

MULTIVARIATE FUZZY-RANDOM AND STOCHASTIC GENERAL SIGMOID ACTIVATION FUNCTION INDUCED NEURAL NETWORK APPROXIMATIONS

George A. Anastassiou

Department of Mathematical Sciences, University of Memphis, Memphis, U.S.A.

ganastss@memphis.edu

Abstract In this article we research the degree of approximation of multivariate pointwise and uniform convergences in the q -mean to the Fuzzy-Random unit operator of multivariate Fuzzy-Random Quasi-Interpolation general sigmoid activation function based neural network operators. These multivariate Fuzzy-Random operators arise in a natural way among multivariate Fuzzy-Random neural networks. The rates are given through multivariate Probabilistic-Jackson type inequalities involving the multivariate Fuzzy-Random modulus of continuity of the engaged multivariate Fuzzy-Random function. The plain stochastic extreme analog of this theory is also met in detail for the stochastic analogs of the operators: the stochastic full quasi-interpolation operators, the stochastic Kantorovich type operators and the stochastic quadrature type operators.

Keywords: Fuzzy-Random analysis, Fuzzy-Random neural networks and operators, Fuzzy-Random modulus of continuity, Fuzzy-Random functions, Stochastic processes, Jackson type fuzzy and probabilistic inequalities, general sigmoid function.

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1. FUZZY-RANDOM FUNCTIONS AND STOCHASTIC PROCESSES BACKGROUND

See also [18], Ch. 22, pp. 497-501.

We start with

Definition 1.1. (see [28]) Let $\mu : \mathbb{R} \rightarrow [0, 1]$ with the following properties:

- (i) is normal, i.e., $\exists x_0 \in \mathbb{R} : \mu(x_0) = 1$.
 - (ii) $\mu(\lambda x + (1 - \lambda)y) \geq \min\{\mu(x), \mu(y)\}$, $\forall x, y \in \mathbb{R}$, $\forall \lambda \in [0, 1]$ (μ is called a convex fuzzy subset).
 - (iii) μ is upper semicontinuous on $\mathbb{R}$, i.e., $\forall x_0 \in \mathbb{R}$ and $\forall \varepsilon > 0$, $\exists$ neighborhood $V(x_0) : \mu(x) \leq \mu(x_0) + \varepsilon$, $\forall x \in V(x_0)$.
 - (iv) the set $\text{supp}(\mu)$ is compact in $\mathbb{R}$ (where $\text{supp}(\mu) := \{x \in \mathbb{R}; \mu(x) > 0\}$).
- We call μ a fuzzy real number. Denote the set of all μ with $\mathbb{R}_{\mathcal{F}}$.

E.g., $\chi_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$, for any $x_0 \in \mathbb{R}$, where $\chi_{\{x_0\}}$ is the characteristic function at x_0 .

For $0 < r \leq 1$ and $\mu \in \mathbb{R}_{\mathcal{F}}$ define $[\mu]^r := \{x \in \mathbb{R} : \mu(x) \geq r\}$ and $[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) > 0\}}$.

Then it is well known that for each $r \in [0, 1]$, $[\mu]^r$ is a closed and bounded interval of $\mathbb{R}$. For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \quad \forall r \in [0, 1],$$

where $[u]^r + [v]^r$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $\lambda [u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g., [28]). Notice $1 \odot u = u$ and it holds $u \oplus v = v \oplus u$, $\lambda \odot u = u \odot \lambda$. If $0 \leq r_1 \leq r_2 \leq 1$ then $[u]^{r_2} \subseteq [u]^{r_1}$. Actually $[u]^r = [u_-^{(r)}, u_+^{(r)}]$, where $u_-^{(r)} < u_+^{(r)}$, $u_-^{(r)}, u_+^{(r)} \in \mathbb{R}$, $\forall r \in [0, 1]$.

Define

$$D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \rightarrow \mathbb{R}_+ \cup \{0\}$$

by

$$D(u, v) := \sup_{r \in [0, 1]} \max \left\{ \left| u_-^{(r)} - v_-^{(r)} \right|, \left| u_+^{(r)} - v_+^{(r)} \right| \right\},$$

where $[v]^r = [v_-^{(r)}, v_+^{(r)}]$; $u, v \in \mathbb{R}_{\mathcal{F}}$. We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$. Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [28], with the properties

$$\begin{aligned} D(u \oplus w, v \oplus w) &= D(u, v), \quad \forall u, v, w \in \mathbb{R}_{\mathcal{F}}, \\ D(k \odot u, k \odot v) &= |k| D(u, v), \quad \forall u, v \in \mathbb{R}_{\mathcal{F}}, \forall k \in \mathbb{R}, \\ D(u \oplus v, w \oplus e) &\leq D(u, w) + D(v, e), \quad \forall u, v, w, e \in \mathbb{R}_{\mathcal{F}}. \end{aligned} \quad (1)$$

Let (M, d) metric space and $f, g : M \rightarrow \mathbb{R}_{\mathcal{F}}$ be fuzzy real number valued functions. The distance between f, g is defined by

$$D^*(f, g) := \sup_{x \in M} D(f(x), g(x)).$$

On $\mathbb{R}_{\mathcal{F}}$ we define a partial order by " $\leq$ ": $u, v \in \mathbb{R}_{\mathcal{F}}$, $u \leq v$ iff $u_-^{(r)} \leq v_-^{(r)}$ and $u_+^{(r)} \leq v_+^{(r)}$, $\forall r \in [0, 1]$.

$\sum^*$ denotes the fuzzy summation, $\tilde{0} := \chi_{\{0\}} \in \mathbb{R}_{\mathcal{F}}$ the neutral element with respect to $\oplus$. For more see also [29], [30].

We need

Definition 1.2. (see also [24], Definition 13.16, p. 654) Let $(X, \mathcal{B}, P)$ be a probability space. A fuzzy-random variable is a $\mathcal{B}$ -measurable mapping $g :$

$X \rightarrow \mathbb{R}_{\mathcal{F}}$ (i.e., for any open set $U \subseteq \mathbb{R}_{\mathcal{F}}$, in the topology of $\mathbb{R}_{\mathcal{F}}$ generated by the metric D , we have

$$g^{-1}(U) = \{s \in X; g(s) \in U\} \in \mathcal{B}. \quad (2)$$

The set of all fuzzy-random variables is denoted by $\mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$. Let $g_n, g \in \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $n \in \mathbb{N}$ and $0 < q < +\infty$. We say $g_n(s) \xrightarrow[n \rightarrow +\infty]{\text{"q-mean"}} g(s)$ if

$$\lim_{n \rightarrow +\infty} \int_X D(g_n(s), g(s))^q P(ds) = 0. \quad (3)$$

Remark 1.1. (see [24], p. 654) If $f, g \in \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, let us denote $F : X \rightarrow \mathbb{R}_+ \cup \{0\}$ by $F(s) = D(f(s), g(s))$, $s \in X$. Here, F is $\mathcal{B}$ -measurable, because $F = G \circ H$, where $G(u, v) = D(u, v)$ is continuous on $\mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}}$, and $H : X \rightarrow \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}}$, $H(s) = (f(s), g(s))$, $s \in X$, is $\mathcal{B}$ -measurable. This shows that the above convergence in q -mean makes sense.

Definition 1.3. (see [24], p. 654, Definition 13.17) Let $(T, \mathcal{T})$ be a topological space. A mapping $f : T \rightarrow \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$ will be called fuzzy-random function (or fuzzy-stochastic process) on T . We denote $f(t)(s) = f(t, s)$, $t \in T$, $s \in X$.

Remark 1.2. (see [24], p. 655) Any usual fuzzy real function $f : T \rightarrow \mathbb{R}_{\mathcal{F}}$ can be identified with the degenerate fuzzy-random function $f(t, s) = f(t)$, $\forall t \in T$, $s \in X$.

Remark 1.3. (see [24], p. 655) Fuzzy-random functions that coincide with probability one for each $t \in T$ will be considered equivalent.

Remark 1.4. (see [24], p. 655) Let $f, g : T \rightarrow \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$. Then $f \oplus g$ and $k \odot f$ are defined pointwise, i.e.,

$$\begin{aligned} (f \oplus g)(t, s) &= f(t, s) \oplus g(t, s), \\ (k \odot f)(t, s) &= k \odot f(t, s), \quad t \in T, s \in X, k \in \mathbb{R}. \end{aligned}$$

Definition 1.4. (see also Definition 13.18, pp. 655-656, [24]) For a fuzzy-random function $f : W \subseteq \mathbb{R}^N \rightarrow \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $N \in \mathbb{N}$, we define the (first) fuzzy-random modulus of continuity

$$\begin{aligned} \Omega_1^{(\mathcal{F})}(f, \delta)_{L^q} &= \\ \sup \left\{ \left(\int_X D^q(f(x, s), f(y, s)) P(ds) \right)^{\frac{1}{q}} : x, y \in W, \|x - y\|_{\infty} \leq \delta \right\}, \\ 0 &< \delta, 1 \leq q < \infty. \end{aligned}$$

Definition 1.5. ([16]) Here $1 \leq q < +\infty$. Let $f : W \subseteq \mathbb{R}^N \rightarrow \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $N \in \mathbb{N}$, be a fuzzy random function. We call f a (q -mean) uniformly continuous fuzzy random function over W , iff $\forall \varepsilon > 0 \exists \delta > 0$: whenever $\|x - y\|_{\infty} \leq \delta$, $x, y \in W$, implies that

$$\int_X (D(f(x, s), f(y, s)))^q P(ds) \leq \varepsilon.$$

We denote it as $f \in C_{FR}^{U_q}(W)$.

Proposition 1.1. ([16]) Let $f \in C_{FR}^{U_q}(W)$, where $W \subseteq \mathbb{R}^N$ is convex.

Then $\Omega_1^{(\mathcal{F})}(f, \delta)_{L^q} < \infty$, any $\delta > 0$.

Proposition 1.2. ([16]) Let $f, g : W \subseteq \mathbb{R}^N \rightarrow \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $N \in \mathbb{N}$, be fuzzy random functions. It holds

(i) $\Omega_1^{(\mathcal{F})}(f, \delta)_{L^q}$ is nonnegative and nondecreasing in $\delta > 0$.

(ii) $\lim_{\delta \downarrow 0} \Omega_1^{(\mathcal{F})}(f, \delta)_{L^q} = \Omega_1^{(\mathcal{F})}(f, 0)_{L^q} = 0$, iff $f \in C_{FR}^{U_q}(W)$.

We mention

Definition 1.6. (see also [6]) Let $f(t, s)$ be a random function (stochastic process) from $W \times (X, \mathcal{B}, P)$, $W \subseteq \mathbb{R}^N$, into $\mathbb{R}$, where $(X, \mathcal{B}, P)$ is a probability space. We define the q -mean multivariate first modulus of continuity of f by

$$\Omega_1(f, \delta)_{L^q} := \sup \left\{ \left(\int_X |f(x, s) - f(y, s)|^q P(ds) \right)^{\frac{1}{q}} : x, y \in W, \|x - y\|_{\infty} \leq \delta \right\}, \quad (4)$$

$\delta > 0$, $1 \leq q < \infty$.

The concept of f being (q -mean) uniformly continuous random function is defined the same way as in Definition 1.5, just replace D by $|\cdot|$, etc. We denote it as $f \in C_{\mathbb{R}}^{U_q}(W)$.

Similar properties as in Propositions 1.1, 1.2 are valid for $\Omega_1(f, \delta)_{L^q}$.

Also we have

Proposition 1.3. ([3]) Let $Y(t, \omega)$ be a real valued stochastic process such that Y is continuous in $t \in [a, b]$. Then Y is jointly measurable in (t, ω) .

According to [23], p. 94 we have the following

Definition 1.7. Let $(Y, \mathcal{T})$ be a topological space, with its σ -algebra of Borel sets $\mathcal{B} := \mathcal{B}(Y, \mathcal{T})$ generated by $\mathcal{T}$. If $(X, \mathcal{S})$ is a measurable space, a function $f : X \rightarrow Y$ is called measurable iff $f^{-1}(B) \in \mathcal{S}$ for all $B \in \mathcal{B}$.

By Theorem 4.1.6 of [23], p. 89 f as above is measurable iff

$$f^{-1}(C) \in \mathcal{S} \text{ for all } C \in \mathcal{T}.$$

We mention

Theorem 1.1. (see [23], p. 95) Let $(X, \mathcal{S})$ be a measurable space and (Y, d) be a metric space. Let f_n be measurable functions from X into Y such that for all $x \in X$, $f_n(x) \rightarrow f(x)$ in Y . Then f is measurable. I.e., $\lim_{n \rightarrow \infty} f_n = f$ is measurable.

We need also

Proposition 1.4. ([16]) Let f, g be fuzzy random variables from $\mathcal{S}$ into $\mathbb{R}_{\mathcal{F}}$. Then

- (i) Let $c \in \mathbb{R}$, then $c \odot f$ is a fuzzy random variable.
- (ii) $f \oplus g$ is a fuzzy random variable.

Proposition 1.5. Let $Y(\vec{t}, \omega)$ be a real valued multivariate random function (stochastic process) such that Y is continuous in $\vec{t} \in \prod_{i=1}^N [a_i, b_i]$. Then Y is jointly measurable in $(\vec{t}, \omega)$ and $\int_{\prod_{i=1}^N [a_i, b_i]} Y(\vec{t}, \omega) d\vec{t}$ is a real valued random variable.

Proof. Similar to Proposition 18.14, p. 353 of [7]. ■

2. ABOUT REAL NEURAL NETWORKS BACKGROUND

Here we follow [21].

Let $h : \mathbb{R} \rightarrow [-1, 1]$ be a general sigmoid function, such that it is strictly increasing, $h(0) = 0$, $h(-x) = -h(x)$, $h(+\infty) = 1$, $h(-\infty) = -1$. Also h is strictly convex over $(-\infty, 0]$ and strictly concave over $[0, +\infty)$, with $h^{(2)} \in C(\mathbb{R})$.

We consider the activation function

$$\psi(x) := \frac{1}{4} (h(x+1) - h(x-1)), \quad x \in \mathbb{R}, \quad (5)$$

As in [20], p. 88, we get that $\psi(-x) = \psi(x)$, thus ψ is an even function. Since $x+1 > x-1$, then $h(x+1) > h(x-1)$, and $\psi(x) > 0$, all $x \in \mathbb{R}$.

We see that

$$\psi(0) = \frac{h(1)}{2}. \quad (6)$$

Let $x > 1$, we have that

$$\psi'(x) = \frac{1}{4} (h'(x+1) - h'(x-1)) < 0,$$

by h' being strictly decreasing over $[0, +\infty)$.

Let now $0 < x < 1$, then $1-x > 0$ and $0 < 1-x < 1+x$. It holds $h'(x-1) = h'(1-x) > h'(x+1)$, so that again $\psi'(x) < 0$. Consequently ψ is strictly decreasing on $(0, +\infty)$.

Clearly, ψ is strictly increasing on $(-\infty, 0)$, and $\psi'(0) = 0$.

See that

$$\lim_{x \rightarrow +\infty} \psi(x) = \frac{1}{4} (h(+\infty) - h(+\infty)) = 0, \quad (7)$$

and

$$\lim_{x \rightarrow -\infty} \psi(x) = \frac{1}{4} (h(-\infty) - h(-\infty)) = 0. \quad (8)$$

That is the x -axis is the horizontal asymptote on ψ .

Conclusion, ψ is a bell symmetric function with maximum

$$\psi(0) = \frac{h(1)}{2}.$$

We need

Theorem 2.1. ([21]) *We have that*

$$\sum_{i=-\infty}^{\infty} \psi(x-i) = 1, \quad \forall x \in \mathbb{R}. \quad (9)$$

Theorem 2.2. ([21]) *It holds*

$$\int_{-\infty}^{\infty} \psi(x) dx = 1. \quad (10)$$

Thus $\psi(x)$ is a density function on $\mathbb{R}$.

We give

Theorem 2.3. ([21]) *Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds*

$$\sum_{\substack{k = -\infty \\ : |nx - k| \geq n^{1-\alpha}}}^{\infty} \psi(nx - k) < \frac{(1 - h(n^{1-\alpha} - 2))}{2}. \quad (11)$$

Notice that

$$\lim_{n \rightarrow +\infty} \frac{(1 - h(n^{1-\alpha} - 2))}{2} = 0.$$

Denote by $\lfloor \cdot \rfloor$ the integral part of the number and by $\lceil \cdot \rceil$ the ceiling of the number.

We further give

Theorem 2.4. ([21]) *Let $x \in [a, b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx - k)} < \frac{1}{\psi(1)}, \quad \forall x \in [a, b]. \quad (12)$$

Remark 2.1. ([21]) *i) We have that*

$$\lim_{n \rightarrow \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx - k) \neq 1, \quad (13)$$

for at least some $x \in [a, b]$.

ii) For large enough $n \in \mathbb{N}$ we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$.

In general, by Theorem 2.1, it holds

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx - k) \leq 1. \quad (14)$$

We introduce

$$Z(x_1, \dots, x_N) := Z(x) := \prod_{i=1}^N \psi(x_i), \quad x = (x_1, \dots, x_N) \in \mathbb{R}^N, \quad N \in \mathbb{N}. \quad (15)$$

It has the properties:

- (i) $Z(x) > 0, \quad \forall x \in \mathbb{R}^N$,
- (ii)

$$\sum_{k=-\infty}^{\infty} Z(x - k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \dots \sum_{k_N=-\infty}^{\infty} Z(x_1 - k_1, \dots, x_N - k_N) = 1, \quad (16)$$

where $k := (k_1, \dots, k_N) \in \mathbb{Z}^N, \quad \forall x \in \mathbb{R}^N$,

hence

- (iii)

$$\sum_{k=-\infty}^{\infty} Z(nx - k) = 1, \quad (17)$$

$\forall x \in \mathbb{R}^N; n \in \mathbb{N}$,

and

(iv)

$$\int_{\mathbb{R}^N} Z(x) dx = 1, \quad (18)$$

that is Z is a multivariate density function.

Here denote $\|x\|_\infty := \max\{|x_1|, \dots, |x_N|\}$, $x \in \mathbb{R}^N$, also set $\infty := (\infty, \dots, \infty)$, $-\infty := (-\infty, \dots, -\infty)$ upon the multivariate context, and

$$\begin{aligned} \lceil na \rceil &:= (\lceil na_1 \rceil, \dots, \lceil na_N \rceil), \\ \lfloor nb \rfloor &:= (\lfloor nb_1 \rfloor, \dots, \lfloor nb_N \rfloor), \end{aligned} \quad (19)$$

where $a := (a_1, \dots, a_N)$, $b := (b_1, \dots, b_N)$.

We obviously see that

$$\begin{aligned} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k) &= \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(\prod_{i=1}^N \psi(nx_i - k_i) \right) = \\ \sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \dots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} \left(\prod_{i=1}^N \psi(nx_i - k_i) \right) &= \prod_{i=1}^N \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \psi(nx_i - k_i) \right). \end{aligned} \quad (20)$$

For $0 < \beta < 1$ and $n \in \mathbb{N}$, a fixed $x \in \mathbb{R}^N$, we have that

$$\begin{aligned} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k) &= \\ \sum_{\substack{k=\lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} Z(nx - k) &+ \sum_{\substack{k=\lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} Z(nx - k). \end{aligned} \quad (21)$$

In the last two sums the counting is over disjoint vector sets of k 's, because the condition $\left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}$ implies that there exists at least one $\left| \frac{k_r}{n} - x_r \right| > \frac{1}{n^\beta}$, where $r \in \{1, \dots, N\}$.

(v) As in [18], pp. 379-380, we derive that

$$\begin{aligned} \sum_{\substack{k=\lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} Z(nx - k) &\stackrel{(11)}{<} \frac{1 - h(n^{1-\beta} - 2)}{2}, \quad 0 < \beta < 1, \end{aligned} \quad (22)$$

with $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \prod_{i=1}^N [a_i, b_i]$.

(vi) By Theorem 2.4 we get that

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)} < \frac{1}{(\psi(1))^N} =: \gamma(N), \quad (23)$$

$\forall x \in \left(\prod_{i=1}^N [a_i, b_i] \right), \quad n \in \mathbb{N}.$

It is also clear that

(vii)

$$\sum_{k=-\infty}^{\infty} Z(nx - k) < \frac{1 - h(n^{1-\beta} - 2)}{2} =: c(\beta, n), \quad (24)$$

$$\left\{ \begin{array}{l} k = -\infty \\ \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}} \end{array} \right.$$

$0 < \beta < 1, \quad n \in \mathbb{N} : n^{1-\beta} > 2, \quad x \in \mathbb{R}^N.$

Furthermore it holds

$$\lim_{n \rightarrow \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k) \neq 1, \quad (25)$$

for at least some $x \in \left(\prod_{i=1}^N [a_i, b_i] \right).$

Let $f \in C \left(\prod_{i=1}^N [a_i, b_i] \right)$, and $n \in \mathbb{N}$ such that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor, \quad i = 1, \dots, N.$

We define the multivariate averaged positive linear neural network operators

$(x := (x_1, \dots, x_N) \in \left(\prod_{i=1}^N [a_i, b_i] \right))$:

$$A_n(f, x_1, \dots, x_N) := A_n(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)} = \quad (26)$$

$$\frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} \dots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, \dots, \frac{k_N}{n}\right) \left(\prod_{i=1}^N \psi(nx_i - k_i) \right)}{\prod_{i=1}^N \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \psi(nx_i - k_i) \right)}.$$

For large enough $n \in \mathbb{N}$ we always obtain $\lceil na_i \rceil \leq \lfloor nb_i \rfloor, \quad i = 1, \dots, N.$ Also $a_i \leq \frac{k_i}{n} \leq b_i$, iff $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor, \quad i = 1, \dots, N.$

When $f \in C_B(\mathbb{R}^N)$ we define

$$B_n(f, x) := B_n(f, x_1, \dots, x_N) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx - k) := \quad (27)$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \dots \sum_{k_N=-\infty}^{\infty} f\left(\frac{k_1}{n}, \frac{k_2}{n}, \dots, \frac{k_N}{n}\right) \left(\prod_{i=1}^N \psi(nx_i - k_i) \right),$$

$n \in \mathbb{N}, \forall x \in \mathbb{R}^N, N \in \mathbb{N}$, the multivariate quasi-interpolation neural network operators.

Also for $f \in C_B(\mathbb{R}^N)$ we define the multivariate Kantorovich type neural network operators

$$\begin{aligned} C_n(f, x) := C_n(f, x_1, \dots, x_N) := & \sum_{k=-\infty}^{\infty} \left(n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) dt \right) Z(nx - k) := \\ & \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \dots \sum_{k_N=-\infty}^{\infty} \left(n^N \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}} \dots \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1, \dots, t_N) dt_1 \dots dt_N \right) \\ & \cdot \left(\prod_{i=1}^N \psi(nx_i - k_i) \right), \end{aligned} \quad (28)$$

$n \in \mathbb{N}, \forall x \in \mathbb{R}^N$.

Again for $f \in C_B(\mathbb{R}^N)$, $N \in \mathbb{N}$, we define the multivariate neural network operators of quadrature type $D_n(f, x)$, $n \in \mathbb{N}$, as follows. Let $\theta = (\theta_1, \dots, \theta_N) \in \mathbb{N}^N$, $\bar{r} = (r_1, \dots, r_N) \in \mathbb{Z}_+^N$, $w_{\bar{r}} = w_{r_1, r_2, \dots, r_N} \geq 0$, such that $\sum_{\bar{r}=0}^{\theta} w_{\bar{r}} = \sum_{r_1=0}^{\theta_1} \sum_{r_2=0}^{\theta_2} \dots \sum_{r_N=0}^{\theta_N} w_{r_1, r_2, \dots, r_N} = 1$; $k \in \mathbb{Z}^N$ and

$$\delta_{nk}(f) := \delta_{n, k_1, k_2, \dots, k_N}(f) := \sum_{\bar{r}=0}^{\theta} w_{\bar{r}} f\left(\frac{k}{n} + \frac{\bar{r}}{n\theta}\right) :=$$

$$\sum_{r_1=0}^{\theta_1} \sum_{r_2=0}^{\theta_2} \dots \sum_{r_N=0}^{\theta_N} w_{r_1, r_2, \dots, r_N} f\left(\frac{k_1}{n} + \frac{r_1}{n\theta_1}, \frac{k_2}{n} + \frac{r_2}{n\theta_2}, \dots, \frac{k_N}{n} + \frac{r_N}{n\theta_N}\right), \quad (29)$$

where $\frac{\bar{r}}{\theta} := \left(\frac{r_1}{\theta_1}, \frac{r_2}{\theta_2}, \dots, \frac{r_N}{\theta_N}\right)$.

We put

$$D_n(f, x) := D_n(f, x_1, \dots, x_N) := \sum_{k=-\infty}^{\infty} \delta_{nk}(f) Z(nx - k) := \quad (30)$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \dots \sum_{k_N=-\infty}^{\infty} \delta_{n, k_1, k_2, \dots, k_N}(f) \left(\prod_{i=1}^N \psi(nx_i - k_i) \right),$$

$\forall x \in \mathbb{R}^N$.

For the next we need, for $f \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$ the first multivariate modulus of continuity

$$\omega_1(f, h) := \sup_{\substack{x, y \in \prod_{i=1}^N [a_i, b_i] \\ \|x-y\|_\infty \leq h}} |f(x) - f(y)|, \quad h > 0.$$

A totally similar definition applies to $f \in C_B(\mathbb{R}^N)$.

Above $\|\cdot\|_\infty$ is the supremum norm.

In [20] we studied the basic approximation properties of A_n, B_n, C_n, D_n neural network operators and as well of their iterates for Banach space valued functions. That is, the quantitative pointwise and uniform convergence of these operators to the unit operator I .

We need

Theorem 2.5. *Let $f \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$, $0 < \beta < 1$, $x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$, $N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$. Then*

1)

$$|A_n(f, x) - f(x)| \leq \gamma(N) \left[\omega_1\left(f, \frac{1}{n^\beta}\right) + 2c(\beta, n) \|f\|_\infty \right] =: \lambda_1, \quad (31)$$

and

2)

$$\|A_n(f) - f\|_\infty \leq \lambda_1. \quad (32)$$

We notice that $\lim_{n \rightarrow \infty} A_n(f) = f$, pointwise and uniformly.

Proof. Similar to [20], p. 118. ■

We need

Theorem 2.6. *Let $f \in C_B(\mathbb{R}^N)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$. Then*

1)

$$|B_n(f, x) - f(x)| \leq \omega_1\left(f, \frac{1}{n^\beta}\right) + 2c(\beta, n) \|f\|_\infty =: \lambda_2, \quad (33)$$

2)

$$\|B_n(f) - f\|_\infty \leq \lambda_2. \quad (34)$$

Given that $f \in (C_U(\mathbb{R}^N) \cap C_B(\mathbb{R}^N))$, we obtain $\lim_{n \rightarrow \infty} B_n(f) = f$, uniformly.

Proof. Similar to [20], p. 128. ■

We also need

Theorem 2.7. Let $f \in C_B(\mathbb{R}^N)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$. Then

1)

$$|C_n(f, x) - f(x)| \leq \omega_1\left(f, \frac{1}{n} + \frac{1}{n^\beta}\right) + 2c(\beta, n) \|f\|_\infty =: \lambda_3, \quad (35)$$

2)

$$\|C_n(f) - f\|_\infty \leq \lambda_3. \quad (36)$$

Given that $f \in (C_U(\mathbb{R}^N) \cap C_B(\mathbb{R}^N))$, we obtain $\lim_{n \rightarrow \infty} C_n(f) = f$, uniformly.

Proof. Similar to [20], p. 129. ■

We also need

Theorem 2.8. Let $f \in C_B(\mathbb{R}^N)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$. Then

1)

$$|D_n(f, x) - f(x)| \leq \omega_1\left(f, \frac{1}{n} + \frac{1}{n^\beta}\right) + 2c(\beta, n) \|f\|_\infty = \lambda_3, \quad (37)$$

2)

$$\|D_n(f) - f\|_\infty \leq \lambda_3. \quad (38)$$

Given that $f \in (C_U(\mathbb{R}^N) \cap C_B(\mathbb{R}^N))$, we obtain $\lim_{n \rightarrow \infty} D_n(f) = f$, uniformly.

Proof. Similar to [20], p. 131. ■

In this article we extend Theorems 2.5, 2.6, 2.7, 2.8 to the random level.

We are also motivated by [1] - [16] and continuing [17]. For general knowledge on neural networks we recommend [25], [26], [27].

3. MAIN RESULTS

I) q -mean Approximation by Fuzzy-Random general sigmoid activation function based Quasi-Interpolation Neural Network Operators

All terms and assumptions here as in Sections 1, 2.

Let $f \in C_{\mathcal{FR}}^{U_q}\left(\prod_{i=1}^N [a_i, b_i]\right)$, $1 \leq q < +\infty$, $n, N \in \mathbb{N}$, $0 < \beta < 1$, $\vec{x} \in \left(\prod_{i=1}^N [a_i, b_i]\right)$, $(X, \mathcal{B}, P)$ probability space, $s \in X$.

We define the following multivariate fuzzy random general sigmoid activation function based quasi-interpolation linear neural network operators

$$\left(A_n^{\mathcal{FR}}(f)\right)(\vec{x}, s) := \sum_{\vec{k}=\lceil na \rceil}^{\lfloor nb \rfloor^*} f\left(\frac{\vec{k}}{n}, s\right) \odot \frac{Z\left(n\vec{x} - \vec{k}\right)}{\sum_{\vec{k}=\lceil na \rceil}^{\lfloor nb \rfloor} Z\left(n\vec{x} - \vec{k}\right)}, \quad (39)$$

(see also (26)).

We present

Theorem 3.1. *Let $f \in C_{\mathcal{FR}}^{U_q}\left(\prod_{i=1}^N [a_i, b_i]\right)$, $0 < \beta < 1$, $\vec{x} \in \left(\prod_{i=1}^N [a_i, b_i]\right)$, $n, N \in \mathbb{N}$, with $n^{1-\beta} > 2$, $1 \leq q < +\infty$. Assume that $\int_X (D^*(f(\cdot, s), \tilde{o}))^q P(ds) < \infty$. Then*

1)

$$\left(\int_X D^q\left(\left(A_n^{\mathcal{FR}}(f)\right)(\vec{x}, s), f(\vec{x}, s)\right) P(ds)\right)^{\frac{1}{q}} \leq \quad (40)$$

$$\gamma(N) \left\{ \Omega_1\left(f, \frac{1}{n^\beta}\right)_{L^q} + 2c(\beta, n) \left(\int_X (D^*(f(\cdot, s), \tilde{o}))^q P(ds)\right)^{\frac{1}{q}} \right\} =: \lambda_1^{(\mathcal{FR})},$$

2)

$$\left\| \left(\int_X D^q\left(\left(A_n^{\mathcal{FR}}(f)\right)(\vec{x}, s), f(\vec{x}, s)\right) P(ds)\right)^{\frac{1}{q}} \right\|_{\infty, \left(\prod_{i=1}^N [a_i, b_i]\right)} \leq \lambda_1^{(\mathcal{FR})}, \quad (41)$$

where $\gamma(N)$ as in (23) and $c(\beta, n)$ as in (24).

Proof. We notice that

$$\begin{aligned} D\left(f\left(\frac{\vec{k}}{n}, s\right), f(\vec{x}, s)\right) &\leq D\left(f\left(\frac{\vec{k}}{n}, s\right), \tilde{o}\right) + D(f(\vec{x}, s), \tilde{o}) \\ &\leq 2D^*(f(\cdot, s), \tilde{o}). \end{aligned} \quad (42)$$

Hence

$$D^q\left(f\left(\frac{\vec{k}}{n}, s\right), f(\vec{x}, s)\right) \leq 2^q D^{*q}(f(\cdot, s), \tilde{o}), \quad (43)$$

and

$$\left(\int_X D^q\left(f\left(\frac{\vec{k}}{n}, s\right), f(\vec{x}, s)\right) P(ds)\right)^{\frac{1}{q}} \leq 2 \left(\int_X (D^*(f(\cdot, s), \tilde{o}))^q P(ds)\right)^{\frac{1}{q}}. \quad (44)$$

We observe that

$$D \left(\left(A_n^{\mathcal{FR}}(f) \right) (\vec{x}, s), f(\vec{x}, s) \right) = \quad (45)$$

$$\begin{aligned} & D \left(\sum_{\vec{k}=\lceil na \rceil}^{\lfloor nb \rfloor^*} f \left(\frac{\vec{k}}{n}, s \right) \odot \frac{Z(nx-k)}{\sum_{\vec{k}=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)}, f(\vec{x}, s) \odot 1 \right) = \\ & D \left(\sum_{\vec{k}=\lceil na \rceil}^{\lfloor nb \rfloor^*} f \left(\frac{\vec{k}}{n}, s \right) \odot \frac{Z(nx-k)}{\sum_{\vec{k}=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)}, f(\vec{x}, s) \odot \frac{\sum_{\vec{k}=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)}{\sum_{\vec{k}=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} \right) = \\ & D \left(\sum_{\vec{k}=\lceil na \rceil}^{\lfloor nb \rfloor^*} f \left(\frac{\vec{k}}{n}, s \right) \odot \frac{Z(nx-k)}{\sum_{\vec{k}=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)}, \sum_{\vec{k}=\lceil na \rceil}^{\lfloor nb \rfloor^*} f(\vec{x}, s) \odot \frac{Z(nx-k)}{\sum_{\vec{k}=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} \right) \end{aligned} \quad (46)$$

$$\leq \sum_{\vec{k}=\lceil na \rceil}^{\lfloor nb \rfloor} \left(\frac{Z(nx-k)}{\sum_{\vec{k}=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} \right) D \left(f \left(\frac{\vec{k}}{n}, s \right), f(\vec{x}, s) \right). \quad (47)$$

So that

$$\begin{aligned} & D \left(\left({}_j A_n^{\mathcal{FR}}(f) \right) (\vec{x}, s), f(\vec{x}, s) \right) \leq \\ & \sum_{\vec{k}=\lceil na \rceil}^{\lfloor nb \rfloor} \left(\frac{Z(nx-k)}{\sum_{\vec{k}=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} \right) D \left(f \left(\frac{\vec{k}}{n}, s \right), f(\vec{x}, s) \right) = \quad (48) \\ & \sum_{\vec{k}=\lceil na \rceil}^{\lfloor nb \rfloor} \left(\frac{Z(nx-k)}{\sum_{\vec{k}=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} \right) D \left(f \left(\frac{\vec{k}}{n}, s \right), f(\vec{x}, s) \right) + \\ & \left\| \frac{\vec{k}}{n} - \vec{x} \right\|_{\infty} \leq \frac{1}{n^{\beta}} \end{aligned}$$

$$\sum_{\substack{\vec{k}=[na] \\ \|\frac{\vec{k}}{n}-\vec{x}\|_{\infty}>\frac{1}{n^{\beta}}}}^{\lfloor nb \rfloor} \left(\frac{Z(nx-k)}{\sum_{\vec{k}=[na]}^{\lfloor nb \rfloor} Z(nx-k)} \right) D \left(f \left(\frac{\vec{k}}{n}, s \right), f(\vec{x}, s) \right).$$

Hence it holds

$$\left(\int_X D^q \left(\left(A_n^{\mathcal{FR}}(f) \right) (\vec{x}, s), f(\vec{x}, s) \right) P(ds) \right)^{\frac{1}{q}} \leq \quad (49)$$

$$\sum_{\substack{\vec{k}=[na] \\ \|\frac{\vec{k}}{n}-\vec{x}\|_{\infty} \leq \frac{1}{n^{\beta}}}}^{\lfloor nb \rfloor} \left(\frac{Z(nx-k)}{\sum_{\vec{k}=[na]}^{\lfloor nb \rfloor} Z(nx-k)} \right) \left(\int_X D^q \left(f \left(\frac{\vec{k}}{n}, s \right), f(\vec{x}, s) \right) P(ds) \right)^{\frac{1}{q}} +$$

$$\sum_{\substack{\vec{k}=[na] \\ \|\frac{\vec{k}}{n}-\vec{x}\|_{\infty} > \frac{1}{n^{\beta}}}}^{\lfloor nb \rfloor} \left(\frac{Z(nx-k)}{\sum_{\vec{k}=[na]}^{\lfloor nb \rfloor} Z(nx-k)} \right) \left(\int_X D^q \left(f \left(\frac{\vec{k}}{n}, s \right), f(\vec{x}, s) \right) P(ds) \right)^{\frac{1}{q}} \leq$$

$$\left(\frac{1}{\sum_{\vec{k}=[na]}^{\lfloor nb \rfloor} Z(nx-k)} \right) \cdot \left\{ \Omega_1^{(\mathcal{F})} \left(f, \frac{1}{n^{\beta}} \right)_{L^q} + \quad (50)$$

$$2 \left(\int_X (D^*(f(\cdot, s), \tilde{o}))^q P(ds) \right)^{\frac{1}{q}} \left(\sum_{\substack{\vec{k}=[na] \\ \|\frac{\vec{k}}{n}-\vec{x}\|_{\infty} > \frac{1}{n^{\beta}}}}^{\lfloor nb \rfloor} Z(nx-k) \right)$$

(by (23), (24))

$$\leq \gamma(N) \left\{ \Omega_1^{(\mathcal{F})} \left(f, \frac{1}{n^{\beta}} \right)_{L^q} + 2c(\beta, n) \left(\int_X (D^*(f(\cdot, s), \tilde{o}))^q P(ds) \right)^{\frac{1}{q}} \right\}. \quad (51)$$

We have proved claim. ■

Conclusion 1. *By Theorem 3.1 we obtain the pointwise and uniform convergences with rates in the q -mean and D -metric of the operator $A_n^{\mathcal{FR}}$ to the unit operator for $f \in C_{\mathcal{FR}}^{U_q} \left(\prod_{i=1}^N [a_i, b_i] \right)$.*

II) 1-mean Approximation by Stochastic general sigmoid activation function based full Quasi-Interpolation Neural Network Operators

Let $g \in C_{\mathcal{R}}^{U_1}(\mathbb{R}^N)$, $0 < \beta < 1$, $\vec{x} \in \mathbb{R}^N$, $n, N \in \mathbb{N}$, with $\|g\|_{\infty, \mathbb{R}^N, X} < \infty$, $(X, \mathcal{B}, P)$ probability space, $s \in X$.

We define

$$B_n^{(\mathcal{R})}(g)(\vec{x}, s) := \sum_{\vec{k}=-\infty}^{\infty} g\left(\frac{\vec{k}}{n}, s\right) Z(n\vec{x} - \vec{k}), \quad (52)$$

(see also (27)).

We give

Theorem 3.2. *Let $g \in C_{\mathcal{R}}^{U_1}(\mathbb{R}^N)$, $0 < \beta < 1$, $\vec{x} \in \mathbb{R}^N$, $n, N \in \mathbb{N}$, with $n^{1-\beta} > 2$, $\|g\|_{\infty, \mathbb{R}^N, X} < \infty$. Then*

1)

$$\int_X \left| \left(B_n^{(\mathcal{R})}(g) \right) (\vec{x}, s) - g(\vec{x}, s) \right| P(ds) \leq \left\{ \Omega_1 \left(g, \frac{1}{n^\beta} \right)_{L^1} + 2c(\beta, n) \|g\|_{\infty, \mathbb{R}^N, X} \right\} =: \mu_1^{(\mathcal{R})}, \quad (53)$$

2)

$$\left\| \int_X \left| \left(B_n^{(\mathcal{R})}(g) \right) (\vec{x}, s) - g(\vec{x}, s) \right| P(ds) \right\|_{\infty, \mathbb{R}^N} \leq \mu_1^{(\mathcal{R})}. \quad (54)$$

Proof. Since $\|g\|_{\infty, \mathbb{R}^N, X} < \infty$, then

$$\left| g\left(\frac{\vec{k}}{n}, s\right) - g(\vec{x}, s) \right| \leq 2 \|g\|_{\infty, \mathbb{R}^N, X} < \infty. \quad (55)$$

Hence

$$\int_X \left| g\left(\frac{\vec{k}}{n}, s\right) - g(\vec{x}, s) \right| P(ds) \leq 2 \|g\|_{\infty, \mathbb{R}^N, X} < \infty. \quad (56)$$

We observe that

$$\left(B_n^{(\mathcal{R})}(g) \right) (\vec{x}, s) - g(\vec{x}, s) =$$

$$\begin{aligned} \sum_{\vec{k}=-\infty}^{\infty} g\left(\frac{\vec{k}}{n}, s\right) Z(nx - k) - g(\vec{x}, s) \sum_{\vec{k}=-\infty}^{\infty} Z(nx - k) = \\ \left(\sum_{\vec{k}=-\infty}^{\infty} g\left(\frac{\vec{k}}{n}, s\right) - g(\vec{x}, s) \right) Z(nx - k). \end{aligned} \quad (57)$$

However it holds

$$\sum_{\vec{k}=-\infty}^{\infty} \left| g\left(\frac{\vec{k}}{n}, s\right) - g(\vec{x}, s) \right| Z(nx - k) \leq 2 \|g\|_{\infty, \mathbb{R}^N, X} < \infty. \quad (58)$$

Hence

$$\begin{aligned} & \left| \left(B_n^{(\mathcal{R})}(g) \right) (\vec{x}, s) - g(\vec{x}, s) \right| \leq \\ & \sum_{\vec{k}=-\infty}^{\infty} \left| g\left(\frac{\vec{k}}{n}, s\right) - g(\vec{x}, s) \right| Z(nx - k) = \\ & \sum_{\vec{k}=-\infty}^{\infty} \left| g\left(\frac{\vec{k}}{n}, s\right) - g(\vec{x}, s) \right| Z(nx - k) + \\ & \left\| \frac{\vec{k}}{n} - \vec{x} \right\|_{\infty} \leq \frac{1}{n^\beta} \\ & \sum_{\vec{k}=-\infty}^{\infty} \left| g\left(\frac{\vec{k}}{n}, s\right) - g(\vec{x}, s) \right| Z(nx - k). \\ & \left\| \frac{\vec{k}}{n} - \vec{x} \right\|_{\infty} > \frac{1}{n^\beta} \end{aligned} \quad (59)$$

Furthermore it holds

$$\begin{aligned} & \left(\int_X \left| \left(B_n^{(\mathcal{R})}(g) \right) (\vec{x}, s) - g(\vec{x}, s) \right| P(ds) \right) \leq \\ & \sum_{\vec{k}=-\infty}^{\infty} \left(\int_X \left| g\left(\frac{\vec{k}}{n}, s\right) - g(\vec{x}, s) \right| P(ds) \right) Z(nx - k) + \\ & \left\| \frac{\vec{k}}{n} - \vec{x} \right\|_{\infty} \leq \frac{1}{n^\beta} \\ & \sum_{\vec{k}=-\infty}^{\infty} \left(\int_X \left| g\left(\frac{\vec{k}}{n}, s\right) - g(\vec{x}, s) \right| P(ds) \right) Z(nx - k) \leq \\ & \left\| \frac{\vec{k}}{n} - \vec{x} \right\|_{\infty} > \frac{1}{n^\beta} \end{aligned} \quad (60)$$

$$\Omega_1 \left(g, \frac{1}{n^\beta} \right)_{L^1} + 2 \|g\|_{\infty, \mathbb{R}^N, X} \sum_{\substack{\vec{k} = -\infty \\ \left\| \frac{\vec{k}}{n} - \vec{x} \right\|_\infty > \frac{1}{n^\beta}}}^{\infty} Z(n\vec{x} - \vec{k}) \leq$$

$$\Omega_1 \left(g, \frac{1}{n^\beta} \right)_{L^1} + 2c(\beta, n) \|g\|_{\infty, \mathbb{R}^N, X},$$

proving the claim. ■

Conclusion 2. By Theorem 3.2 we obtain pointwise and uniform convergences with rates in the 1-mean of random operators $B_n^{(\mathcal{R})}$ to the unit operator for $g \in C_{\mathcal{R}}^{U_1}(\mathbb{R}^N)$.

III) 1-mean Approximation by Stochastic general sigmoid activation function based multivariate Kantorovich type neural network operator

Let $g \in C_{\mathcal{R}}^{U_1}(\mathbb{R}^N)$, $0 < \beta < 1$, $\vec{x} \in \mathbb{R}^N$, $n, N \in \mathbb{N}$, with $\|g\|_{\infty, \mathbb{R}^N, X} < \infty$, $(X, \mathcal{B}, P)$ probability space, $s \in X$.

We define

$$C_n^{(\mathcal{R})}(g)(\vec{x}, s) := \sum_{\vec{k} = -\infty}^{\infty} \left(n^N \int_{\frac{\vec{k}}{n}}^{\frac{\vec{k}+1}{n}} g(\vec{t}, s) d\vec{t} \right) Z(n\vec{x} - \vec{k}), \quad (61)$$

(see also (28).

We present

Theorem 3.3. Let $g \in C_{\mathcal{R}}^{U_1}(\mathbb{R}^N)$, $0 < \beta < 1$, $\vec{x} \in \mathbb{R}^N$, $n, N \in \mathbb{N}$, with $n^{1-\beta} > 2$, $\|g\|_{\infty, \mathbb{R}^N, X} < \infty$. Then

1)

$$\int_X \left| \left(C_n^{(\mathcal{R})}(g) \right) (\vec{x}, s) - g(\vec{x}, s) \right| P(ds) \leq$$

$$\left[\Omega_1 \left(g, \frac{1}{n} + \frac{1}{n^\beta} \right)_{L^1} + 2c(\beta, n) \|g\|_{\infty, \mathbb{R}^N, X} \right] =: \gamma_1^{(\mathcal{R})}, \quad (62)$$

2)

$$\left\| \int_X \left| \left(C_n^{(\mathcal{R})}(g) \right) (\vec{x}, s) - g(\vec{x}, s) \right| P(ds) \right\|_{\infty, \mathbb{R}^N} \leq \gamma_1^{(\mathcal{R})}. \quad (63)$$

Proof. Since $\|g\|_{\infty, \mathbb{R}^N, X} < \infty$, then

$$\left| n^N \int_{\frac{\vec{k}}{n}}^{\frac{\vec{k}+1}{n}} g(\vec{t}, s) d\vec{t} - g(\vec{x}, s) \right| = \left| n^N \int_{\frac{\vec{k}}{n}}^{\frac{\vec{k}+1}{n}} (g(\vec{t}, s) - g(\vec{x}, s)) d\vec{t} \right| \leq$$

$$n^N \int_{\frac{\vec{k}}{n}}^{\frac{\vec{k}+1}{n}} \left| g\left(\frac{\vec{t}}{n}, s\right) - g\left(\frac{\vec{x}}{n}, s\right) \right| d\vec{t} \leq 2 \|g\|_{\infty, \mathbb{R}^N, X} < \infty. \quad (64)$$

Hence

$$\int_X \left| n^N \int_{\frac{\vec{k}}{n}}^{\frac{\vec{k}+1}{n}} g\left(\frac{\vec{t}}{n}, s\right) d\vec{t} - g\left(\frac{\vec{x}}{n}, s\right) \right| P(ds) \leq 2 \|g\|_{\infty, \mathbb{R}^N, X} < \infty. \quad (65)$$

We observe that

$$\begin{aligned} & \left(C_n^{(\mathcal{R})}(g) \right) \left(\frac{\vec{x}}{n}, s \right) - g\left(\frac{\vec{x}}{n}, s\right) = \\ & \sum_{\vec{k}=-\infty}^{\infty} \left(n^N \int_{\frac{\vec{k}}{n}}^{\frac{\vec{k}+1}{n}} g\left(\frac{\vec{t}}{n}, s\right) d\vec{t} \right) Z\left(n\frac{\vec{x}}{n} - \frac{\vec{k}}{n}\right) - g\left(\frac{\vec{x}}{n}, s\right) = \\ & \sum_{\vec{k}=-\infty}^{\infty} \left(n^N \int_{\frac{\vec{k}}{n}}^{\frac{\vec{k}+1}{n}} g\left(\frac{\vec{t}}{n}, s\right) d\vec{t} \right) Z\left(n\frac{\vec{x}}{n} - \frac{\vec{k}}{n}\right) - g\left(\frac{\vec{x}}{n}, s\right) \sum_{\vec{k}=-\infty}^{\infty} Z\left(n\frac{\vec{x}}{n} - \frac{\vec{k}}{n}\right) = \\ & \sum_{\vec{k}=-\infty}^{\infty} \left[\left(n^N \int_{\frac{\vec{k}}{n}}^{\frac{\vec{k}+1}{n}} g\left(\frac{\vec{t}}{n}, s\right) d\vec{t} \right) - g\left(\frac{\vec{x}}{n}, s\right) \right] Z\left(n\frac{\vec{x}}{n} - \frac{\vec{k}}{n}\right) = \\ & \sum_{\vec{k}=-\infty}^{\infty} \left[n^N \int_{\frac{\vec{k}}{n}}^{\frac{\vec{k}+1}{n}} \left(g\left(\frac{\vec{t}}{n}, s\right) - g\left(\frac{\vec{x}}{n}, s\right) \right) d\vec{t} \right] Z\left(n\frac{\vec{x}}{n} - \frac{\vec{k}}{n}\right). \end{aligned} \quad (66)$$

However it holds

$$\sum_{\vec{k}=-\infty}^{\infty} \left[n^N \int_{\frac{\vec{k}}{n}}^{\frac{\vec{k}+1}{n}} \left| g\left(\frac{\vec{t}}{n}, s\right) - g\left(\frac{\vec{x}}{n}, s\right) \right| d\vec{t} \right] Z\left(n\frac{\vec{x}}{n} - \frac{\vec{k}}{n}\right) \leq 2 \|g\|_{\infty, \mathbb{R}^N, X} < \infty. \quad (67)$$

Hence

$$\begin{aligned} & \left| \left(C_n^{(\mathcal{R})}(g) \right) \left(\frac{\vec{x}}{n}, s \right) - g\left(\frac{\vec{x}}{n}, s\right) \right| \leq \\ & \sum_{\vec{k}=-\infty}^{\infty} \left[n^N \int_{\frac{\vec{k}}{n}}^{\frac{\vec{k}+1}{n}} \left| g\left(\frac{\vec{t}}{n}, s\right) - g\left(\frac{\vec{x}}{n}, s\right) \right| d\vec{t} \right] Z\left(n\frac{\vec{x}}{n} - \frac{\vec{k}}{n}\right) = \end{aligned} \quad (68)$$

$$\begin{aligned} & \sum_{\vec{k}=-\infty}^{\infty} \left[n^N \int_{\frac{\vec{k}}{n}}^{\frac{\vec{k}+1}{n}} \left| g\left(\frac{\vec{t}}{n}, s\right) - g\left(\frac{\vec{x}}{n}, s\right) \right| d\vec{t} \right] Z\left(n\frac{\vec{x}}{n} - \frac{\vec{k}}{n}\right) + \\ & \left\| \frac{\vec{k}}{n} - \frac{\vec{x}}{n} \right\|_{\infty} \leq \frac{1}{n^{\beta}} \end{aligned} \quad (69)$$

$$\begin{aligned}
& \sum_{\substack{\vec{k}=-\infty \\ \|\frac{\vec{k}}{n}-\vec{x}\|_\infty > \frac{1}{n^\beta}}}^{\infty} \left[n^N \int_{\frac{\vec{k}}{n}}^{\frac{\vec{k}+1}{n}} \left| g\left(\vec{t}, s\right) - g\left(\vec{x}, s\right) \right| d\vec{t} \right] Z\left(n\vec{x} - \vec{k}\right) = \\
& \sum_{\substack{\vec{k}=-\infty \\ \|\frac{\vec{k}}{n}-\vec{x}\|_\infty \leq \frac{1}{n^\beta}}}^{\infty} \left[n^N \int_0^{\frac{1}{n}} \left| g\left(\vec{t} + \frac{\vec{k}}{n}, s\right) - g\left(\vec{x}, s\right) \right| d\vec{t} \right] Z\left(n\vec{x} - \vec{k}\right) + \quad (70) \\
& \sum_{\substack{\vec{k}=-\infty \\ \|\frac{\vec{k}}{n}-\vec{x}\|_\infty > \frac{1}{n^\beta}}}^{\infty} \left[n^N \int_0^{\frac{1}{n}} \left| g\left(\vec{t} + \frac{\vec{k}}{n}, s\right) - g\left(\vec{x}, s\right) \right| d\vec{t} \right] Z\left(n\vec{x} - \vec{k}\right).
\end{aligned}$$

Furthermore it holds

$$\begin{aligned}
& \left(\int_X \left| \left(C_n^{(\mathcal{R})}(g) \right) (\vec{x}, s) - g(\vec{x}, s) \right| P(ds) \right) \stackrel{\leq}{\text{(by Fubini's theorem)}} \\
& \sum_{\substack{\vec{k}=-\infty \\ \|\frac{\vec{k}}{n}-\vec{x}\|_\infty \leq \frac{1}{n^\beta}}}^{\infty} \left[n^N \int_0^{\frac{1}{n}} \left(\int_X \left| g\left(\vec{t} + \frac{\vec{k}}{n}, s\right) - g(\vec{x}, s) \right| P(ds) \right) d\vec{t} \right] Z\left(n\vec{x} - \vec{k}\right) + \\
& \sum_{\substack{\vec{k}=-\infty \\ \|\frac{\vec{k}}{n}-\vec{x}\|_\infty > \frac{1}{n^\beta}}}^{\infty} \left[n^N \int_0^{\frac{1}{n}} \left(\int_X \left| g\left(\vec{t} + \frac{\vec{k}}{n}, s\right) - g(\vec{x}, s) \right| P(ds) \right) d\vec{t} \right] Z\left(n\vec{x} - \vec{k}\right) \leq \quad (71) \\
& \Omega_1 \left(g, \frac{1}{n} + \frac{1}{n^\beta} \right)_{L^1} + 2 \|g\|_{\infty, \mathbb{R}^N, X} \sum_{\substack{\vec{k}=-\infty \\ \|\frac{\vec{k}}{n}-\vec{x}\|_\infty > \frac{1}{n^\beta}}}^{\infty} Z\left(n\vec{x} - \vec{k}\right) \leq \\
& \Omega_1 \left(g, \frac{1}{n} + \frac{1}{n^\beta} \right)_{L^1} + 2c(\beta, n) \|g\|_{\infty, \mathbb{R}^N, X}, \quad (72)
\end{aligned}$$

proving the claim. ■

Conclusion 3. *By Theorem 3.3 we obtain pointwise and uniform convergences with rates in the 1-mean of random operators $C_n^{(\mathcal{R})}$ to the unit operator for $g \in C_{\mathcal{R}}^{U_1}(\mathbb{R}^N)$.*

IV) 1-mean Approximation by Stochastic general sigmoid activation function based multivariate quadrature type neural network operator

Let $g \in C_{\mathcal{R}}^{U_1}(\mathbb{R}^N)$, $0 < \beta < 1$, $\vec{x} \in \mathbb{R}^N$, $n, N \in \mathbb{N}$, with $\|g\|_{\infty, \mathbb{R}^N, X} < \infty$, $(X, \mathcal{B}, P)$ probability space, $s \in X$.

We define

$$D_n^{(\mathcal{R})}(g)(\vec{x}, s) := \sum_{\vec{k}=-\infty}^{\infty} \left(\delta_{n\vec{k}}(g) \right)(s) Z\left(n\vec{x} - \vec{k}\right), \quad (73)$$

where

$$\left(\delta_{n\vec{k}}(g) \right)(s) := \sum_{\vec{r}=0}^{\vec{\theta}} w_{\vec{r}} g\left(\frac{\vec{k}}{n} + \frac{\vec{r}}{n\vec{\theta}}, s\right), \quad (74)$$

(see also (29), (30)).

We finally give

Theorem 3.4. Let $g \in C_{\mathcal{R}}^{U_1}(\mathbb{R}^N)$, $0 < \beta < 1$, $\vec{x} \in \mathbb{R}^N$, $n, N \in \mathbb{N}$, with $n^{1-\beta} > 2$, $\|g\|_{\infty, \mathbb{R}^N, X} < \infty$. Then

1)

$$\int_X \left| \left(D_n^{(\mathcal{R})}(g) \right)(\vec{x}, s) - g(\vec{x}, s) \right| P(ds) \leq \left\{ \Omega_1 \left(g, \frac{1}{n} + \frac{1}{n^\beta} \right)_{L^1} + 2c(\beta, n) \|g\|_{\infty, \mathbb{R}^N, X} \right\} =: \gamma_1^{(\mathcal{R})}, \quad (75)$$

2)

$$\left\| \int_X \left| \left(D_n^{(\mathcal{R})}(g) \right)(\vec{x}, s) - g(\vec{x}, s) \right| P(ds) \right\|_{\infty, \mathbb{R}^N} \leq \gamma_1^{(\mathcal{R})}. \quad (76)$$

Proof. Notice that

$$\begin{aligned} & \left| \left(\delta_{n\vec{k}}(g) \right)(s) - g(\vec{x}, s) \right| = \\ & \left| \sum_{\vec{r}=0}^{\vec{\theta}} w_{\vec{r}} g\left(\frac{\vec{k}}{n} + \frac{\vec{r}}{n\vec{\theta}}, s\right) - g(\vec{x}, s) \right| \leq \\ & \sum_{\vec{r}=0}^{\vec{\theta}} w_{\vec{r}} \left| g\left(\frac{\vec{k}}{n} + \frac{\vec{r}}{n\vec{\theta}}, s\right) - g(\vec{x}, s) \right| \leq 2 \|g\|_{\infty, \mathbb{R}^N, X} < \infty. \end{aligned} \quad (77)$$

Hence

$$\int_X \left| \left(\delta_{n\vec{k}}(g) \right)(s) - g(\vec{x}, s) \right| P(ds) \leq 2 \|g\|_{\infty, \mathbb{R}^N, X} < \infty. \quad (78)$$

We observe that

$$\begin{aligned}
& \left(D_n^{(\mathcal{R})}(g) \right) (\vec{x}, s) - g(\vec{x}, s) = \\
& \sum_{\vec{k}=-\infty}^{\infty} \left(\delta_{n\vec{k}}(g) \right) (s) Z \left(n\vec{x} - \vec{k} \right) - g(\vec{x}, s) = \\
& \sum_{\vec{k}=-\infty}^{\infty} \left(\left(\delta_{n\vec{k}}(g) \right) (s) - g(\vec{x}, s) \right) Z \left(n\vec{x} - \vec{k} \right). \tag{79}
\end{aligned}$$

Thus

$$\begin{aligned}
& \left| D_n^{(\mathcal{R})}(g) (\vec{x}, s) - g(\vec{x}, s) \right| \leq \\
& \sum_{\vec{k}=-\infty}^{\infty} \left| \left(\delta_{n\vec{k}}(g) \right) (s) - g(\vec{x}, s) \right| Z \left(n\vec{x} - \vec{k} \right) \leq 2 \|g\|_{\infty, \mathbb{R}^N, X} < \infty. \tag{80}
\end{aligned}$$

Hence it holds

$$\begin{aligned}
& \left| \left(D_n^{(\mathcal{R})}(g) \right) (\vec{x}, s) - g(\vec{x}, s) \right| \leq \\
& \sum_{\vec{k}=-\infty}^{\infty} \left| \left(\delta_{n\vec{k}}(g) \right) (s) - g(\vec{x}, s) \right| Z \left(n\vec{x} - \vec{k} \right) = \\
& \sum_{\vec{k}=-\infty}^{\infty} \left| \left(\delta_{n\vec{k}}(g) \right) (s) - g(\vec{x}, s) \right| Z \left(n\vec{x} - \vec{k} \right) + \\
& \left\| \frac{\vec{k}}{n} - \vec{x} \right\|_{\infty} \leq \frac{1}{n^{\beta}} \\
& \sum_{\vec{k}=-\infty}^{\infty} \left| \left(\delta_{n\vec{k}}(g) \right) (s) - g(\vec{x}, s) \right| Z \left(n\vec{x} - \vec{k} \right). \tag{81} \\
& \left\| \frac{\vec{k}}{n} - \vec{x} \right\|_{\infty} > \frac{1}{n^{\beta}}
\end{aligned}$$

Furthermore we derive

$$\begin{aligned}
& \left(\int_X \left| \left(D_n^{(\mathcal{R})}(g) \right) (\vec{x}, s) - g(\vec{x}, s) \right| P(ds) \right) \leq \\
& \sum_{\vec{k}=-\infty}^{\infty} \sum_{\vec{r}=0}^{\vec{\theta}} w_{\vec{r}} \left(\int_X \left| g \left(\frac{\vec{k}}{n} + \frac{\vec{r}}{n\vec{\theta}}, s \right) - g(\vec{x}, s) \right| P(ds) \right) Z \left(n\vec{x} - \vec{k} \right) \\
& \left\| \frac{\vec{k}}{n} - \vec{x} \right\|_{\infty} \leq \frac{1}{n^{\beta}} \tag{82}
\end{aligned}$$

$$\begin{aligned}
& + \left(\sum_{\substack{\vec{k}=-\infty \\ \left\| \frac{\vec{k}}{n} - \vec{x} \right\|_{\infty} > \frac{1}{n^{\beta}}}}^{\infty} Z \left(n \vec{x} - \vec{k} \right) \right) 2 \|g\|_{\infty, \mathbb{R}^N, X} \leq \\
& \Omega_1 \left(g, \frac{1}{n} + \frac{1}{n^{\beta}} \right)_{L^1} + 2c(\beta, n) \|g\|_{\infty, \mathbb{R}^N, X}, \quad (83)
\end{aligned}$$

proving the claim. ■

Conclusion 4. From Theorem 3.4 we obtain pointwise and uniform convergences with rates in the 1-mean of random operators $D_n^{(\mathcal{R})}$ to the unit operator for $g \in C_{\mathcal{R}}^{U_1}(\mathbb{R}^N)$.

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