtseva, L., *On XML Standards to Present Heterogeneous Data and Documents*, Proceedings of Workshop on Intelligent Information Systems WIIS2021, October 14-15, 2021, Chişinău, Republic of Moldova, pp.112-117.

[7] Cojocaru, S.; Gaindric, C.; Verlan, T., *Artificial Intelligence Strategies: Republic of Moldova relative to European Union countries*, Computer Science Journal of Moldova, vol. **29**, N.3 (87), 2021, ISSN 1561-4042. pp.299-319.

[8] Burtseva, L.; Cojocaru, S.; Malahov, L.; Colesnicov, A., *Heterogeneous document processing: Case study of mathematical texts*, Abstracts of International Conference Mathematics & It: Research and Education (MITRE-2021) dedicated to the 75th anniversary of Moldova State University, July 1-3, 2021, pp. 96–97. ISBN 978-9975-158-19-0.

[9] Bumbu, T., *On Classification of 17th Century Fonts using Neural Networks*, Computer Science Journal of Moldova, vol. **29**, N.3(87), 2021, ISSN 1561-4042. pp.291-298

[10] Bumbu, T., *On Classification of 17th Century Fonts using Neural Networks*, Proceedings of the Workshop on Intelligent Information Systems WIIS2021, October 14-15, 2021, Chişinău, Republic of Moldova. pp. 58-64.

[11] Titchiev, I., *Collaborative learning modelled by High-Level Petri nets*, Proceedings WIIS2021, Workshop on Intelligent Information Systems, October 14-15, 2021, Chisinau, Republic of Moldova, pp. 178-184.

[12] Parahonco, A.; Petic, M. *Generation and use of educational content within adaptive learning*, Proceedings of Workshop on Intelligent Information Systems WIIS2021, October 14-15, 2021, Chişinău, ISBN 978-9975-68-438-5, pp. 156 – 167.

[13] Petic, M.; Gorea, A.; Titchiev, I. *Approaches in Assessing the Credibility of Online Information.* In: Proceedings of the 15 the International Conference "Linguistic Re-

sources and Tools for Natural language processing", online, 14-15 December 2020, pp. 151-160. ISSN 1843-911X.

[14] Titchiev, I.; Iamandi, V., *Analysis of learning management systems features*, International Conference Mathematics & IT: Research and education (MITRE -2021) dedicate to the 75 th anniversary of Moldova State University. Abstracts, July 01-03, pp. 116-117, Chişinău, 2021, ISBN 978-9975-158-19-0.

[15] Parahonco, A., Petic, M., *Steps in content generation for e-cources*, Abstracts of International Conference Mathematics & It: Research and Education (MITRE-2021) July 1-3, 2021, pp. 109-110. ISBN 978-9975-158-19-0.

[16] Gorea, A.; Petic, M., *Some particularities in assessment of online information credibility.* Abstracts of International Conference Mathematics & IT: Research and Education (MITRE-2021) July 1-3, 2021, 105-106. ISBN 978-9975-158-19-0.

[17] Bogatencov, P.; Secrieru, G.; Hincu, B.; Degteariov, N., *Development of computing infrastructure for support of Open Science in Moldova*, In Workshop on Intelligent Information Systems (WIIS2021) Proceedings, Chişinău, IMI, 2020, pp. 34-45.

[18] Ciubotaru, C., *Backtracking algorithm for lexicon generation*, Computer Science Journal of Moldova, vol.**29**, no.1(85)(2021), pp. 135-152.

[19] Ciubotaru, C., *Generation and visualization of graphical representations of finite automata*, Proceedings of the Workshop on Intelligent Information Systems WIIS2020, December 04-05, 2020, Chisinau, Republic of Moldova, pp.52-60.

[20] *Artificial Intelligence for Europe*, COM (2018) 237-final, [online] https://knowww.eu/nodes/5cdbf48ddb38f26fcdf0d3a3.

[21] Albu, V., *The Plirophoria: The Missing Puzzle of the Ultimate Picture of the Universal Information*, Proceedings of the Workshop on Intelligent Information Systems WIIS2021, October 14-15, 2021, Chisinau, Republic of Moldova, pp. 26-33.