ROMAI J., v.18, no.2(2022), 89–106

IMPLICIT FRACTIONAL DIFFERENTIAL INCLUSIONS WITH CAPUTO FRACTIONAL DERIVATIVE AND NONLOCAL CONDITION

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Abstract In this paper, we establish sufficient conditions for the existence of solutions for implicit fractional differential inclusions with nonlocal conditions. Both cases of convex and nonconvex valued right hand side are considered.

Keywords: Existence, Caputo derivative, implicit, fractional differential inclusion, convex, nonconvex.

2020 MSC: 26A33, 34A08, 34A60, 34B15.

1. INTRODUCTION

In this paper, we are concerned with the existence of solution for implicit fractional differential inclusion

$$^cD^\alpha y(t) \in F(t, y(t), ^c D^\alpha y(t)), \text{ for a.e. } t \in J = [0, T], \ 0 < \alpha \leq 1 \qquad (1)$$

$$\sum_1^m a_k y(t_k) = y_0, \qquad (2)$$

where $^cD^\alpha$ is the Caputo fractional derivative, $F : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}.$,$y_0 \in \mathbb{R}$ $a_k \in \mathbb{R}$ and $0 < t_1 < t_2 < ... < t_m < T, \ k = 1, 2, ..., m.$

Differential equations and inclusions of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed we can find numerous applications in viscoelasticity, electrochemistry, electromagnetism, and so forth. For details, including some applications and recent results, see the monographs of Kilbas et al. [21], Podlubny [23], and the papers of Agarwal et al [4, 5], Momani et al.[22], Guerraiche et *al.* [18, 19], and the references therein.

Implicit differential equations involving the regularized fractional derivative were analyzed by many authors, in the last year, see for instance [1, 2, 3] and the references therein [10, 11, 12].

To our knowledge, the literature on integral solutions for fractional differential equations is very limited. El-Sayed and Hashem [16] studied the existence of integral and continuous solutions for quadratic integral equations. El-Sayed and Abd El Salam considered Lp-solutions for a weighted Cauchy problem for differential equations involving the Riemann-Liouville fractional derivative.

In this paper, we present an existence result for the problem (1)-(2) when the right hand side is convex valued by using nonlinear alternative of Leray Schauder type. The second results are given for nonconvex valued right hand sides, which are based upon a fixed point theorem for contraction multivalued maps due to Covitz and Nadler[13] . Finally, we present an example to demonstrate the application of our main results.
Let us mention that most of the existing results for fractional order differential inclusions are devoted to continuous or Caratheodory solutions. Thus, the main results of the present paper constitute a contribution to this emerging field.

2. PRELIMINARIES

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper.
Let $C(J,\mathbb{R})$ be the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_{\infty} = \sup\{|y(t)| : t \in J\}.$$

and let $L^1(J;\mathbb{R})$ be the Banach Lebesgue integrale functions $y : J \to R$ with the norm

$$\|y\|_{L^1} = \int_J |y(t)|dt.$$

For any Banach space $(X, \|.\|)$, we set:

$P_{cl}(X)$=$\{Y \in P(X)$:Y closed$\}$.

$P_b(X)$=$\{Y \in P(X)$:Y bounded$\}$.

$P_{cp}(X)$=$\{Y \in P(X)$:Y compact$\}$.

$P_{cp,c}(X)$=$\{Y \in P(X)$:Y compact and convex$\}$.

A multivalued map $G : X \to \mathcal{P}(x)$ is:

- convex(closed)valued if $G(X)$ is convex(closed) for all $x \in X$;
- bounded on bounded sets if $G(B) = U_{x\in B}G(x)$ is bounded in X, for all $B \in P_p(X)$ (i,e $sup_{x\in B}sup|y|; y \in G(x)$;

- upper semi-continuous (u.s.c) on X if for each $x_0 \in X$,the set $G(x_0)$ is a there nonempty closed subset of X , and for each open set N of X containing $G(x_0)$ there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subset N$;
- completely continuous if $G(B)$ is relatively compact for every $B \in P_b(x)$.

If the multivalued map G is completely continuous with nonempty compact values,then G is u.s.c if and only if G has a closed graph (i.e $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in X$ such that $x \in G(X)$. The fixed point set of the multivalued operator G will be denoted by $FixG$.A multivalued map $G : J \to P_{cl}(\mathbb{R})$is said to be measurable if for every $y \in (R)$, the function:

$$t \to d(y, G(t)) = inf\{|y - z| : z \in G(t)\}$$

is measurable.

Definition 2.1. *A multi-valued map* $F : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is said to be Carathéodory if*

(1) $t \to F(t, u, v)$*is measurable for each* $u, v \in \mathbb{R}$*;*

(2) $u \to F(t, u, v)$ *is upper semicontinuous for almost all* $t \in J$*.*
Further, a Carathéodory function is called L^1*-Carathéodory, if*

(3) for each $\rho > 0$*,there exists* $\phi_\rho \in L^1([0, T], \mathbb{R}^+)$ *such that*
$\|F(t, u, v)\| = \sup\{|v|, v \in F(t, u, v)\} < \phi_\rho(t)$*, for all* $|v|$*,* $|u| < \rho$*.*

Let (X, d) be a metric space induced from the normed space $(X, \|.\|)$. The function $H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}$ given by :

$$H_d(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, B)\}$$

is known as the Hausdorff-Pompeiu metric.

Definition 2.2. *A multivalued operator* $N : X \to P_{cl}(X)$ *is called*

(1) Lipschitz if there exists $\gamma > 0$ *such that*

$$H_d(N(x), N(y)) < \gamma d(x, y) \text{ for each } x, y \in X;$$

(2) a contraction if it isγ*-Lipschitz with* $\gamma < 1$*.*

The following fixed point result for contraction multivalued maps is due to Covitz and Nadler [13].

Lemma 2.1. *Let (X, d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

For more details on multivalued maps see the books of Aubin and Cellina [6], Aubin and Frankowska [7] and Castaing and Valadier [14].

Definition 2.3. *([21]) Let $h \in L^1([a, b], \mathbb{R})$. The left sided fractional integral of Riemann-Liouville of order α is defined by*

$$(I_a^\alpha h)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s) ds,$$

where $\alpha > 0$. When $a = 0$, we write

$$I^\alpha h(t) = h(t) * \varphi_\alpha(t)$$

where

$$\varphi_\alpha(t) = \begin{cases} \dfrac{t^{\alpha-1}}{\Gamma(\alpha)} & \text{for } t > 0 \\ 0 & \text{for } t \leq 0, \end{cases}$$

and

$$\varphi_\alpha \to \delta(t) \text{ as } \alpha \to 0$$

where δ is the delta function.

Definition 2.4. *Suppose that $\alpha > 0$, $t > a$, $\alpha, a, t \in \mathbb{R}$. The fractional operator*

$$^C D^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^t \frac{f^n(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, & n-1 < \alpha < n \in \mathbb{N} \\ \dfrac{d^n}{dt^n} f(t), & \alpha = n \in \mathbb{N}, \end{cases} \tag{3}$$

is called the Caputo fractional derivative or Caputo fractional differential operator of order α.

Example 2.1.

Let $a = 0, \alpha = \frac{1}{2}, (n = 1), f(t) = t$.
Then, applying formula(3), we have

$$^C D^{\frac{1}{2}} t = \frac{1}{\Gamma(\frac{1}{2})} \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}} d\tau.$$

Taking into account the properties of the Gamma function and using substitution $u := (t-\tau)^{\frac{1}{2}}$ the final result for the Caputo fractional derivative of the function $f(t) = t$ is obtained as

$$\begin{aligned} {}^{C}D^{\frac{1}{2}}t &= \frac{1}{-\sqrt{\pi}} \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}} d(t-\tau) \\ &= \frac{1}{-\sqrt{\pi}} \int_{\sqrt{t}}^0 \frac{1}{u} du^2 \\ &= \frac{1}{\sqrt{\pi}} \int_0^{\sqrt{t}} \frac{2u}{u} du \\ &= \frac{2}{\sqrt{\pi}} (\sqrt{t} - 0). \end{aligned}$$

Thus:

$${}^{C}D^{\frac{1}{2}}t = \frac{2\sqrt{t}}{\sqrt{\pi}}.$$

Lemma 2.2. *For the Caputo fractional derivative:*

$$D^{\alpha}C = 0, \quad C = const.$$

Theorem 2.1. *(Kolmogorov compactness criterion* [15] *). Let $\Omega \subseteq Lp(J,\mathbb{R})$, $1 \le p \le +\infty$. If*

(i) Ω is bounded in $Lp(J,\mathbb{R})$ and

(ii) $u_h \to u$ as $h \to 0$ uniformly with respect to $u \in \Omega$ then Ω is relatively compact in $Lp(J,\mathbb{R})$, where

$$u_h(t) = \frac{1}{h} \int_t^{t+h} u(s)ds.$$

Proposition 1. [21] *Let α,$\beta > 0$. Then we have*

(1) $I^{\alpha} : L^1(J,\mathbb{R}) \to L^1(J,\mathbb{R})$, and if $f \in L^1(J,\mathbb{R})$, then

$$I^{\alpha} I^{\beta} f(t) = I^{\beta} I^{\alpha} f(t) = I^{\alpha+\beta} f(t).$$

(2) If $f \in L^p(J,\mathbb{R})$, $1 < p < \infty$, then $\| I^{\alpha} f(t) \|_{L^p} \le \frac{T^{\alpha}}{\Gamma(\alpha+1)} \| f(t) \|_{L^p}$.

(3) The fractional integration operator I^{α} is linear.

(4) The fractional order integral operator I^{α} maps $L^1(J,\mathbb{R})$ into itself.

(5) When $\alpha = n$, I_0^{α} is the n-fold integration.

(6) The Caputo and Riemann-Liouville fractional derivative are linear.

Theorem 2.2. [17]*(Nonlinear alternative of Leray Schauder type)*
Let X be a Banach space and C a nonempty closed convex subset of X. Let U be a nonempty convex subset of C with $0 \in U$ and $T : \bar{U} \to P_{cp,c}(X)$ is a upper semicontinuous compact map. Then either

(i) T has a fixed point in U, or

(ii) there exist $u \in \partial U$ and $\lambda \in [0,1]$ for which $u \in \lambda T(u)$.

Lemma 2.3. [24] *Let $\alpha \geqslant 0$. Then the differential equation*

$$^cD^\alpha h(t) = 0$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 ... + c_{n-1} t^{n-1}$, $c_i \in \mathbb{R}$, $i = 0, 1, ..., n-1$, $n = [\alpha] + 1$.

Lemma 2.4. [24] *Let $\alpha \geqslant 0$. Then the differential equation*

$$^cD^\alpha h(t) = h(t)$$

has solutions $I^{\alpha c}D^\alpha h(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 ... + c_{n-1} t^{n-1} + h(t)$, $c_i \in \mathbb{R}$, $i = 0, 1, 2, 3, ..., n-1$, $n = [\alpha] + 1$.

We define the set of all measurable selections of F that belong to the Banach space $L^1([0,T];\mathbb{R})$ that is

$$S^1_{F,y} = \{v \in L^1([0,T];\mathbb{R}),\ v(t) \in F(t, y(t), ^C D^r y(t)) \text{ a.e. } t \in [0,T]\}.$$

3. MAIN RESULTS

Let us start by defining what we mean by a solution of the problem (1)-(2).

Definition 3.1. *A function $y \in L^1([0,T],\mathbb{R})$ is said to be a solution of (1)-(2) if there exists a function $x \in L^1([0,T],\mathbb{R})$ with $x(t) \in F(t, y(t), ^c D^\alpha y(t))$ for a.e. $t \in [0,T]$ such that $^cD^\alpha y(t) = x(t)$ and the function ysatisfies conditions (2).*

Let us start by defining what we mean by an integrable solution of the nonlocal problem (1)-(2).
We assume that $\sum_{k=1}^m a_k \neq 0$ and

$$a = \frac{1}{\sum_{k=1}^m a_k}.$$

For the existence of solutions for the nonlocal problem (1)-(2) we need the following auxiliary lemma.

Lemma 3.1. *The nonlocal problem(1)-(2) is equivalent to the integral equation*

$$y(t) = ay_0 - a\sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds, \tag{4}$$

where x is the solution of the functional integral equation

$$x(t) \in F(t, ay_0 - a\sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds, x(t)). \tag{5}$$

Proof. Let ${}^cD^\alpha y(t) = x(t)$ in equation (1)

$$x(t) \in F(t, y(t), x(t)) \tag{6}$$

and

$$\begin{aligned} y(t)) &= y(0) + I^\alpha x(t) \\ &= y(0) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds. \end{aligned} \tag{7}$$

Let $t = t_k$ in (7), we obtain

$$y(t_k) = y(0) + \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds$$

and

$$\sum_{k=1}^{m} a_k y(t_k) = \sum_{k=1}^{m} a_k y(0) + \sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds. \tag{8}$$

Substitute from (2) into (8)

$$y_0 = \sum_{k=1}^{m} a_k y(0) + \sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds$$

and

$$y(0) = a(y_0 - \sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds). \tag{9}$$

By substituting (9) into (7) and (6), we obtain (4) and (8). In order to complete the proof, we prove that equation (4) satisfies the nonlocal problem (1)-(2). Differentiating (4), we get

$${}^cD^\alpha y(t) = x(t) \in F(t, y(t), {}^c D^\alpha y(t)).$$

Let $t = t_k$ in, we obtain (4)

$$y(t_k) = ay_0 - a\sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma\alpha} x(s)ds + \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds$$

$$= ay_0 + a(1 - a\sum_{k=1}^{m} a_k) \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds.$$

Then

$$\sum_{k=1}^{m} a_k y(t_k) = \sum_{k=1}^{m} a_k a y_0 + \sum_{k=1}^{m} a_k a(1 - a\sum_{k=1}^{m} a_k) \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds = y_0.$$

This completes the proof of the equivalence between the nonlocal problem (1)-(2) and the integral equation (4). ∎

Theorem 3.1. *Assume the following hypotheses hold :*

(H1) $F : J \times \mathbb{R} \times \mathbb{R} \to P_{cp,c}(\mathbb{R})$ *is a Carathéodory multi-valued map;*

(H2) There exist $p \in L^1(J, \mathbb{R}^+)$ *continuous and nondecreasing such that*

$$\|F(t, u_1, u_2)\|_P = \sup\{|v| : v \in F(t, u_1, u_2)\} \leq p(t)(1 + |u_1| + |u_2|) \text{ for } t \in J \text{ and each } u_1, u_2 \in \mathbb{R};$$

(H3) There exist $l_1, l_2 \in L^1([0;T];\mathbb{R})$, *with* $I^r l < \infty$, *such that*

$$H_d(F(t, x, y), F(t, \bar{x}, \bar{y})) < l_1(t)|x - \bar{x}| + l_2(t)|y - \bar{y}| \text{ for every } x, \bar{x}, y, \bar{y} \in \mathbb{R},$$

and

$$d(0, F(t, 0, 0)) \leq l \text{ a,e } t \in J.$$

Then the problem (1)-(2) has at least one solution on J.

Remark 3.1. *Note that for an* L^1- *Carathéodory multifunction* $F : J \times \mathbb{R} \times \mathbb{R} \to P_{cp}(\mathbb{R})$, *the set* $S^1_{F,y}$ *is not empty.*

Proof. Transform the problem (1)-(2) into a fixed point problem. Consider the multivalued operator,

$$N : L^1(J, \mathbb{R}) \to \mathcal{P}(L^1(J, \mathbb{R}))$$

$$N(x) = \left\{ h \in L^1(J,\mathbb{R}) \quad \begin{array}{rcl} h(t) & = & ay_0 - a\sum_{k=1}^{m} a_k \displaystyle\int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \\ & + & \displaystyle\int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \ \ v \in S^1_{F,y}. \end{array} \right\}$$

Clearly, from Lemma (3.1), the fixed points of N are solutions to (1)-(2). We shall show that N satisfies the assumptions of nonlinear alternative of Leray-Schauder fixed point theorem. The proof will be given in several steps.
Let

$$r \geq \frac{|ay_0|T + \dfrac{2T^\alpha}{\Gamma(\alpha+1)}\|p\|_{L^1} + \dfrac{2T^{\alpha+1}}{\Gamma(\alpha+1)}|ay_0|\|p\|_{L^1}}{1 - \left[\dfrac{2T^\alpha}{\Gamma(\alpha+1)} + \dfrac{4T^{2\alpha}}{\Gamma(2\alpha+1)}\right]\|p\|_{L^1}},$$

and consider the bounded set

$$B_r = \{x \in L^1(J,\mathbb{R}) : \|x\|_{L^1} \leq r\}.$$

Step 1: *$N(x)$ is convex for each $y \in B_r$.*

Indeed, if h_1 , h_2 belong to $N(y)$ then there exist v_1, v_2 such that for each $t \in J$ we have, for $i = 1, 2$

$$\begin{array}{rcl} h_i(t) & = & ay_0 - a\sum_{k=1}^{m} a_k \displaystyle\int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} v_i(s)ds \\ & + & \displaystyle\int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_i(s)ds \end{array}$$

Let $0 \leq d \leq 1$. Then, for each $t \in J$ we have

$$\begin{array}{rcl} (dh_1 + (1-d)h_2)(t) & = & ay_0 - a\sum_{k=1}^{m} a_k \displaystyle\int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} (dv_1 + (1-d)v_2)(s)ds \\ & + & \displaystyle\int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (dv_1 + (1-d)v_2)(s)ds. \end{array}$$

Since $S_{F,y}$ is convex (because F has convex values), we have

$$dh_1 + (1-d)h_2 \in N(x).$$

Step 2: *$N(B_r)$ is relatively compact.*

(a) $N(B_r)$ *is bounded.* Let $y \in B_r$, for each $h \in N(x)$ and $t \in J$, we have by (H2),

$$\begin{aligned} h(t) &= ay_0 - a\textstyle\sum_{k=1}^m a_k \displaystyle\int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds. \end{aligned}$$

By (H2), we have, for each $t \in J$:

$$\begin{aligned} \|h\|_{L^1} &= \int_0^T |h(t)|dt \\ &= \int_0^T \left| ay_0 - a\sum_{k=1}^m a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \right| \\ &\le |ay_0|T + \int_0^T \left(a\sum_{k=1}^m a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} |v(s)|ds \right) dt \\ &+ \int_0^T \int_0^t \left(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |v(s)|ds \right) dt \\ &\le |ay_0|T + \int_0^T \frac{a\sum_{k=1}^m a_k (t_k)^\alpha}{\Gamma(\alpha+1)} |v(s)|ds + \int_0^T \frac{T^\alpha}{\Gamma(\alpha+1)} |v(s)|ds \\ &\le |ay_0|T + \int_0^T \frac{2T^\alpha}{\Gamma(\alpha+1)} |v(s)|ds \\ &\le |ay_0|T + \frac{2T^\alpha}{\Gamma(\alpha+1)} \int_0^T [|p(t)| \\ &+ \left| ay_0 - a\textstyle\sum_{k=1}^m a_k \displaystyle\int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds \right| |p(t)| \\ &+ |x(t)||p(t)|]dt \\ &\le |ay_0|T + \frac{2T^\alpha}{\Gamma(\alpha+1)} \|p\|_{L^1} + \frac{2T^{\alpha+1}}{\Gamma(\alpha+1)} |ay_0| \|p\|_{L^1} \\ &+ \frac{4T^{2\alpha}}{\Gamma(2\alpha+1)} \|p\|_{L^1} \|x\|_{L^1} + \frac{2T^\alpha}{\Gamma(\alpha+1)} \|p\|_{L^1} \|x\|_{L^1} \\ &\le |ay_0|T + \left(\frac{2T^\alpha + 2T^{\alpha+1}|ay_0|}{\Gamma(\alpha+1)} + \left[\frac{2T^\alpha}{\Gamma(\alpha+1)} + \frac{4T^{2\alpha}}{\Gamma(2\alpha+1)} \right] \|x\|_{L^1} \right) \|p\|_{L^1}. \end{aligned}$$

Thus

$$\|h\|_{L^1} \le |ay_0|T + \left(\frac{2T^\alpha + 2T^{\alpha+1}|ay_0|}{\Gamma(\alpha+1)} + \left[\frac{2T^\alpha}{\Gamma(\alpha+1)} + \frac{4T^{2\alpha}}{\Gamma(2\alpha+1)} \right] r \right) \|p\|_{L^1}. \le r$$

Then the above inequalities show that

$$\|N(x)\| = \sup\{\|h\|_{L^1} : h \in N(x)\} \le r$$

which shows that $N(B_r) \subset B_r$ is bounded, then $N(B_r)$ is bounded.

(b) $N(x)_r \to N(y)$, in $L^1(J,\mathbb{R})$ for each $y \in B_r$.

Let $y \in N(y)$ then we have

$$\begin{aligned}
\|h_r - h\|_{L^1} &= \int_0^T |h_r(t) - h(t)|dt \\
&= \int_0^T \left| \frac{1}{r}\int_t^{t+r} h(s)ds - h(t)\right| dt \\
&\leq \int_0^T \left(\frac{1}{r}\int_t^{t+r} |h(s) - h(t)|ds\right) dt \\
&\leq \int_0^T \left(\frac{1}{r}\int_t^{t+r} |I^\alpha v(s) - I^\alpha v(t)|ds\right) dt \\
&\leq \int_0^T \frac{1}{r}\int_t^{t+r} |I^\alpha v(s) - I^\alpha v(t)|dsdt.
\end{aligned}$$

Since $v \in L^1(J,\mathbb{R})$ and by Proposition1 it follows that $I^\alpha v \in L^1(J,\mathbb{R})$, then we have

$$\frac{1}{r}\int_t^{t+r} |I^\alpha v(s) - I^\alpha v(t)|ds \to 0, \text{when}\, r \to 0.$$

Hence

$$N(y)_r \to N(y) \text{ uniformaly } r \to 0.$$

As a consequence of (a) and (b) together with the Kolmogorov compactness criterion, we can conclude that $N(B_r)$ is relatively compact.

Step 3: *Nhas a closed graph.*

Let $y_n \to y_*$, $h_n \in N(y_n)$ and $h_n \to h_*$. We need to show that $h_n \in N(y_n)$ means that there exists $v_n \in S^1_{F,y}$, such that, for each$t \in J$

$$\begin{aligned}
h_n(t) &= ay_0 - a\textstyle\sum_{k=1}^m a_k \displaystyle\int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)ds \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)ds.
\end{aligned}$$

We must show that there exists $v_* \in S^1_{F,y}$ such that, for each $t \in J$

$$\begin{aligned}
h_*(t) &= ay_0 - a\textstyle\sum_{k=1}^m a_k \displaystyle\int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} v_*(s)ds \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_*(s)ds.
\end{aligned}$$

Since $F(t,.,.)$ is upper semi-continuous, then for every $\epsilon > 0$ there exists $n_0(\epsilon) > 0 >$ such that, for every $n > n_0$, we have

$v_n(t) \in F(t, y(t), x(t)) \subset F(t, y_*(t), x_*(t)) + \epsilon B(0,1)$, a.e. $t \in J$.

Since $F(.,.,.)$ has compact values, then there exists a subsequence $v_{n_m}(.)$ such that

$v_{n_m}(.) \to v_*$ as $m \to \infty$,

and $v_*(t) \in F(t, y_*(t), x_*(t))$ a.e.$t \in J$.

For every $w \in F(t, y_*(t), x_*(t)))$, we have

$$|v_{n_m} - v_*| \leq |v_{n_m} - w| + |w - v_*|.$$

Then

$$|v_{n_m} - v_*| \leq d(v_{n_m}, F(t, y_*(t), x_*(t)).$$

We obtain an analogous relation by interchanging the roles of v_{n_m} and v_*, and it follows that

$$\begin{array}{rcl} |v_{n_m} - v_*| & \leq & H_d(F(t, y_{n_m}(t), x_{n_m}(t)), F(t, y_*(t), x_*(t)))) \\ & \leq & l_1(t)|y_{n_m}(t) - y_*(t)| + l_2(t)|x_{n_m}(t) - x_*(t)|. \end{array}$$

Then

$$\begin{array}{rcl} |h_n(t) - h_*(t)| & \leq & a\sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)}|v_{n_m} - v_*|ds \\ & + & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|v_{n_m} - v_*|ds \\ & \leq & \frac{2T^\alpha}{\Gamma(\alpha+1)}\int_0^T (l_1(t)|y(t) - \bar{y}(t)| + l_2(t)|x(t) - \bar{x}(t)|)ds \\ & \leq & (\frac{4T^{2\alpha}}{\Gamma(2\alpha+1)}\|l_1\|_{L^1} + \frac{2T^\alpha}{\Gamma(\alpha+1)}\|l_2\|_{L^1})\|x - \bar{x}\|_{L^1} \end{array}.$$

Hence

$$\|h_n(t) - h_*(t)\|_{L^1} \to 0, \text{as} \quad m \to \infty.$$

Step 4: *A priori bounds of solutions.*

Let y be such that $y \in \lambda N(y)$ with $\lambda \in (0,1]$ Then there exists $v \in S^1_{F,y}$ such that, for each $t \in J$,

$$\begin{array}{rcl} h(t) & = & ay_0 - a\sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds \\ & + & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)ds. \end{array}$$

By (H2), we have, for each $t \in J$

$$\begin{aligned}
\|h\|_{L^1} &= \int_0^T |h(t)|dt \\
&= \int_0^T \left| ay_0 - a\sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \right| \\
&\leq |ay_0|T + \int_0^T \left(a\sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} |v(s)|ds \right) dt \\
&+ \int_0^T \int_0^t \left(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |v(s)|ds \right) dt \\
&\leq |ay_0|T + \int_0^T \frac{a\sum_{k=1}^{m} a_k (t_k)^{\alpha}}{\Gamma(\alpha+1)} |v(s)|ds + \int_0^T \frac{T^{\alpha}}{\Gamma(\alpha+1)} |v(s)|ds \\
&\leq |ay_0|T + \int_0^T \frac{2T^{\alpha}}{\Gamma(\alpha+1)} |v(s)|ds \\
&\leq |ay_0|T + \frac{2T^{\alpha}}{\Gamma(\alpha+1)} \int_0^T [|p(t)| \\
&+ \left| ay_0 - a\sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds \right| |p(t)| \\
&+ |x(t)||p(t)|]dt \\
&\leq |ay_0|T + \frac{2T^{\alpha}}{\Gamma(\alpha+1)} \|p\|_{L^1} + \frac{2T^{\alpha+1}}{\Gamma(\alpha+1)} |ay_0| \|p\|_{L^1} \\
&+ \frac{4T^{2\alpha}}{\Gamma(2\alpha+1)} \|p\|_{L^1} \|x\|_{L^1} + \frac{2T^{\alpha}}{\Gamma(\alpha+1)} \|p\|_{L^1} \|x\|_{L^1} \\
&\leq |ay_0|T + \left(\frac{2T^{\alpha} + 2T^{\alpha+1}|ay_0|}{\Gamma(\alpha+1)} + \left[\frac{2T^{\alpha}}{\Gamma(\alpha+1)} + \frac{4T^{2\alpha}}{\Gamma(2\alpha+1)} \right] \|x\|_{L^1} \right) \|p\|_{L^1} \\
&:= r.
\end{aligned}$$

Let $U = \{x \in L^1(J, \mathbb{R}) : \|x\|_{L^1} < r+1\}$ The operator $N : \bar{U} \to \mathcal{P}(L^1(J, \mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x \in \lambda N(x)$, for some $\lambda \in (0, 1]$. As a consequence of the nonlinear alternative of Leray-Shauder, we deduce that N has a fixed point $x \in \bar{U}$ which is a solution of the problem (1)-(2).This completes the proof. ∎

We present now a result for the problem (1)-(2) with a nonconvex valued right hand side. Our considerations are based on the fixed point result in Lemma(2.1)

Theorem 3.2. *Assume (H3) and the following hypothesis hold:*

(H4) $F : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp}(X)$ *has the property that* $F(., u, v) : J \to \mathcal{P}_{cp}(X)$ *is measurable for each* $u, v \in \mathbb{R}$

if

$$\frac{4T^{2\alpha}}{\Gamma(2\alpha+1)}\|l_1\|_{L^1} + \frac{2T^{\alpha}}{\Gamma(\alpha+1)}\|l_2\|_{L^1} < 1. \tag{10}$$

Then the problem (1)-(2) has at least one solution on J.

Remark 3.2. *By (H4), we can see that $S^1_{F,y}$ is nonempty for each $y \in L^1(J,\mathbb{R})$, so F has a measurable selection (see* [14]*, Theorem III.6).*

Proof. We shall show that N satisfies the assumptions of Lemma(2.1). The proof will be given in two steps.

Step1: *$N(x) \in \mathcal{P}_{cl}(L^1(J,\mathbb{R}))$ for each $x \in L^1(J,\mathbb{R})$.*

Indeed, let $(h_n)_{n\geq 0} \subset N(x)$ be such that $h_n \to \widetilde{h}$ in $L^1(J,\mathbb{R})$, then $\widetilde{h}$ in $L^1(J,\mathbb{R})$ and there exists $v_n \in S_{F,y}$ such that for each $t \in J$

$$\begin{aligned} h_n(t) &= ay_0 - a\sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)ds. \end{aligned}$$

Using the fact that F has compact values and from (H3) we may pass to a subsequence if necessary to get that v_n converges weakly to v in $L^1_w C(J,\mathbb{R})$ (the space endowed with the weak topology). An application of Mazurs theorem implies that v_n converges strongly to v and hence $v \in S^1_{F,y}$. Then for each $t \in J$

$$\begin{aligned} h_n(t) \to \widetilde{h}(t) &= ay_0 - a\sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds. \end{aligned}$$

So, $\widetilde{h} \in N(x)$.

Step2: *There exists $\gamma < 1$ such that $H_d(N(x), N(\bar{x})) < \gamma\|x-\bar{x}\|_{L^1}$ for each $x, \bar{x} \in L^1(J,\mathbb{R})$.*

Let $x, \bar{x} \in L^1(J,\mathbb{R})$ and $h_1 \in N(x)$. Then there exists $v_1 \in F(t, y(t), x(t))$ such that, for each $t \in J$,

$$\begin{aligned} h_1(t) &= ay_0 - a\sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} v_1(s)ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_1(s)ds. \end{aligned}$$

From (H3) it follows that

$$H_d(F(t,y(t),x(t)),F(t,\bar{y}(t),\bar{x}(t)) \le l(t)|y(t)-\bar{y}(t)| + l(t)|x(t)-\bar{x}(t)|.$$

Hence, there exists $w \in F(t,\bar{y}(t),\bar{x}(t))$ such that

$$|v_1(t)-w| \le l_1(t)|y(t)-\bar{y}(t)| + l_2(t)|x(t)-\bar{x}(t)|, t \in J.$$

Consider $U : J \to \mathcal{P}(\mathbb{R})$ given by

$$U(t) = \{w \in \mathbb{R} : |v_1(t)-w| \le l_1(t)|y(t)-\bar{y}(t)| + l_2(t)|x(t)-\bar{x}(t)|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t,\bar{y}(t),\bar{x}(t))$ is measurable, there exists a function $v_2(t)$ which is a measurable selection for V. Then $v_2 \in F(t,\bar{y}(t),\bar{x}(t))$, and for each $t \in J$,

$$|v_1(t)-v_2(t)| \le l_1(t)|y(t)-\bar{y}(t)| + l_2(t)|x(t)-\bar{x}(t)|, t \in J.$$

Let us define for each $v_2 \in J$

$$\begin{aligned} h_2(t) &= ay_0 - a\sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} v_2(s)ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_2(s)ds. \end{aligned}$$

Then for each $t \in J$,

$$\begin{aligned} |h_1(t)-h_2(t)| &\le a\sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} |v_1-v_2|ds \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |v_1-v_2|ds \\ &\le \frac{2T^\alpha}{\Gamma(\alpha+1)} \int_0^T |v_1-v_2|ds \\ &+ \frac{T^\alpha}{\Gamma(\alpha+1)} \int_0^t |v_1-v_2|ds \\ &\le \frac{2T^\alpha}{\Gamma(\alpha+1)} \int_0^T |v_1-v_2|ds \\ &\le \frac{2T^\alpha}{\Gamma(\alpha+1)} \int_0^T (l_1(t)|y(t)-\bar{y}(t)| + l_2(t)|x(t)-\bar{x}(t)|)ds \\ &\le \frac{4T^{2\alpha}}{\Gamma(2\alpha+1)} \|l_1\|_{L^1}\|x-\bar{x}\|_{L^1} + \frac{2T^\alpha}{\Gamma(\alpha+1)} \|l_2\|_{L^1}\|x-\bar{x}\|_{L^1}. \end{aligned}$$

Thus

$$\|h_1-h_2\|_{L^1} \le \frac{4T^{2\alpha}}{\Gamma(2\alpha+1)} \|l_1\|_{L^1}\|x-\bar{x}\|_{L^1} + \frac{2T^\alpha}{\Gamma(\alpha+1)} \|l_2\|_{L^1}\|x-\bar{x}\|_{L^1}.$$

For an analogous relation, obtained by interchanging the roles of x and $\bar{x}$ it follows that

$$H_d(N(x), N(\bar{x})) \leq (\frac{4T^{2\alpha}}{\Gamma(2\alpha+1)}\|l_1\|_{L^1} + \frac{2T^{\alpha}}{\Gamma(\alpha+1)}\|l_2\|_{L^1})\|x-\bar{x}\|_{L^1}.$$

So by (10), N is a contraction and thus, by Lemma(2.1), N has a fixed point x which is solution to (1)-(2). The proof is complete. ∎

4. AN EXAMPLE

We conclude this paper with an example to illustrate our main result. We apply Theorem 3.4 to the the following fractional differential inclusion

$$^cD^{\alpha}y(t) \in F(t, y(t), ^cD^{\alpha}y(t)), \text{ for a.e. } t \in J = [0,1],\ 0 < \alpha \leq 1 \tag{11}$$

$$\sum_{1}^{m} a_k y(t_k) = 1, \tag{12}$$

where

$$F(t, y(t), ^cD^{\alpha}y(t)) = \{v \in \mathbb{R} : f_1(t, y(t), ^cD^{\alpha}y(t)) \leq v \leq f_2(t, y(t), ^cD^{\alpha}y(t))\}$$

where $f_1, f_2 : J \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$. We assume that for each $t \in [0,1]$, $f_1(t, \cdot, \cdot)$ is lower semi-continuous (i.e., the set $\{y \in \mathbb{R} : f_1(t, y(t), ^cD^{\alpha}y(t)) > \mu\}$ is open for each $\mu \in \mathbb{R}$), and assume that for each $t \in [0,1]$, $f_2(t, \cdot, \cdot)$ is upper semi-continuous (i.e., the set $\{y \in \mathbb{R} : f_2(t, y(t), ^cD^{\alpha}y(t)) < \mu\}$ is open for each $\mu \in \mathbb{R}$). Assume that there are $p \in L^1([0,1], \mathbb{R}^+)$ continuous and nondecreasing such that

$$\begin{aligned}
\|F(t, u_1, u_2)\|_P &= \sup\{|v| : v \in F(t, u_1, u_2)\} \\
&= \max(|f_1(t, y(t), x(t))|, |f_2(t, y(t), x(t))|) \\
&\leq p(t) \times [1 + |x| + |y|],
\end{aligned}$$

for $t \in J$ and each $x, y \in \mathbb{R}$.

It is clear that F is compact and convex-valued, and it is upper semi-continuous.

Since all the conditions of Theorem (3.1) are satisfied, problem (11)-(12) has at least one solution y on $[0,1]$.

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