ROMAI J., v.19, no.1(2023), 73–105

FLUID FLOW ON VEGETATED HILLSLOPE: A MATHEMATICAL MODEL

Stelian Ion, Dorin Marinescu, Ştefan-Gicu Cruceanu

"Gheorghe Mihoc-Caius Iacob" Institute of Mathematical Statistics and Applied Mathematics of Romanian Academy, Bucharest, Romania

stelian.ion@ismma.ro, marinescu.dorin@ismma.ro, stefan.cruceanu@ismma.ro

Abstract In this paper, we present a deduction of swallow water equations in the presence of vegetation based on spatial averaging techniques starting from the general principles of conservation of mass and momentum. For this purpose, we worked in the hydrostatic approximation of the pressure field and we considered certain hypotheses of kinematic and topographical nature and assumptions on the structure of the vegetation. Some elements of differential geometry necessary to facilitate the reading of the paper can be found in the Appendix.

Keywords: shallow water equation, numerical approximation.
2020 MSC: 35Q35, 35K55, 35L60, 76S99, 53Z05.

1. INTRODUCTION

The presence of plants on the hill creates a resistance force to the water flow and influences the process of water accumulation on the soil surface. The large diversity of plants growing on a hill makes the elaboration of an unitary model of the water flow over a soil covered by vegetation very difficult. Here, we present a model based on water mass and momentum balance equations that takes into account the presence of certain type of plants.

More precisely, the plants form a dense net of rigid vertical tubes and the water fills the "voided" space up to a level not higher than these plant tubes, see Figure 1. The figure 1 explaines the representative element of the volume P_δ used for mediation. The bottom surface of P_δ has a representative width δ along two orthogonal directions on this surface. The water depth h associated to P_δ is the averaged value of the physical water depth $\widetilde{h}$ inside P_δ.

The article is structured as follows. A full hyperbolic PDE model obtained by averaging the equations for the conservation of mass and momentum is presented in Section 2. Some closure relations for these balance equations can be found in

In the Section 3 we introduce some closure relations conceerning the water-plant and water-soil interaction and we anlyse some mathematical properties of the model. We note that, by suitable assumptions, different simplified models can be obtained from the general model.

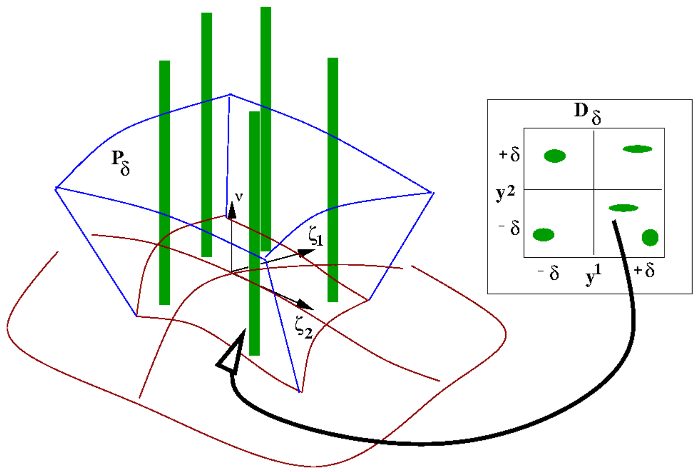


Fig. 1. The representative element of the volume used for mediation

The Appendix is dedicated to some elements of differential geometry used throughout the paper.

2. SPACE AVERAGING MODELS

Space averaging is a method to define a unique continuous model associated to a heterogeneous fluid-solid mechanical system. The method is largely used in porous soil media models [2, 5, 12]. For the fluid-plant physical system, the porous analogy was also used in [1, 6, 8], especially in the case of submerged vegetation.

At a hydrographic basin scale, there are variations in the geometrical properties of the terrain (curvature, orientation, slope) and vegetation density or vegetation type etc. Assume there is a map that models the terrain surface

$$x^i = b^i(\xi^1, \xi^2), \quad (\xi^1, \xi^2) \in D \subset \mathbb{R}^2, \quad i = 1, 2, 3. \tag{1}$$

Denote the tangent vectors to the coordinate curves on this surface by

$$\varsigma_a = \partial_a \boldsymbol{b} := \frac{\partial \boldsymbol{b}}{\partial \xi^a}, \quad a = 1, 2. \tag{2}$$

Using this fixed surface, one introduces a new coordinate y^3 along the normal direction $\boldsymbol{\nu}$ to the surface. A point in the neighborhood of this surface is defined in this new system of coordinates $Y = (\xi^1, \xi^2, y^3)$ by

$$x^i = b^i(\xi^1, \xi^2) + y^3 \nu^i, \quad (\xi^1, \xi^2) \in D \subset \mathbb{R}^2, \quad y^3 \in J \in \mathbb{R}, \quad i = 1, 2, 3, \tag{3}$$

where $\boldsymbol{\nu} = (\nu^1, \nu^2, \nu^3)$ represents the unit normal to the surface.

We introduce the tangent vectors to the coordinate curves defined by Y

$$\boldsymbol{\zeta}_I := \partial_I \boldsymbol{x}, \quad I = 1, 2, 3. \tag{4}$$

One has

$$\boldsymbol{\zeta}_3 = \boldsymbol{\nu}, \quad \boldsymbol{\zeta}_a = (\delta_a^b - y^3 \kappa_a^b)\boldsymbol{\varsigma}_b, \quad a = 1, 2, \tag{5}$$

where $\boldsymbol{\kappa}$ is the curvature tensor of the terrain surface.

In the presence of vegetation on the hill slope, the fluid occupies the free space between plant bodies and the mechanical characteristics of the fluid flow are defined only in the domain occupied by the fluid.

We adopt the following
General convention: *any variable bearing a tilde over it designates a micro-local physical quantity, while the absence of tilde indicates the corresponding averaged quantity. Also, when the micro-local quantity does not differ from the corresponding averaged quantity, we denote the micro-local quantity without tilde.*

Denote by Ω_f and Ω_p the spatial domain occupied by fluid and plants, respectively. Consider $\widetilde{\psi}$ to be some microscopic quantity that refers to the fluid. Let $\boldsymbol{y} = (y^1, y^2)$ be a point in D. One introduces the rectangular domain

$$D_\delta = D_\delta(\boldsymbol{y}) := [y^1 - \delta, y^1 + \delta] \times [y^2 - \delta, y^2 + \delta]. \tag{6}$$

Define the spatial averaging volume

$$P = P(\boldsymbol{y}) = \big\{(x^1, x^2, x^3) \,|\, x^i = b^i(\xi^1, \xi^2) + y^3 \nu^i, \\ 0 < y^3 < \bar{h}(\xi^1, \xi^2),\ (\xi^1, \xi^2) \in D_\delta(\boldsymbol{y}),\ i = 1, 2, 3\big\}.$$

Here, $\bar{h}$ is some extension of $\widetilde{h}$ to the domain D, where $\widetilde{h}$ is the function describing the free water surface outside the domain occupied by plants.

Denote by P^f the fluid domain inside P,

$$P^f := P \cap \Omega^f.$$

The boundary of P^f can be partitioned as

$$\partial P^f = \Sigma^{fp} \cap \Sigma^{ff} \cap \Sigma^{fa} \cap \Sigma^{fs},$$

where Σ^{fp} is the fluid-plant contact surface inside P^f, Σ^{fa} is the free surface of the fluid inside P^f, Σ^{fs} is the fluid-soil contact surface inside P^f, and Σ^{ff} is the boundary surface separating the fluid inside and outside P^f.

The general form of a balance equation, [7] is

$$\partial_t \int\limits_{P^f} \widetilde{\rho}\widetilde{\psi} \mathrm{d}V + \int\limits_{\partial P^f} \widetilde{\rho}\widetilde{\psi}(\widetilde{\boldsymbol{v}} \cdot \boldsymbol{n} - u_n)\mathrm{d}\sigma = \int\limits_{\partial P^f} \widetilde{\boldsymbol{\Phi}}_\psi \cdot \boldsymbol{n}\mathrm{d}\sigma + \int\limits_{P^f} \widetilde{\rho}\widetilde{\boldsymbol{\phi}}_\psi \mathrm{d}V. \tag{7}$$

Here, the significance of the above quantities are:
- $\widetilde{\rho}$ – the micro-local mass density of the fluid;
- $\widetilde{\boldsymbol{v}}$ – the micro-local velocity of the fluid;
- $\boldsymbol{n}$ – the exterior unit normal on ∂P^f;
- $\widetilde{\boldsymbol{\Phi}}_\psi$ – the micro-local flux density of $\widetilde{\psi}$;
- $\widetilde{\boldsymbol{\phi}}_\psi$ – the micro-local mass density of supply $\widetilde{\psi}$;
- u_n – the normal surface velocity;
- $\mathrm{d}V$ – the volume element;
- $\mathrm{d}\sigma$ – the surface element.

To obtain a mathematical treatable model, one needs to make some assumptions concerning the complex fluid-plant-soil system. The first assumption refers to the plant cover.

Assumption 2.1 (Vegetation structure)**.** *The plant cover satisfies:*
A1. *The plants are almost normal to the terrain surface and they behave like rigid sticks.*
A2. *The water depth is smaller than the height of the plants.*

We remark that A1 is often used in the porous model of the vegetation and A2 is proper to the overland flow.

The soil-fluid $\mathcal{I}_{fs}$ and fluid-air $\mathcal{I}_{fa}$ interfaces can be represented as

$$\mathcal{I}_{fs} := \{\boldsymbol{x} \,\Big|\, x^i = b^i(\xi^1, \xi^2), \quad (\xi^1, \xi^2) \in D^f,\ i = 1, 2, 3\}$$

and

$$\mathcal{I}_{fa} := \{\boldsymbol{x} \,\Big|\, x^i = b^i(\xi^1, \xi^2) + \widetilde{h}(\xi^1, \xi^2)\delta_3^i, \quad (\xi^1, \xi^2) \in D^f,\ i = 1, 2, 3\},$$

respectively, where $D^f := \big\{(\xi^1, \xi^2) \in D \,\big|\, \boldsymbol{b}(\xi^1, \xi^2) \in \Omega^f\big\}$.

Define the averaged water depth by

$$h(y^1, y^2, t) := \frac{1}{\omega_f} \int\limits_{D^f_\delta} \widetilde{h}(\xi^1, \xi^2, t)\beta(\xi^1, \xi^2)\mathrm{d}\xi^1\mathrm{d}\xi^2, \tag{8}$$

where ω_f measures the area of Σ^{fs},

$$\omega_f := \int\limits_{D^f_\delta} \beta(\xi^1, \xi^2)\mathrm{d}\xi^1\mathrm{d}\xi^2. \tag{9}$$

The volume of the fluid inside the elementary domain P is given by

$$\mathrm{vol}(P^f) = \omega_f h. \tag{10}$$

A pure geometrical result which refers to the flux of $\widetilde{\psi}$ through the boundary Σ^{ff} is formulated as:

Lemma 1.

$$\int\limits_{\Sigma^{ff}} \widetilde{\rho}\,\widetilde{\psi}\,\widetilde{\boldsymbol{v}}\cdot\boldsymbol{n}\mathrm{d}\sigma = \partial_a \int\limits_{D^f} \int\limits_{0}^{\widetilde{h}(\xi^1,\xi^2,t)} \widetilde{\rho}\,\widetilde{\psi}\,\widetilde{v}^a \Delta \mathrm{d}y^3 \beta(\xi^1,\xi^2)\mathrm{d}\xi^1\mathrm{d}\xi^2, \tag{11}$$

where $\Delta = 1 - y^3 K_M + (y^3)^2 K_G$, with K_M and K_G the mean and Gauss curvature respectively, and $\beta \mathrm{d}\xi \mathrm{d}\eta$ is the area element of the terrain surface. The quantities $\widetilde{v}^a$, with $a = 1, 2$ stand for the contravariant components of the velocity fields in the local basis $\{\boldsymbol{\zeta}_I\}_{I=\overline{1,3}}$

$$\widetilde{\boldsymbol{v}} = \widetilde{v}^a \boldsymbol{\zeta}_a + \widetilde{v}^3 \boldsymbol{\nu}.$$

In Lemma 1, the partial differentiation ∂_a stands for

$$\partial_a := \frac{\partial}{\partial y^a}.$$

2.1. AVERAGED MASS BALANCE EQUATION

Although the water density is considered to be a constant function, we keep it in the mass balance formulation for emphasizing the physical meaning of the equations. Define the averaged water flux by

$$\rho v^a(\boldsymbol{x}, t) := \frac{1}{\mathrm{vol}(P^f)} \int\limits_{D^f_\delta} \int\limits_{0}^{\widetilde{h}(\xi^1,\xi^2,t)} \widetilde{\rho}\,\widetilde{v}^a \Delta \mathrm{d}y^3 \beta \mathrm{d}\xi^1 \mathrm{d}\xi^2. \tag{12}$$

The mass balance equation results from (7) by taking $\widetilde{\psi} = 1$, $\widetilde{\boldsymbol{\Phi}}_\psi = 0$ and $\widetilde{\phi}_\psi = 0$. Since the plants are treated as solid bodies and the water does not penetrate the plant bodies, the water flux through the boundary of the elementary volume P^f reduces to

$$\int\limits_{\partial P^f} \widetilde{\rho}(\widetilde{\boldsymbol{v}}\cdot\boldsymbol{n} - u_n)\mathrm{d}\sigma = \int\limits_{\Sigma^{ff}} \widetilde{\rho}\,\widetilde{\boldsymbol{v}}\cdot\boldsymbol{n}\mathrm{d}\sigma + \int\limits_{\Sigma^{fa}} \widetilde{\rho}(\widetilde{\boldsymbol{v}}\cdot\boldsymbol{n} - u_n)\mathrm{d}\sigma + \int\limits_{\Sigma^{fs}} \widetilde{\rho}\,\widetilde{\boldsymbol{v}}\cdot\boldsymbol{n}\mathrm{d}\sigma.$$

The second integral in the r.h.s. of the above relation represents the water flux due to the rain which leads to the water mass gain inside P^f. The third term corresponds to the water flux due to the infiltration which contributes to the water loss inside P^f. Using Lemma 1 and the definition of the averaged quantities, one can write the mass balance:

$$\frac{\partial}{\partial t}(\omega_f h) + \partial_a(\omega_f h v^a) = \omega r - \omega_f i, \tag{13}$$

with

$$\int\limits_{\Sigma^{fa}} \widetilde{\rho}(\widetilde{\boldsymbol{v}} \cdot \boldsymbol{n} - u_n)\mathrm{d}\sigma = -\rho\omega r \quad \text{and} \quad \int\limits_{\Sigma^{fs}} \widetilde{\rho}\,\widetilde{\boldsymbol{v}} \cdot \boldsymbol{n}\mathrm{d}\sigma = \rho\omega_f i \tag{14}$$

representing the rain and the infiltration rates, respectively. Here, as in (9), ω is defined as

$$\omega := \int_{D_\delta} \beta(\xi^1, \xi^2)\mathrm{d}\xi^1\mathrm{d}\xi^2.$$

2.2. AVERAGED MOMENTUM BALANCE EQUATIONS

The momentum balance equation results from (7) with $\widetilde{\psi} = \widetilde{\boldsymbol{v}}$, $\widetilde{\boldsymbol{\Phi}}_\psi = \widetilde{\boldsymbol{T}}$, where $\widetilde{\boldsymbol{T}}$ is the stress tensor and $\widetilde{\phi}_\psi = \widetilde{\boldsymbol{f}}$, with $\widetilde{\boldsymbol{f}}$ denoting the body forces. Here, we only consider the gravitational force.

In contrast to the planar case, there are some difficulties in writing componentwise the space averaging balance momentum equations. These difficulties appear due to the point dependence of the local basis. In the euclidean basis of X, the momentum of the elementary volume P^f is given by

$$\mathcal{H}^i(P^f) = \int\limits_{P^f} \widetilde{\rho}\,\widetilde{v}^i\mathrm{d}V.$$

Using the components of $\widetilde{\boldsymbol{v}}$ in the basis of Y coordinates, we obtain

$$\mathcal{H}^i(P^f) = \int\limits_{\Sigma^{fs}} \int\limits_0^{\widetilde{h}} \widetilde{\rho}\,\zeta^i_a\,\widetilde{v}^a \Delta\mathrm{d}y^3\mathrm{d}\sigma + \int\limits_{\Sigma^{fs}} \int\limits_0^{\widetilde{h}} \widetilde{\rho}\,\nu^i\,\widetilde{v}^3 \Delta\mathrm{d}y^3\mathrm{d}\sigma, \tag{15}$$

which can be rewritten as

$$\mathcal{H}^i(P^f) = \varsigma^i_a \int\limits_{\Sigma^{fs}} \int\limits_0^{\widetilde{h}} \widetilde{\rho}\,\widetilde{v}^a \Delta\mathrm{d}y^3\mathrm{d}\sigma + \nu^i \int\limits_{\Sigma^{fs}} \int\limits_0^{\widetilde{h}} \widetilde{\rho}\,\widetilde{v}^3 \Delta\mathrm{d}y^3\mathrm{d}\sigma + \mathcal{E}^i_1(\widetilde{\boldsymbol{v}}, P^f). \tag{16}$$

Here and in what follows, we make the following convention: $\boldsymbol{\varsigma}_a = \boldsymbol{\varsigma}_a(\boldsymbol{y})$, where $\boldsymbol{y} = (y^1, y^2)$ is the point defining the domain $D_\delta(\boldsymbol{y})$ from (6). When it appears inside the integral, the unit normal $\boldsymbol{\nu}$ is a variable quantity depending on the current point from the domain D_δ, but when it appears outside the integral, it is the unit normal defined by the same $\boldsymbol{y}$ as ς_a.

The term

$$\mathcal{E}_1^i(\widetilde{\boldsymbol{v}}, P^f) := \int\limits_{\Sigma^{fs}} \int\limits_0^{\widetilde{h}} \widetilde{\rho}(\zeta_a^i - \varsigma_a^i)\widetilde{v}^a \Delta \mathrm{d}y^3 \mathrm{d}\sigma$$

represents an error introduced by neglecting the variation of the basis $\boldsymbol{\zeta}_I$ along the domain P^f.

By averaging, from (16) one has

$$\mathcal{H}(P^f) = \rho h \omega_f v^a \boldsymbol{\varsigma}_a + \rho h \omega_f v^3 \boldsymbol{\nu} + \mathcal{E}_1(\widetilde{\boldsymbol{v}}, P^f). \tag{17}$$

If one neglects the momentum transfer on the fluid-air and fluid-soil interfaces, then the flux of the momentum through the boundary ∂P^f can be reduced to

$$\mathcal{F}(\widetilde{\rho}\,\widetilde{\boldsymbol{v}}, \partial P^f) := \int\limits_{\partial P^f} \widetilde{\rho}\,\widetilde{\boldsymbol{v}}(\widetilde{\boldsymbol{v}} \cdot \boldsymbol{n} - u_n)\mathrm{d}\sigma = \int\limits_{\Sigma^{ff}} \widetilde{\rho}\,\widetilde{\boldsymbol{v}}(\widetilde{\boldsymbol{v}} \cdot \boldsymbol{n})\mathrm{d}\sigma.$$

Using Lemma 1, one has

$$\mathcal{F}(\widetilde{\rho}\,\widetilde{\boldsymbol{v}}, \partial P^f) = \partial_a \int\limits_{D^f} \int\limits_0^{\widetilde{h}(\xi^1,\xi^2,t)} \widetilde{\rho}\,\widetilde{\boldsymbol{v}}\,\widetilde{v}^a \Delta \mathrm{d}y^3 \beta(\xi^1,\xi^2)\mathrm{d}\xi^1\mathrm{d}\xi^2,$$

and then,

$$\begin{aligned}\mathcal{F}(\widetilde{\rho}\,\widetilde{\boldsymbol{v}}, \partial P^f) = &\\ &\partial_a(\rho\omega_f h v^b v^a \boldsymbol{\varsigma}_b) + \partial_a(\rho\omega_f h w^{ba}\boldsymbol{\varsigma}_b) + \partial_a(\rho\omega_f h v^3 v^a \boldsymbol{\nu}) + \\ &\mathcal{E}_2(\widetilde{v}^2, P^f),\end{aligned} \tag{18}$$

where the fluctuation

$$\rho w^{ab} := \frac{1}{\omega_f h}\int_{\Sigma^f}\int_0^{\widetilde{h}(\xi^1,\xi^2,t)} \widetilde{\rho}(\widetilde{v}^b - v^b)\widetilde{v}^a y^3 \beta(\xi^1,\xi^2)\mathrm{d}\xi^1\mathrm{d}\xi^2.$$

The quantity $\mathcal{E}_2(\widetilde{v}^2, P^f)$ (as $\mathcal{E}_1(\widetilde{v}, P^f)$ appearing above), represents the error introduced by approximating the variable local basis $(\boldsymbol{\zeta}_1(\xi^1,\xi^2,y^3), \boldsymbol{\zeta}_2(\xi^1,\xi^2,y^3), \boldsymbol{\nu}(\xi^1,\xi^2,0))$ with the fixed local basis $(\boldsymbol{\varsigma}_1, \boldsymbol{\varsigma}_2, \boldsymbol{\nu})$ at $(y^1, y^2, 0)$. The quantities $\mathcal{E}_3$, $\mathcal{E}_4$ and $\mathcal{E}_5$ introduced in what follows are errors of the same nature.

Rel. (18) can be rewritten as

$$
\begin{aligned}
&\mathcal{F}(\widetilde{\rho}\,\widetilde{\boldsymbol{v}}, \partial P^f) = \\
=&\partial_a(\rho\omega_f h v^b v^a)\boldsymbol{\varsigma}_b + \rho\omega_f h v^b v^a \partial_a \boldsymbol{\varsigma}_b + \partial_a(\rho\omega_f h w^{ba})\boldsymbol{\varsigma}_b + \rho\omega_f h w^{ba}\partial_a\boldsymbol{\varsigma}_b + \\
&\partial_a(\rho\omega_f h v^3 v^a)\boldsymbol{\nu} + \rho\omega_f h v^3 v^a \partial_a\boldsymbol{\nu} + \mathcal{E}_2(\widetilde{v}^2, P^f) \\
=&\partial_a(\rho\omega_f h v^b v^a)\boldsymbol{\varsigma}_b + \rho\omega_f (h v^b v^a + w^{ba})(\gamma^c_{ab}\boldsymbol{\varsigma}_{\boldsymbol{c}} + \kappa_{ab}\boldsymbol{\nu}) + \\
&\partial_a(\rho\omega_f h w^{ba})\boldsymbol{\varsigma}_b + \partial_a(\rho\omega_f h v^3 v^a)\boldsymbol{\nu} - \rho\omega_f h v^3 v^a \kappa^b_a \boldsymbol{\varsigma}_b + \mathcal{E}_2(\widetilde{v}^2, P^f) \\
=&\partial_a(\rho\omega_f h(v^b v^a + w^{ba}))\boldsymbol{\varsigma}_b - \rho\omega_f h v^3 v^a \kappa^b_a\boldsymbol{\varsigma}_b + \rho\omega_f(h v^b v^a + w^{ba})\gamma^c_{ab}\boldsymbol{\varsigma}_{\boldsymbol{c}} + \\
&\rho\omega_f(h v^b v^a + w^{ba})\kappa_{ab}\boldsymbol{\nu} + \partial_a(\rho\omega_f h v^3 v^a)\boldsymbol{\nu} + \mathcal{E}_2(\widetilde{v}^2, P^f),
\end{aligned}
\tag{19}
$$

where γ^c_{ab} are the Christoffel symbols.

To express the contribution of the stress forces to the momentum balance, we decompose the stress tensor field $\widetilde{\boldsymbol{T}}$ in two components: the pressure field $\widetilde{p}$ and the viscous part of the stress tensor field $\widetilde{\boldsymbol{\tau}}$

$$\widetilde{\boldsymbol{T}} = -\widetilde{p}\boldsymbol{I} + \widetilde{\boldsymbol{\tau}}.$$

The flux of the stress vector can now be written as

$$\mathcal{F}(\widetilde{\boldsymbol{T}}, \partial P_f) = \mathcal{F}(-p\boldsymbol{I}, \partial P_f) + \mathcal{F}(\widetilde{\boldsymbol{\tau}}, \partial P_f).$$

An elementary calculation show that

$$\mathcal{F}(-p\boldsymbol{I}, \partial P_f) = -\int\limits_{D^f}\int\limits_0^{\widetilde{h}(\xi^1,\xi^2,t)} \left(\partial_a p g^{ab}\boldsymbol{\zeta}_b + \partial_3 p\boldsymbol{\nu}\right)\Delta \mathrm{d}y^3\beta\mathrm{d}\xi^1\mathrm{d}\xi^2 \tag{20}$$

The pressure field is determined up to a constant value. If we subtract the atmospheric pressure from the water pressure, on the interface fluid-air the pressure must be zero. We assume the pressure field to be hydrostatically distributed.

Let $\boldsymbol{g} = -g\boldsymbol{i}_3$ be the gravitational force acting on the mass unit. In the local frame of coordinates related to the free surface of the fluid, this force has the representation

$$\boldsymbol{g} = \widetilde{f}^a\boldsymbol{\zeta}_a - \widetilde{f}^3\boldsymbol{\nu}.$$

Assumption 2.2 (Hydrostatic approximation)**.** *One assumes that*
A3. *The hydrostatic pressure field has the form*

$$\widetilde{p}(\xi^1, \xi^2, y^3) = \widetilde{\rho}\,\widetilde{f}^3(\widetilde{h}(\xi^1, \xi^2) - y^3).$$

We neglect the shear forces on the fluid-air interface, i.e.

$$\mathcal{F}(\widetilde{\boldsymbol{\tau}}, \Sigma^{fa}) = 0.$$

On the fluid-soil interface, the stress vector $\widetilde{\boldsymbol{t}} := \widetilde{\boldsymbol{\tau}} \cdot \boldsymbol{n}$ can be written as

$$\widetilde{\boldsymbol{t}} = \widetilde{t}^a \boldsymbol{\zeta}_a + \widetilde{t}^3 \boldsymbol{\nu}.$$

On the soil-water interface, we can write

$$\mathcal{F}(\widetilde{\boldsymbol{\tau}}, \Sigma^{fs}) = \boldsymbol{\varsigma}_a \int\limits_{\Sigma^{fs}} \widetilde{t}^a \mathrm{d}\sigma + \boldsymbol{\nu} \int\limits_{\Sigma^{fs}} \widetilde{t}^3 \mathrm{d}\sigma + \mathcal{E}_3(\widetilde{\boldsymbol{T}}, \Sigma^{fs}). \tag{21}$$

Introducing the shear force at the fluid-soil interface

$$\sigma_s^a = \frac{1}{\rho \omega_f} \int\limits_{\Sigma^{fs}} \widetilde{t}^a \mathrm{d}\sigma,$$

relation (21) takes the form

$$\mathcal{F}(\widetilde{\boldsymbol{\tau}}, \Sigma^{fs}) = \boldsymbol{\varsigma}_a \rho \omega_f \sigma_s^a + \boldsymbol{\nu} \int\limits_{\Sigma^{fs}} \widetilde{t}^3 \mathrm{d}\sigma + \mathcal{E}_3(\widetilde{\boldsymbol{T}}, \Sigma^{fs}). \tag{22}$$

On the fluid-plant interface

$$\mathcal{F}(\widetilde{\boldsymbol{\tau}}, \Sigma^{fp}) = \int\limits_{\Sigma^{fp}} \widetilde{\boldsymbol{\tau}} \cdot \boldsymbol{n} \mathrm{d}\sigma = \sum_l \int\limits_{\Sigma_l^{fp}} \widetilde{\boldsymbol{\tau}} \cdot \boldsymbol{n} \mathrm{d}\sigma, \tag{23}$$

where Σ_l^{fp} is the fluid-plant surface corresponding to the plant l. Obviously, $\bigcup\limits_l \Sigma_l^{fp} = \Sigma^{fp}$. Since the plant stems are supposed to be perpendicular to the ground surface, (23) becomes

$$\mathcal{F}(\widetilde{\boldsymbol{\tau}}, \Sigma^{fp}) = \boldsymbol{\varsigma}_a \sum_l \int\limits_{\Sigma_l^{fp}} \widetilde{t}^a \mathrm{d}\sigma + \mathcal{E}_4(\widetilde{\boldsymbol{\tau}}, \Sigma^{fp}) \tag{24}$$

and introducing the plant resistance force

$$\sigma_p^a = \frac{1}{\rho \omega} \sum_l \int\limits_{\Sigma_l^{fp}} \widetilde{t}^a \mathrm{d}\sigma,$$

relation (24) becomes

$$\mathcal{F}(\widetilde{\boldsymbol{\tau}}, \Sigma^{fp}) = \boldsymbol{\varsigma}_a \rho \omega \sigma_p^a + \mathcal{E}_4(\widetilde{\boldsymbol{\tau}}, \Sigma^{fp}). \tag{25}$$

On the fluid interface of P^f, invoking again Lemma 1, the contribution of the viscous part of the stress tensor on the interface fluid-fluid takes the form

$$\mathcal{F}(\widetilde{\boldsymbol{\tau}}, \Sigma^{ff}) = \partial_a \int\limits_{\Sigma^{fs}} \int\limits_0^{\widetilde{h}} \widetilde{\tau}^{ba} \boldsymbol{\zeta}_b \Delta \mathrm{d}y^3 \mathrm{d}\sigma + \partial_a \int\limits_{\Sigma^{fs}} \int\limits_0^{\widetilde{h}} \widetilde{\tau}^{3a} \boldsymbol{\nu} \Delta \mathrm{d}y^3 \mathrm{d}\sigma.$$

Then, we write the above quantity as,

$$\mathcal{F}(\widetilde{\boldsymbol{\tau}}, \Sigma^{ff}) = \partial_a(\omega_f h \tau^{ba} \boldsymbol{\varsigma}_b) + \partial_a(\omega_f h \tau^{3a} \boldsymbol{\nu}) + \mathcal{E}_5(\widetilde{\boldsymbol{\tau}}_v, P^f). \tag{26}$$

Rel. (26) implies that

$$\begin{aligned}
\mathcal{F}(\widetilde{\boldsymbol{\tau}}, \Sigma^{ff}) = & \\
&= \partial_a(\omega_f h \tau^{ba}) \boldsymbol{\varsigma}_b + \omega_f h \tau^{ba} \partial_a \boldsymbol{\varsigma}_b + \partial_a(\omega_f h \tau^{3a}) \boldsymbol{\nu} + \omega_f h \tau^{3a} \partial_a \boldsymbol{\nu} \\
&+ \mathcal{E}_5(\widetilde{\boldsymbol{\tau}}_v, P^f) \\
&= \partial_a(\omega_f h \tau^{ba}) \boldsymbol{\varsigma}_b + \omega_f h \tau^{ba} (\gamma_{ab}^c \boldsymbol{\varsigma}_c + \kappa_{ab} \boldsymbol{\nu}) + \partial_a(\omega_f h \tau^{3a}) \boldsymbol{\nu} \\
&- \omega_f h \tau^{3a} \kappa_a^b \boldsymbol{\varsigma}_b + \mathcal{E}_5(\widetilde{\boldsymbol{\tau}}_v, P^f) \\
&= \partial_a(\omega_f h \tau^{ba}) \boldsymbol{\varsigma}_b - \omega_f h \tau^{3a} \kappa_a^b \boldsymbol{\varsigma}_b + \omega_f h \tau^{ba} \gamma_{ab}^c \boldsymbol{\varsigma}_c + \omega_f h \tau^{ba} \kappa_{ab} \boldsymbol{\nu} \\
&+ \partial_a(\omega_f h \tau^{3a}) \boldsymbol{\nu} + \mathcal{E}_5(\widetilde{\boldsymbol{\tau}}_v, P^f).
\end{aligned} \tag{27}$$

For the supply $\widetilde{\Phi}_\psi$, we only consider the contribution of the gravitational force. Proceeding by components as in (16), the second term in the r.h.s. of (7) is finally expressed as

$$\int\limits_{P^f} \widetilde{\rho} \widetilde{\boldsymbol{\phi}}_\psi \mathrm{d}V = \int\limits_{D^f} \int\limits_0^{\widetilde{h}(\xi^1, \xi^2, t)} \left(\widetilde{f}^a \boldsymbol{\zeta}_a - \widetilde{f}^3 \boldsymbol{\nu} \right) \Delta \mathrm{d}y^3 \beta \mathrm{d}\xi^1 \mathrm{d}\xi^2 \tag{28}$$

The relations (17, 19, 20, 22, 25, 27) and some order assumptions are the basis for averaged momentum equations.

The porosity θ of the plant cover is defined by

$$\theta = \frac{\omega_f}{\omega}.$$

Let $\beta_0 = \beta(y_1, y_2)$, where $\boldsymbol{y} = (y^1, y^2)$ is the point defining the domain $D_\delta(\boldsymbol{y})$ from (6).

Let ϵ be a small parameter.

Assumption 2.3 (Kinematical and topographical assumptions)**.** *Suppose that the physical processes satisfy the following properties:*

A4. The water depth. $\widetilde{h} = O(\epsilon)$.
A5. The velocity. $v^3 = O(\epsilon)$.
A6. Geometric assumptions:
A6.1. Curvature. *The terrain surface curvatures and the curvature of the coordinate curves are of order of ϵ. This means that locally the surface is almost planar.*
A6.2. Metric tensor. $\beta = \beta_0 + O(\epsilon)$.
A7. The averaged dimension δ. $d_p \ll \delta \ll L$ *and* $\delta K_M = O(\epsilon)$.

In what follows, by abuse of notations, we denote β_0 by β.

The shallow water type approximation of the averaged momentum balance for an incompressible fluid results by an asymptotic analysis.

Theorem 2.1 (Averaged momentum equations)**.** *Under assumptions* A1–A7*, the first order approximation for the momentum equations is given by*

$$\partial_t(h\beta\theta v^a) + \partial_b \mathfrak{F}^{ab}(h, \boldsymbol{v}) + h\beta\theta\beta^{ab}\partial_a w = \mathfrak{G}^a(h, \boldsymbol{v}), \quad a = 1, 2, \tag{29}$$

where

$$w = g(b^3 + h\nu^3), \quad (g - \textit{the gravitational acceleration})$$

$$\mathfrak{F}^{ab}(h, \boldsymbol{v}) = h\beta\theta\left(v^a v^b + w^{ab} - \frac{1}{\rho}\tau^{ab}\right),$$

$$\mathfrak{G}^a(h, \boldsymbol{v}) = \beta\theta\sigma_p^a + \beta\theta\sigma_s^a - \gamma_{bc}^a\eta^{bc}$$

and

$$\eta^{ac} = h\beta\theta\left(v^a v^b + w^{ab} - \frac{1}{\rho}\tau^{ab}\right).$$

Sketch of proof. Using Assumption 2.3 and relations (17, 19, 22, 25, 27) one can prove that the terms $\mathcal{E}_1, \ldots, \mathcal{E}_5$ are of order ϵ^2. For $\epsilon \ll 1$ these terms as well as the terms containing the factors $v^3 h$, $h\kappa$ or h^2 (which are of same order ϵ^2) can be neglected.

The equations (29) must be supplemented by empirical laws concerning the *averaged stress tensor* $\boldsymbol{\tau}$, the *averaged vegetation force resistance* $\boldsymbol{\sigma}_p$, the *averaged shear fluid-soil force* $\boldsymbol{\sigma}_s$ and the *averaged fluctuation* w^{ab}. These empirical laws are expressed by functions depending on the averaged velocity $\boldsymbol{v}$, the averaged water depth h and a set of parameters $\boldsymbol{\lambda}$ defined by the characteristics of the plant cover.

$$\begin{cases} \tau^{ab} = \mathfrak{T}^{ab}(\nabla \boldsymbol{v}, h, \boldsymbol{\lambda}), \\ \sigma_p^b = \mathfrak{S}_p^b(\boldsymbol{v}, h, \boldsymbol{\lambda}), \\ \sigma_s^b = \mathfrak{S}_s^b(\boldsymbol{v}, h, \boldsymbol{\lambda}), \\ w^{ab} = \mathfrak{W}^{ab}(\boldsymbol{v}, h, \boldsymbol{\lambda}). \end{cases} \tag{30}$$

3. SHALOW WATER EQUATIONS WITH VEGETATION (SWE-VEG) MODELS

The averaged models of water flow on a vegetated hillslope consists of mass balance equation (13), momentum balance equations (29) and a set of empirical relations (30). The empirical relations are generaly obtained by experiments or in situ mesurements of hydrodinamic variables.

The models we will present here quantify the interactions water-plant, σ_p^b and water-soil, σ_p^b. One assumes that the viscosity of fluid and the fluctuation of the velocity field have a small effect as compared with the bed friction and plant resistance. We set

$$\boldsymbol{\tau} = 0, \quad \boldsymbol{w} = 0$$

The averaged vegetation force resistance

The most used empirical relations that relate the vegetation resistance and fluid velocity have the form [8, 1]

$$\sigma_p^a = -\frac{1}{2} C_d m h d |\boldsymbol{v}| v^a, \tag{31}$$

where m is the number of stems on the surface ω and d is the averaged diameters of the stems.

The bed shear stress

One uses the experimental relations of Manning or Ch´ezy, or the Darcy–Weisbach formula:

$$\sigma_b^a = -\frac{g}{C_b^2} |\boldsymbol{v}| v^a, \tag{32}$$

$|\boldsymbol{v}|$ being the magnitude of the averaged velocity *i.e.*

$$|\boldsymbol{v}|^2 = \beta_{ab} v^a v^b.$$

Generally, C_b depends on h, see [11], [13] Therefore the base model is given by

$$\begin{aligned} \frac{\partial}{\partial t}\left(h\beta\theta\right)+\partial_a\left(h\beta\theta v^a\right) =& \beta(\mathfrak{m}_r-\theta\mathfrak{m}_i), \\ \frac{\partial}{\partial t}h\theta\beta v^c+\frac{\partial}{\partial y^a}\theta\beta\, hv^c v^a+h\theta\beta\gamma^c_{ab}v^a v^b+h\beta\theta\beta^{ca}\partial_a w =& -\beta\mathcal{K}(h,\theta)|\boldsymbol{v}|v^c. \end{aligned} \tag{33}$$

The parameter function $\mathcal{K}(h,\theta)$ is given by

$$\mathcal{K}(h,\theta)=\frac{1}{2}C_d m(\boldsymbol{y})hd+\frac{g\theta}{C_b^2}$$

here m stands for the density number of the stems on surface area. In our model, the porosity θ and the density number m are related by

$$\theta=1-m\frac{\pi d^2}{4}.$$

such that one can write

$$\mathcal{K}(h,\theta)=\alpha_p h(1-\theta)+\alpha_s\theta,$$

where the new parameters are given by

$$\alpha_p=\frac{2C_d}{\pi d},\quad \alpha_s=\frac{g}{C_b^2}.$$

Note that the system equations modeling the water flow on an unvegetated hill can be obtained from the model (33) by simply considering the porosity $\theta=1$.

The full PDE model for the water flow on vegetated hill is given by (33). The system is hyperbolic with source terms and there is an energy function that is a conserved quantity in the absence of plants and water-soil friction. Also, the model preserves the steady state of the lake.

Proposition 2. *The model* (33) *is of hyperbolic type with source terms.*
(a) *The conservative form of the system is given by*

$$\partial_t\mathcal{H}^i(\boldsymbol{y},t,\boldsymbol{u})+\partial_a\mathcal{F}^{ia}(\boldsymbol{y},t,\boldsymbol{u})=\mathcal{P}^i(\boldsymbol{y},t,\boldsymbol{u}), \tag{34}$$

where

$$\boldsymbol{u}=\begin{pmatrix} h\\ v^1\\ v^2\end{pmatrix},\quad \mathcal{H}(\boldsymbol{y},t,\boldsymbol{u})=\begin{pmatrix}\beta\theta h\\ \beta\theta h v^1\\ \beta\theta h v^2\end{pmatrix},$$

$$\mathcal{F}(\boldsymbol{y},t,\boldsymbol{u})=\begin{pmatrix}\beta\theta h v^1 & \beta\theta h v^2\\ \beta\theta(hv^1v^1+g\nu^3\beta^{11}h^2/2) & \beta\theta(hv^1v^2+g\nu^3\beta^{12}h^2/2)\\ \beta\theta(hv^2v^1+g\nu^3\beta^{21}h^2/2) & \beta\theta(hv^2v^2+g\nu^3\beta^{22}h^2/2)\end{pmatrix},$$

and

$\mathcal{P}(\boldsymbol{y}, t, \boldsymbol{u}) =$

$$\begin{pmatrix} \beta(\mathfrak{m}_r - \theta(y)\mathfrak{m}_i) \\ -\beta\theta h\gamma^1_{ab}v^a v^b - gh\left[\beta\theta\beta^{1a}\left(\partial_a x^3 + \frac{h}{2}\partial_a\nu^3\right) - \frac{h}{2}\nu^3\partial_a\beta\theta\beta^{1a}\right] - \beta\mathcal{K}|v|v^1 \\ -\beta\theta h\gamma^1_{ab}v^a v^b - gh\left[\beta\theta\beta^{2a}\left(\partial_a x^3 + \frac{h}{2}\partial_a\nu^3\right) - \frac{h}{2}\nu^3\partial_a\beta\theta\beta^{2a}\right] - \beta\mathcal{K}|v|v^2 \end{pmatrix}.$$

(b) *For any unitary vector $\boldsymbol{n} \in \mathbb{R}^3$, the eigenvalue problem [17]*

$$\left(\frac{\partial}{\partial u^i}\mathcal{F}^{ja}n_a - \lambda\frac{\partial}{\partial u^i}\mathcal{H}^j\right) r^i = 0 \tag{35}$$

has three solutions:

$$\lambda_- = v^a n_a - \sqrt{g\nu^3 h}, \quad \lambda_0 = v^a n_a, \quad \lambda_+ = v^a n_a + \sqrt{g\nu^3 h}. \tag{36}$$

Proof. In order to prove the existence of the solution for (35), it is sufficient to show that

$$\frac{\partial}{\partial u^i}\mathcal{F}^{ja}n_a - \lambda\frac{\partial}{\partial u^i}\mathcal{H}^j = \beta\theta\begin{pmatrix} \delta & hn_1 & hn_2 \\ v^1\delta + g\nu^3 h\beta^{1a}n_a & h\delta + hv^1 n_1 & hv^1 n_2 \\ v^2\delta + g\nu^3 h\beta^{2a}n_a & hv^2 n_1 & h\delta + hv^2 n_2 \end{pmatrix},$$

where $\delta = v^a n_a - \lambda$. The solutions (36) results then from straightforward calculations.

Proposition 3. *The following properties hold for system* (33)*:*

(a) it preserves the steady state of a lake

$$x^3 + h\nu^3 = \text{constant};$$

(b) there is a conservative equation for the energy

$$\frac{\partial}{\partial t}h\beta\theta\mathcal{E} + \frac{\partial}{\partial y^a}h\beta\theta v^a\left(\mathcal{E} + g\frac{h}{2}\nu^3\right) = \beta\left(\left(\mathfrak{M}(-\frac{1}{2}|\boldsymbol{v}|^2 + w\right) - \mathcal{K}|\boldsymbol{v}|^3\right), \tag{37}$$

where

$$\mathcal{E} := \frac{1}{2}|\boldsymbol{v}|^2 + g(x^3 + \frac{h}{2}\nu^3), \quad \mathfrak{M} = \mathfrak{m}_r - \theta\mathfrak{m}_i$$

(c) Bernoulli's law. At a steady state, in the absence of mass source and friction force, the total energy

$$\mathcal{E}^{\mathrm{t}} = \frac{1}{2}|\boldsymbol{v}|^2 + gx^3 + p(\boldsymbol{y}, h)$$

is constant along a current line

$$v^a\partial_a\mathcal{E}^{\mathrm{t}} = 0. \tag{38}$$

3.1. FLOW ON ALMOST LOCAL FLAT SURFACE

The model equations (33) is a too complicate mathematical for many practical applications. It is a good base model to generate simplified models of certain realistics problem. A simplified version of the full model correspond to a given soil surface topograhy and a given structure of the plant cover. In the sequell we introduce a simplified variant of the full model that yet allows variation in the soil topography and plant porosity.

Let the soil surface be given by

$$x^1 = y^1, x^2 = y^2, x^3 = z(y^1, y^2), \boldsymbol{y} \in D \subset \mathbb{R}^2 \tag{39}$$

We denote the euclidian norm of gradient of surface by

$$|\nabla z|^2 = (\partial_1 z)^2 + (\partial_2 z)^2$$

The geometrical characteristics of the surface can be written as(see the Anexa):

$$\begin{aligned} &\beta_{ab} = \delta_{ab} + \partial_a z \partial_b z, \beta = \sqrt{1 + |\nabla z|^2} \\ &\nu^a = \frac{-\partial_a z}{\beta}, \nu^3 = \frac{1}{\beta} \\ &\gamma^c_{ab} = \frac{\partial_c z \partial^2_{ab} z}{\beta^2}, \kappa^a_b = \frac{\beta^{bc} \partial^2_{cb} z}{\beta} \end{aligned} \tag{40}$$

An almost local flat surface is one characterized by:

$$\partial^2_{ab} z \approx 0, a, b = 1, 2.$$

For such surface one assumes that:

$$\beta = \text{constant}, \gamma^a_{bc} = 0, \kappa^a_b = 0, a, b, c = 1, 2$$

One these ground the equations (33) can be approximate as:

$$\begin{aligned} \frac{\partial}{\partial t} \theta h + \partial_a (\theta h v^a) &= \mathfrak{M}, \\ \frac{\partial}{\partial t} \theta h v_a + \partial_b \theta h v_a v^b + \theta h \partial_a w &= -\mathcal{K}(h, \theta) |v| v_a. \end{aligned} \tag{41}$$

where

$$\mathcal{K}(h, \theta) = \alpha_p h (1 - \theta) + \theta \alpha_s, \mathfrak{M} = \mathfrak{m}_r - \mathfrak{m}_i \theta, \tag{42}$$

Note that the water depth is measured along the normal direction to the base flow surface in the case of slity inclined surface the vertical components of the unitary normal to the surface can approximat by, $\nu^3 = 1$ so that the potential of free water surface is given by

$$w = g(z(y^1, y^2) + h). \tag{43}$$

The model (41, 43) is most used model in the practical applications. It preserve the main properties of the full model.

Proposition 4. *The reduce model* (39) *equations of the water flow on vegetated hill is of the hyperbolic type with source terms.*
(a) *The conservative form of it is given by*

$$\begin{aligned} \frac{\partial}{\partial t}\theta h + \partial_a\left(\theta h v^a\right) &= \mathfrak{M}, \\ \frac{\partial}{\partial t}\theta h v_a + \partial_b\left(\theta h v_a v^b + \delta_a^b \theta g \frac{h^2}{2}\right) &= -hg\partial_a z - g\tfrac{h^2}{2}\partial_a\theta - \mathcal{K}(h,\theta)|v|v_a. \end{aligned} \tag{44}$$

(b) *For any unitary vectors $\boldsymbol{n} \in \mathbb{R}^2$ the eginvalues are given by*

$$\lambda_- = v^a n_a - \sqrt{gh}, \lambda_0 = v^a n_a, \lambda_+ = v^a n_a + \sqrt{gh}. \tag{45}$$

Proposition 5. *The system (39) has the properties:*
(a) *it preserve the steady state of a lake*

$$x^3 + h = \text{constant},$$

(b) *there exists a conservative form equation of the energy disipation*

$$\frac{\partial}{\partial t}\theta h\mathcal{E} + \frac{\partial}{\partial y^a}\theta h v^a\left(\mathcal{E} + g_{\text{gravit}}\frac{h}{2}\right) = \left(\left(\mathfrak{M}(-\frac{1}{2}|v|^2 + w\right) - \mathcal{K}|v|^3\right), \tag{46}$$

where

$$\mathcal{E} := \frac{1}{2}|v|^2 + g(x^3 + \frac{h}{2})$$

(c) *Bernoulli law. In a steady state in the absence of the mass source and without friction force the total energy, i.e*

$$\mathcal{E}^{\mathrm{t}} = \frac{1}{2}|v|^2 + gx^3 + p(y,h)$$

is constant along of a current line

$$v^a\partial_a\mathcal{E}^{\mathrm{t}} = 0. \tag{47}$$

The presence of the plants and the exitence of the frictional ineraction between water and soil induce and energetic lost. To put in evidence such phenomenon let us consider a domain Ω whos and let $\boldsymbol{n}$ be the normal unitary to the $\partial\Omega$ outward orientated. One assume that the $\partial\Omega$ consists in an impermeables portion and an exit portion $\partial\Omega = \Gamma_1 \cup \Gamma_2$ $\boldsymbol{n}\cdot\boldsymbol{v} = 0$ on Γ_1 and $\boldsymbol{n}\cdot\boldsymbol{v} > 0$ on the Γ_2, one of the two portion can be a void set.

Proposition 6 (Energy disipation)**.** *Assume that there is no mass production. Then the energy of* Ω *is a deacreasing function whith respect to time*

$$\partial_t \int\limits_{\Omega} h\beta\theta\mathcal{E}\mathrm{d}x < 0 \tag{48}$$

To prove the assertion one integrates the energy dissipation equation (46)

$$\partial_t \int\limits_{\Omega} h\beta\theta\mathcal{E}\mathrm{d}x + \int\limits_{\partial\Omega} h\beta\theta\boldsymbol{v}\cdot\boldsymbol{n}\mathcal{E}^t\mathrm{d}s = -\int\limits_{\Omega} \mathcal{K}|v|^3\mathrm{d}x$$

and and one observes that the second integrals in the left hand side is a positive quantity.

4. APPLICATIONS

We will presents three applications of the SWE-Veg model given by 41, 43. The first application deals with the Riemann problem and the next two applications reffer to the ability of the model to accuratelly predict the real phenomena.

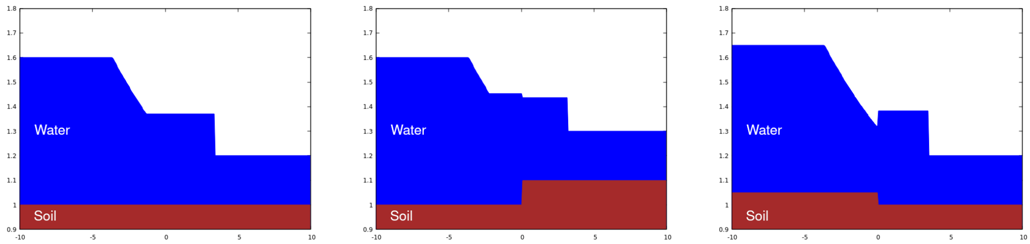
Riemann problem. The Riemann Problem is a central topic in the theory of the hyperbolic systems, [14], [15], [16], . When solvable, the solution of it consists in a superposition of shock and rarefaction waves. This very special solutions can be used to define a class of numerical schemes, Riemann solver: [18], [19], [22], [24].

In the case of the SWE-Veg model a shock wave solution is defined as a measure solution that satisfies certain requirements, [20], [23], [9]. The Riemann solver is still composed by picewise smooth solutions, but this time the solver include a new steady shock wave located at the point discontinuity of the soil or porosity function [25], [26].

The Riemann problem for the shallow water equations with topography and vegetation consists in finding a solution in the class of functions with bounded variation for the equations 41 with the following initial conditions:

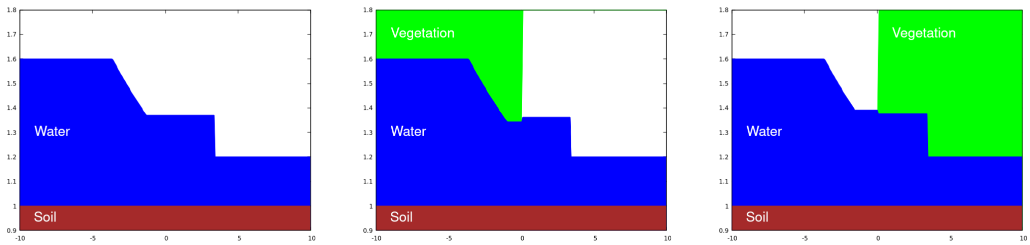
$$(h,u,z,\theta) = \begin{cases} (h^L,u^L,z^L,\theta^L) & x<0, \\ (h^R,u^R,z^R,\theta^R) & x>0 \end{cases} \tag{49}$$

In the papper [26] was proved that Riemann problem is locally solvable. In the figures (2), (3 and (4) we illustrate the solution of the RP for different initial data. All pictures contain the h profiles at the moment of time $t = 0.7s$. In all cases illustrate here the solutions include rarefaction wave, that propagate to the left and a shock wave that propagate to the right. If the data terrain



(a) (b) (c)

Fig. 2. Solutions of Riemann Probelm: $(u^L, \theta^L) = (u^R, \theta^R)$; (a) $(h^L, z^L) = (0.6, 1), (h^R, z^R) = (0.2, 1)$; (b) $(h^L, z^L) = (0.6, 1), (h^R, z^R) = (0.2, 1.1)$; (c) $(h^L, z^L) = (0.6, 1.05), (h^R, z^R) = (0.2, 1)$.



(a) (b) (c)

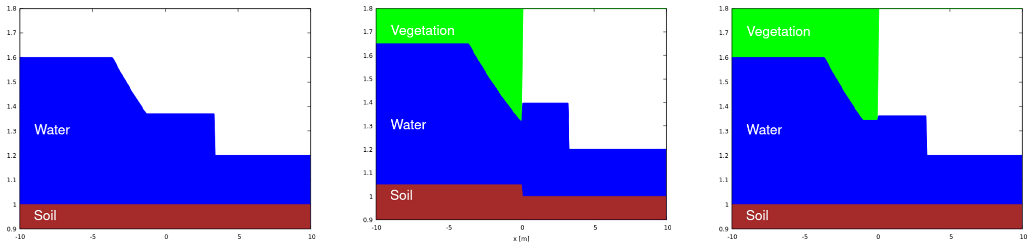
Fig. 3. Solutions of Riemann Probelm: $(u^L, z^L) = (u^R, z^R)$; (a) $(h^L, \theta^L) = (0.6, 1), (h^R, \theta^R) = (0.2, 1)$; (b) $(h^L, \theta^L) = (0.6, 0.9), (h^R, \theta^R) = (0.2, 1)$; (c) $(h^L, \theta^L) = (0.6, 1), (h^R, \theta^R) = (0.2, 0.9)$.

present a jump then a new shock wave is generated that is located at the discontiuity point, $x = 0$.

The global solvability of the Riemann problem for SWE-veg equations is an open problem. The problem was discused in the paper [26].

Comparision of the model prediction with the experimental data. Generally speaking, a mathematical model is a metaphor of the reality that it refers. He cannot quantify all the state variables but only a part of them, the variables that dominate and control the evolution or state of the system. With necessity the model must retain the dominant forces that govern the physical phenomenon and must ignore others that induce small effects in the state of it.

The SWE-Veg models are intended to predict the dynamics of water flow on the soil surface. Apparently this is not so complicated process, but really it is difficult to mathematically model it. The water-plant and water soil



(a) (b) (c)

Fig. 4. Solutions of Riemann Probelm: $u^L = u^R$; (a) $(h^L, z^L, \theta^L) = (0.6, 1, 1), (h^R, z^R, \theta^R) = (0.2, 1, 1)$; (b) $(h^L, z^L, \theta^L) = (0.6, 1., 1), (h^R, z^R, \theta^R) = (0.2, 1.1, 1)$; (c) $(h^L, z^L, \theta^L) = (0.6, 1.05, 1), (h^R, z^R, \theta^R) = (0.2, 0, 1, 1)$.

interactions forces are hardly quantifiables and in certain circumstances new processes can become relevant, erosion and water infiltration, for examples

In spite of this difficulties the SWE-Veg model can predict the evolution of water dynamics variables with a satisfactory accuracy.

The model is versatil enough to cope with a large class of physical processes. By a proper choice of the model parameters α_p and α_s and a proper determination of the porosity function θ and soil altitude function z it can be used to simulate water flow for different, real or imaginary, scenarios.

We will illustrate the model ability to predict main water flow charcteristics by comparing its prediction with some experimental data. All results furnished by the model was obtained by using a numerical scheme exposed in [27].

We consider two experiments, one is the dam break simulation and another one is the water flow on a vegetated slope.

Water flow on vegetated slope

Briefly, the experimental installation consists of an 18m long and 1m width laboratory flume with a longitudinal bottom slope S = 1.05mm/m, see figure (5). = 1.1738. Figure (6) includes the numerical and experimental data for forth different steady configurations. The experimental data are extracted from the graphics The experimental results was reported in [28].

Dam break flow in an L-shaped channel

We consider the CADAM test case of the dam break flow propaga- tion in an impermeable L-shaped channel, [29], [30]. The layout of this experiment and the initial state of the water at rest are presented in Figure (7). The water level in reservoir is $h = 0.53[m]$ and $h = 0$ in the channel. The experimental data for bare soil are for [30]. We simulate the same problem but considering in addition a vegetated channel with $\theta = 0.99$, the numerical and experimental results are given in the figures (8) . One notes the vegetation effect to dammped the water oscilation and to slow down the speed of the propagation of the direct wave, see the data from gauge P4.

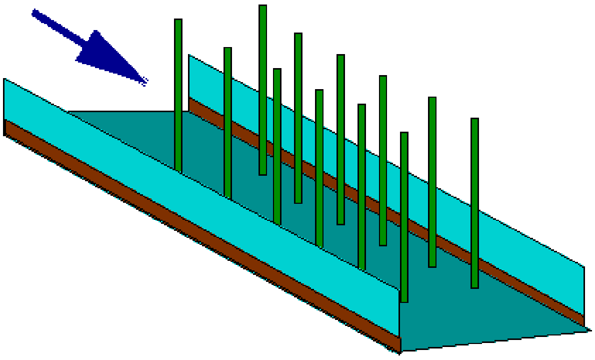


Fig. 5. Experimental instalation for water flow on vegetated slope

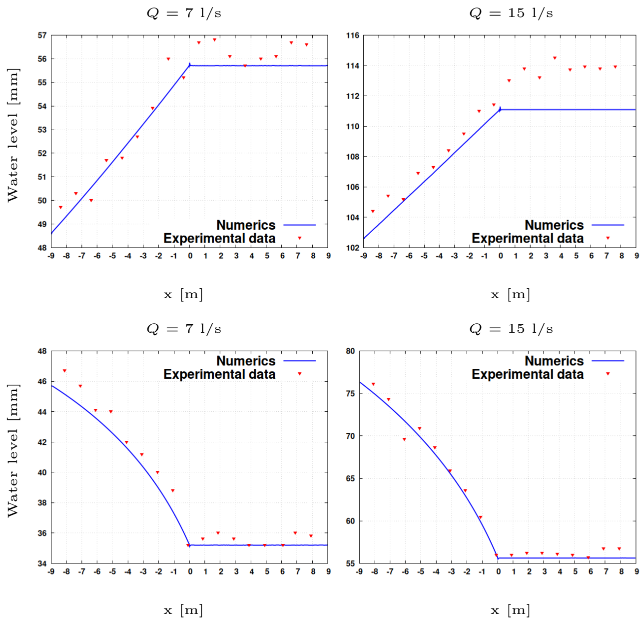


Fig. 6. Flow over a slope with vegetation

5. FINALY REMARKS AND FURTHER RESEARCH

The flow of water is a natural phenomenon that interests everyone. To predict the flow main characteristcs like water depth or water velocity of water coming from rain or generated by the floods is of the great important for hydrogist or agriculters. To predict the flow main characteristics like water depth or water velocity of water coming from rain or generated by the floods is

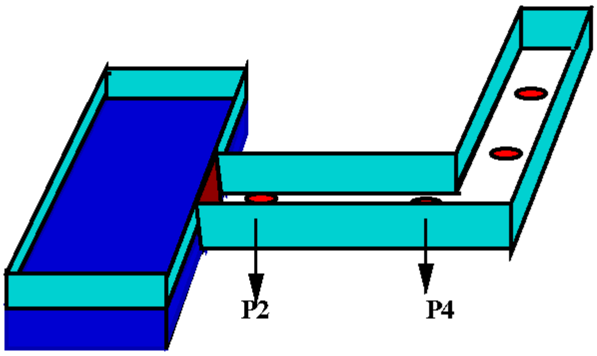


Fig. 7. Experimental instalation for dam break flow

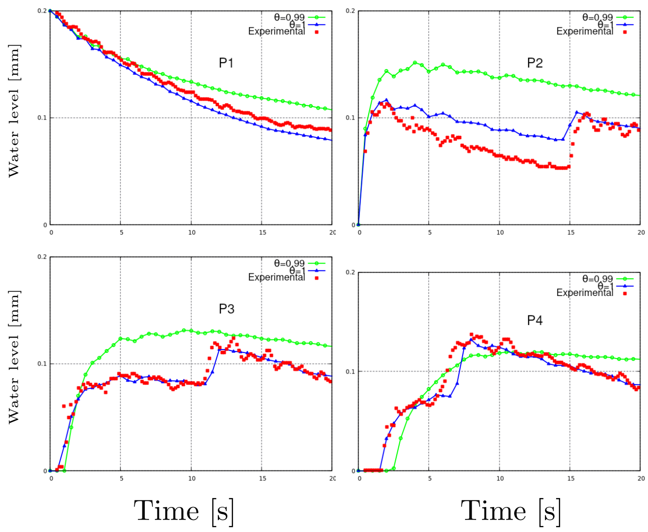


Fig. 8. Dam break flow in L-shaped channel

of the great important for hydrologist, agriculturists or or officials responsable with soil and water management. Given that there are a large variety of geographical context where the water flow is of the interes, there are a plethora of mathematical model used to study it . Some of them are empirical models and some of them are physical processes based models.

Independently of their character, all includes many simplified hypothesis in order to be solvable or to fit to a particular case.

In the paper we has try to show how one can obtain a class of shallow water model equations by using the general continuum mechanical principles.

This mathematical dedudicting way revels what was assumed as important and what was less important and was dropped out from the model. In this way a person interesed in applications can judge in advance if a simplified version of the model is adequat for a given problem or one must consider a more complicated model.

A comparating study presented in the section Application show thta the simplified model (41), (43) is able to accurate predict the main characteristics of the water flow and it can be used for many application in hydrology or agrucukture.

It is very versatil and it can be relativelly ease programated.

The elements of differential geometry presented in the Annexa faciltate the readers to understend the mathematical tools used in the paper.

We belive that there are two main directions to extend the area of applications of the model: to develope more accurate and more economically, in time and computer memory, numerical schemes and to incorporate the surface curvature effects in the model.

References

[1] M.J. Baptist, V. Babovic, J. Rodriguez Uthurburu, M. Keijzer, R.E. Uittenbogaard, A. Mynett and A. Verwey, *On inducing equations for vegetation resistance*, Journal of Hydraulic Research, **45**(4), pp. 435–450, 2007.

[2] J. Bear, *Dynamics of Fluids in Porous Media*, Dover, 1988.

[3] R.A. Bagnold, *An Approach to the Sediment Transport Problem from General Physics*, Geological Survey Prof. Paper 422-I, Wash., 1966.

[4] L.P. Eisenhart, *An Introduction to Differential Geometry - With the Use of Tensor Calculus*, Kessinger Publishing, 2010.

[5] S.M. Hassanizadeh, W.G. Gray, *Mechanics and thermodynamics of multiphase flow in porous media including interphase boundaries*, Adv. Water Resources, **13**(4), pp. 169–186, 1990.

[6] R.J. Lowe, U. Shavit, J.L. Falter, J.R. Koseff and S.G. Monismith *Modeling flow in coral communities with and without waves: A synthesis of porous media and canopy flow approaches*, Limnol. Oceanogr., **53**(6), pp. 2668–2680, 2008.

[7] I. Müller, *Thermodynamics*, Boston : Pitman, 1985.

[8] H.M. Nepf, *Drag, turbulence, and diffusion in flow through emergent vegetation*, Water Resource Research, **35**(2), pp. 479–489, 1999. Resource Research, **35**(2), pp. 479–489, 1999.

[9] L. Cozzolino, V. Pepe, L. Cimorelli, A. D'Aniello, R. D. Morte, D. Pianese, *The solution of the dam-break problem in the porous shallow water equations*, Advances in Water Resources 114 (2018)

83 – 101. doi:https://doi.org/10.1016/j.advwatres.2018.01.026. URL http://www.sciencedirect.com/science/article/pii/S0309170817306966

[10] Kamel Mohamed, *A finite volume method for numerical simulation of shallow water models with porosity*, Computers&Fluids 104 (2014) 9–19.

[11] L. Cea, M.E. Vázquez-Cendón, *Unstructured finite volume discretisation of bed friction and convective flux in solute transport models linked to the shallow water equations*, Journal of Computational Physics 231 (2012) 3317–3339.

[12] S. Whitaker, *Flow in Porous Media I: A Theoretical Derivation of Darcy's Law*, Transport in Porous Media, **1**, pp. 3–25, 1986.

[13] S.L. Neitsch, J.G. Arnold, J.R. Kiniry, J.R. Williams, *Soil and Water Assessment Tool, Theoretical Documentation*, Texas Water Resources Institute Technical Report No. 406, 2009.

[14] Tai-Ping Liu, *The Deterministic Version of the Glimm Scheme*, Commun. math. Phys. 57, 135—148 (1977).

[15] Glimm, J., *Solutions in the large for nonlinear hyperbolic systems of equations*, Commun. Pure Appl.Math. **18**(1965), 697—-715.

[16] Lax,P.D., *Hyperbolic systems of conservation laws and the mathematical theory of shock waves*, Philadelphia SIAM Regional Conf. Ser. in Appl. Math. **11**(1973)

[17] C. M. Dafermos, *Solution of the Riemann problem for a class of hyperbolic systems of conservation laws by the viscosity method*, Arch. Rational Mech. Anal., **52**, pp. 1–9, 1973.

[18] P. L. Roe. "Approximate Riemann solvers, parameter vectors and difference schemes". Journal of Computational Physics. **43**(2)(1981), 357–372.

[19] Toro, E. F. (1999), *Riemann Solvers and Numerical Methods for Fluid Dynamics*, Springer-Verlag, 1999.

[20] Dal Maso, Gianni and G. LeFloch, Philippe and Murat, François, *Definition and weak stability of nonconservative products*, J. de Math. Pures Appl., **74** (1995), 483-548

[21] David L. George, *Augmented Riemann solvers for the shallow water equations over variable topography with steady states and inundation*, Journal of Computational Physics **227**(2008) 3089–3113.

[22] D. I. Ketcheson, R. J. LeVeque, M. J. del Razo, *Riemann Problems and Jupyter Solutions*, SIAM, 2020.

[23] P. G. LeFloch, M. D. Thanh, *The Riemann problem for the shallow water equations with discontinuous topography*, Commun. Math. Sci. **5**(4) (2007) 865–885.

[24] F. B. Francisco Alcrudo, *Exact solution of the Riemann problem for the shallow water equations with a bottom step*, Computers& Fluids, **30**(2001) 643 – 671.

[25] S. Ion, D. Marinescu, S. G. Cruceanu, *Riemann problem for shallow water equations with vegetation*, An. St. Univ. Ovidius **26** (2) (2018) 83 – 114.

[26] S. Ion, D. Marinescu, S. G. Cruceanu, *Constructive Approach of the Solution of Riemann Problem for Shallow Water Equations with Topography and Vegetation* , An. St. Univ. Ovidius **28** (2) (2020) 145 – 173.

[27] S. Ion, D. Marinescu, S. G. Cruceanu,*Numerical scheme for solving a porous Sain-Venant type model for water flow on vegetated hill slope*, Applied Numerical Mathematics, **172**(2020), 67-98.

[28] V. Dupuis, S. Proust, C. Berni, A. Paquier, *Combined effects of bed friction and emergent cylinder drag in open channel flow*, Environmental Fluid Mechanics **16**(6) (2016) 1173–11

[29] S. Soares Frazao, Y. Zech, *Dam break in a channel with 90^1 bend*, J. Hydraul. Eng., **128**(2002), 956-968.

[30] S. Soares Frazao, X. Sillen and Y. Zech, *Dam break flow through sharp bends-physical model and 2D Boltzman model validation*, CADAM Meeting Wallingford, United Kingdom, March 1998; Belgium, 1999.

[31] V. Guinot, *Multiple porosity shallow water models for macroscopic modelling of urban floods*, Advances in Water Resources 37 (2012) 40 – 72. doi:https://doi.org/10.1016/j.advwatres.2011.11.002.

6. BASICS OF DIFFERENTIAL GEOMETRY IN EUCLIDEAN SPACE

6.1. CURVILINEAR COORDINATE

Let $O\boldsymbol{x}$ be a Cartesian coordinate system in the reference Euclidean space $\mathbb{E}^3$. Let $\{y^I\}_{I=\overline{1,3}}$ be another coordinate system and let

$$x^i = x^i(y^1, y^2, y^3), \quad \boldsymbol{y} \in D \tag{50}$$

be the transformation rule. By coordinate line, one understands the curves generated by the variation of a single variable y^I, while the rest are kept constants. The tangent vectors at the coordinate lines are defined by

$$\boldsymbol{e}_I = \partial_I \boldsymbol{x}. \tag{51}$$

The set of vectors $\{\boldsymbol{e}_I\}_{I=\overline{1,3}}$ give rise to a new base of tensor fields. For the vectors and tensors of rank 2, one writes

$$\boldsymbol{v} = v^I \boldsymbol{e}_I, \quad \mathbf{t} = \mathrm{t}^{IJ} \boldsymbol{e}_I \boldsymbol{e}_J.$$

In the new coordinate system, the components of the metric tensor $\boldsymbol{g}$ are given by

$$g_{IJ} = \delta_{ij} e^i_I e^j_J \tag{52}$$

and

$$g^{IJ} = \delta^{ij} h^I_i h^J_j, \tag{53}$$

where

$$h^I_j = \partial_j y^I. \tag{54}$$

One has

$$e^j_I h^I_i = \delta^j_i, \quad e^j_I h^J_j = \delta^J_I \tag{55}$$

and then

$$g^{IK} g_{KJ} = \delta^I_J.$$

The volume element is

$$J = \varepsilon_{ijk} e^i_1 e^j_2 e^k_3, \tag{56}$$

with ε_{ijk} representing the Levi-Civita symbol. From (56) and (52), one obtains

$$\det \boldsymbol{g} = J^2, \tag{57}$$

where $\boldsymbol{g}$ is the matrix with the elements g_{IJ}.

The variation of the basis $\{\boldsymbol{e}_I\}_I$ with respect to the y coordinate is stored inside Christoffel's symbols Γ

$$\partial_I \boldsymbol{e}_J = \Gamma^L_{IJ} \boldsymbol{e}_L. \tag{58}$$

Alternatively, one can calculate the Γ coefficients by

$$\begin{aligned} \Gamma^L_{IJ} &= h^L_i \partial_J e^i_I, \\ \Gamma^L_{IJ} &= -e^i_I e^j_J \partial_i h^L_j, \\ \Gamma^L_{IJ} &= \frac{1}{2} g^{LK} \left(\partial_I g_{KJ} + \partial_J g_{KI} - \partial_K g_{IJ} \right). \end{aligned} \tag{59}$$

The first relation here results from the definition (58) and (55), the second relation results from the first one, and the last relation results from (58) and (52). Define now the covariant derivative of a vector by

$$v^I_{;L} = \partial_L v^I + v^K \Gamma^I_{LK} \tag{60}$$

and the covariant derivative of tensor by

$$t^{IJ}_{;L} = \partial_L t^{IJ} + t^{KJ} \Gamma^I_{LK} + + t^{IK} \Gamma^J_{LK}. \tag{61}$$

An elementary way to introduce the covariant derivative is to estimate the difference of vector fields between two neighbor points

$$\begin{aligned} \boldsymbol{v}(\boldsymbol{y} + \triangle \boldsymbol{y}) - \boldsymbol{v}(\boldsymbol{y}) =& v^I(\boldsymbol{y} + \triangle \boldsymbol{y}) \boldsymbol{e}_I(\boldsymbol{y})(\boldsymbol{y} + \triangle \boldsymbol{y}) - v^I \boldsymbol{e}_I(\boldsymbol{y}) \\ =& \left(\partial_L v^I(\boldsymbol{y}) + v^K(\boldsymbol{y}) \Gamma^I_{LK}(\boldsymbol{y}) \right) \boldsymbol{e}_I(\boldsymbol{y}) \triangle y^L + O(\triangle \boldsymbol{y}^2). \end{aligned}$$

6.2. BASIC NOTIONS OF DIFFERENTIAL GEOMETRY ON A SURFACE IN $\mathbb{E}^3$

For completeness, we present here the essential facts about the differential geometry of the surface in the euclidean space E^3; as a reference, one can consult the classical books [4]. Let $O\boldsymbol{x}$ be a Cartesian coordinate system in the reference Euclidean space $\mathbb{E}^3$. Let $\mathcal{S}$ be a surface in E^3 and let

$$x^i = b^i(y^1, y^2), \quad (y^1, y^2) \in D \in \mathbb{R}^2 \tag{62}$$

be a parameterization of $\mathcal{S}$. One defines the tangent vectors to the surface by

$$\tau^i_a = \frac{\partial b^i}{\partial y^a} \tag{63}$$

and the oriented normal direction to the surface by

$$\mathcal{N}_i = \varepsilon_{j\,k\,i}\tau^j_1\tau^k_2. \tag{64}$$

The unitary normal $\boldsymbol{\nu}$ to the surface is given by

$$\nu_i = \frac{\mathcal{N}_i}{||\boldsymbol{\mathcal{N}}||}. \tag{65}$$

Metric tensor $\boldsymbol{\beta}$ of the surface. The covariant components of $\boldsymbol{\beta}$ are given by

$$\beta_{ab} = \delta_{ij}\tau^i_a\tau^j_b \tag{66}$$

and the contravariant components β^{ab} of it are defined by the relations

$$\delta^a_b = \beta^{ac}\beta_{cb} = \beta_{bc}\beta^{ca}. \tag{67}$$

The area element of the surface is defined by

$$\mathrm{d}\sigma(y) = \beta(y)\mathrm{d}y^1\mathrm{d}y^2, \tag{68}$$

where

$$\beta = \sqrt{\varepsilon^{ab}\beta_{a1}\beta_{b2}}, \tag{69}$$

with ε^{ab} being the Levi-Civita symbol.

Note that

$$||\boldsymbol{\mathcal{N}}|| = \beta.$$

The curvature tensor $\boldsymbol{\kappa}$. The curvature tensor κ and the affine connection γ can be defined by the Gauss-Wiengarten equations

$$\begin{aligned} \frac{\partial \boldsymbol{\tau}_a}{\partial y^b} &= \gamma^c_{a\,b}\boldsymbol{\tau}_c + \kappa_{a\,b}\boldsymbol{\nu}, && \text{(Gauss)} \\ \frac{\partial \boldsymbol{\nu}}{\partial y^a} &= -\kappa^b_a\boldsymbol{\tau}_b. && (Wiengarten) \end{aligned} \tag{70}$$

6.3. SURFACE BASED CURVILINEAR COORDINATE SYSTEM

A surface $\mathcal{S}$ based coordinate system in the space $\mathbb{E}^3$ is introduced as follows. Given a parameterization (62) of the surface, one defines the applications

$$x^i = b^i(y^1, y^2) + y^3\nu^i, \quad (y^1, y^2) \in \widetilde{D} \subset \mathbb{R}^2, \quad y^3 \in \widetilde{I} \in \mathbb{R}, \tag{71}$$

where $\widetilde{I}$ is an open neighborhood of zero. Assume that (71) defines a coordinate transformation from $\widetilde{D} \times \widetilde{I}$ to a space neighborhood Ω of the surface $\mathcal{S}$. The surface $\mathcal{S}$ in the new coordinate system is given by $y^3 = 0$. Furthermore, we have:

• the tangent vectors to the coordinate lines

$$\boldsymbol{e}_I = \frac{\partial \boldsymbol{x}}{\partial y^I} \Longrightarrow \left\{ \begin{array}{l} \boldsymbol{e}_a = q_a^b \boldsymbol{\tau}_b, \quad q_a^b := \delta_a^b - y^3\kappa_a^b, \quad a = \overline{1,2} \\ \boldsymbol{e}_3 = \boldsymbol{\nu} \end{array} \right. ; \tag{72}$$

• the coefficients of the metric tensor

$$g_{IJ} = \delta_{ij} e_I^i e_J^j \Longrightarrow \left\{ \begin{array}{ll} g_{ab} = q_a^c q_b^d \beta_{cd}, & g_{a3} = 0, \\ g_{3a} = 0, & g_{33} = 1, \end{array} \right. \tag{73}$$

with

$$\sqrt{\det \boldsymbol{g}} = \beta\Delta, \quad \Delta := 1 - 2y^3 K_M + (y^3)^2 K_G, \tag{74}$$

where $K_M = 1/2\kappa_a^a$ and $K_G = \epsilon_{a,b}\kappa_1^a\kappa_2^b$ are the mean curvature and the Gauss curvature of the surface, respectively; • the affine connection

$$\frac{\partial \boldsymbol{e}_I}{\partial y^J} = \Gamma_{IJ}^L \boldsymbol{e}_L \Longrightarrow \left\{ \begin{array}{ll} \Gamma_{ab}^c = \left(\gamma_{ab}^d - y^3\left(\partial_a\kappa_b^d + \kappa_b^f\gamma_{af}^d\right)\right) Q_d^c, & \Gamma_{a3}^c = -\kappa_a^e Q_e^c, \\ \Gamma_{ab}^3 = (\delta_a^c - y^3\kappa_a^c)\kappa_{cb}, & \Gamma_{a3}^3 = 0, \end{array} \right. \tag{75}$$

where Q is defined by

$$\boldsymbol{\tau}_a = Q_a^b \boldsymbol{e}_b \Longrightarrow \left\{ \begin{array}{ll} Q_1^1 = \dfrac{1 - y^3\kappa_2^2}{\Delta(y)}, & Q_1^2 = \dfrac{y^3\kappa_1^2}{\Delta(y)}, \\ Q_2^1 = \dfrac{y^3\kappa_2^1}{\Delta(y)}, & Q_2^2 = \dfrac{1 - y^3\kappa_1^1}{\Delta(y)}. \end{array} \right. \tag{76}$$

Obs. For any $y^3 \in I$, the tangent vectors $\boldsymbol{e}_a$, $a = \overline{1,2}$ belong to the tangent plane at the surface $y^3 =$ const and they are orthogonal to the normal $\boldsymbol{e}_3 = \boldsymbol{\nu}$. In the new coordinate system, the volume element is $\vartheta(y)\mathrm{d}y^1\mathrm{d}y^2\mathrm{d}y^3$, where

$$\vartheta(y) = \epsilon_{ijk} e_1^i e_2^j e_3^k = \sqrt{\det \boldsymbol{g}} = \left(1 - 2y^3 K_M + (y^3)^2 K_G\right)\beta. \tag{77}$$

6.4. INTEGRALS OF VECTORS AND SECOND ORDER TENSORS

Let V be a domain in $\mathbb{E}^3$ defined by

$$\boldsymbol{x} = \boldsymbol{b}(y^1, y^2) + y^3\boldsymbol{\nu}, \quad (y^1, y^2) \in D, \quad u(y^1, y^2) < y^3 < w(y^1, y^2)$$

where D is a open closed domain with boundary ∂D, $u(y^1, y^2)$ and $w(y^1, y^2)$ are two functions that define some surfaces in $\mathbb{E}^3$. We are interested in calculating the flux of vectors or tensors through the boundary of V, to evaluate integral of vectors in V or to calculate integrals of vectors on surfaces. In $\mathbb{E}^3$, such integrals define global quantities of the same type with the integrands: scalars define scalars, vectors define vectors and second order tensors define second order tensors. If one uses curvilinear coordinates, such invariant properties are lost for vectors and tensors.

Let $\mathcal{S}$ and V be a surface and a domain in $\mathbb{E}^3$, respectively. Define the flux of $\boldsymbol{f}$ and $\boldsymbol{\Phi}$ through a surface by

$$\mathcal{F}_{\boldsymbol{f}}(S) := \int\limits_S f^i\, n_i \mathrm{d}\sigma,$$
$$\mathcal{F}^i_{\boldsymbol{\Phi}}(S) := \int\limits_S \Phi^{ij}\, n_j \mathrm{d}\sigma,$$

where $\boldsymbol{n}$ stands for outward oriented unitary normal to the surface.

Define by components the integral of a vector field $\boldsymbol{f}$ on V

$$\mathcal{I}^j_{\boldsymbol{f}}(V) := \int\limits_V f^j \mathrm{d}x$$

and the integral on the surface S

$$\mathcal{I}^j_{\boldsymbol{f}}(S) := \int\limits_S f^j \mathrm{d}\sigma.$$

Let $\mathcal{S}_r$ be the surface defined by some function $r(y^1, y^2)$

$$\boldsymbol{x} = \boldsymbol{b}(y^1, y^2) + r(y^1, y^2)\boldsymbol{\nu}, \quad (y^1, y^2) \in D.$$

One denotes the "vertical" boundary of V by

$$\begin{aligned} \Sigma = \{\, &\boldsymbol{x} \in \mathbb{E}^3 | \boldsymbol{x} = \boldsymbol{b}(y^1(s), y^2(s)) + y^3\boldsymbol{\nu}(y^1(s), y^2(s)), \\ &s \in (0, L),\ u(y^1(s), y^2(s)) < y^3 < w(y^1(s), y^2(s))\} \end{aligned} \tag{78}$$

where $\big(y^1(s), y^2(s)\big)$, $s \in (0, L)$ is a parameterization of ∂D.

Let $\boldsymbol{f}$ and $\boldsymbol{\Phi}$ be a vector field and a second order tensor field in $\mathbb{E}^3$, respectively. Using the law of transformation of the coordinate system of a tensor field under coordinate transformation, one can write

$$f^i = f^I e^i_I, \quad \Phi^{ij} = e^i_I e^j_J \Phi^{IJ}.$$

Next lemma refers to various integrals.

Lemma 7. *Let $\boldsymbol{f}$ and $\boldsymbol{\Phi}$ be some smooth fields on a domain $\Omega \subset \mathbb{E}^3$. Let S_r, V and Σ be a surface, domain and portion of ∂V, respectively, as previously defined. Then:*

$$\begin{aligned}
\mathcal{I}^i_f(V) &= \iint\limits_D \left(\tau^i_a \int\limits_u^w q^a_b f^b \vartheta \mathrm{d}y^3 + \nu^i \int\limits_u^w f^3 \vartheta \mathrm{d}y^3 \right) \mathrm{d}y^1 \mathrm{d}y^2, \\
\mathcal{F}_f(S_r) &= \iint\limits_D \vartheta(y) \left(f^3 - f^a \frac{\partial r}{\partial y^a} \right) \Bigg|_{y^3=r} \mathrm{d}y^1 \mathrm{d}y^2, \\
\mathcal{F}_f(\Sigma) &= \iint\limits_D \frac{\partial r}{\partial y^a} \int\limits_u^w \vartheta f^a \mathrm{d}y^3 \mathrm{d}y^1 \mathrm{d}y^2, \\
\mathcal{F}^i_\Phi(S_r) &= \iint\limits_D \Bigg[\Bigg(\tau^i_c q^c_b \left(\Phi^{b3} - \frac{\partial r}{\partial y^a} \Phi^{ba} \right) \\
&\quad + \nu^i \left(\Phi^{33} - \frac{\partial r}{\partial y^a} \Phi^{3a} \right) \Bigg) \vartheta(y) \Bigg] \Bigg|_{y^3=r} \mathrm{d}y^1 \mathrm{d}y^2, \\
\mathcal{F}^i_\Phi(\Sigma) &= \iint\limits_D \tau^i_c \Bigg(\frac{\partial}{\partial y^a} \int\limits_u^w q^c_b \vartheta(y) \Phi^{ba} \mathrm{d}y^3 \\
&\quad + \gamma^c_{ae} \int\limits_u^w q^e_b \vartheta(y) \Phi^{ba} \mathrm{d}y^3 - \kappa^c_a \int\limits_u^w \vartheta(y) \Phi^{3a} \mathrm{d}y^3 \Bigg) \mathrm{d}y^1 \mathrm{d}y^2 \\
&\quad + \iint\limits_D \nu^i \Bigg(\kappa_{ca} \int\limits_u^w q^c_b \vartheta(y) \Phi^{ba} \mathrm{d}y^3 \\
&\quad + \frac{\partial}{\partial y^a} \int\limits_u^w \vartheta(y) \Phi^{3a} \mathrm{d}y^3 \Bigg) \mathrm{d}y^1 \mathrm{d}y^2.
\end{aligned} \tag{79}$$

Proof. Let $(y^1(s), y^2(s))$, $s \in (0, L)$ be a parameterization of the boundary ∂D. On Σ, the tangent directions are given by

$$\begin{aligned}\boldsymbol{t}_s &= \boldsymbol{e}_a w^a,\\ \boldsymbol{e}_3 &= \boldsymbol{\nu},\end{aligned}$$

where $w^a = \dfrac{\mathrm{d}y^a}{\mathrm{d}s}$ and the outward normal direction is given by

$$N_i := \epsilon_{jki} e_3^j t_s^k = \epsilon_{jki} \nu^j e_a^k w^a.$$

Thus, one can evaluate the flux as

$$\mathcal{F}_f(\Sigma) := \int_\Sigma f^i n_i \mathrm{d}\sigma = \int\limits_0^L \int\limits_{\widetilde{u}(s)}^{\widetilde{w}(s)} f^i N_i \mathrm{d}y^3 \mathrm{d}s,$$

with $\widetilde{w}(s) = w(y^1(s), y^2(s))$, $\widetilde{u}(s) = u(y^1(s), y^2(s))$. Then, one writes $\boldsymbol{f}$ in the local basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ and obtains

$$f^i N_i = (f^b e_b^i + f^3 \nu^i) N_i = \epsilon_{jki} \nu^j e_a^k e_b^i w^a f^b = \vartheta(y) \epsilon_{ab} w^a f^b$$

and

$$\mathcal{F}_f(\Sigma) = \int\limits_0^L \int\limits_{\widetilde{u}(s)}^{\widetilde{w}(s)} \vartheta(y) \epsilon_{ab} w^a f^b \mathrm{d}y^3 \mathrm{d}s = \int\limits_0^L \epsilon_{ab} w^a \int\limits_{\widetilde{u}(s)}^{\widetilde{w}(s)} \vartheta(y) f^b \mathrm{d}y^3 \mathrm{d}s.$$

Observe that $\epsilon_{ab} w^a = \epsilon_{ab} \dfrac{\partial y^a}{\partial s}$ is the normal direction to the boundary ∂D and use the flux-divergence theorem and to obtain

$$\mathcal{F}_f(\Sigma) = \iint\limits_D \frac{\partial}{\partial y^a} \int\limits_{u(y^1,y^2)}^{w(y^1,y^2)} \vartheta(y) f^a \mathrm{d}y^3 \mathrm{d}y^1 \mathrm{d}y^2. \tag{80}$$

On S_r, one has the tangent vectors

$$\boldsymbol{\zeta}_a = \frac{\partial \boldsymbol{x}}{\partial y^a} = \boldsymbol{e}_a + \frac{\partial r}{\partial y^a} \boldsymbol{\nu} \tag{81}$$

and normal direction

$$N_i = \epsilon_{jki} \left(e_1^j + \frac{\partial r}{\partial y^1} \nu^j \right) \left(e_2^k + \frac{\partial r}{\partial y^2} \nu^k \right). \tag{82}$$

Then, we obtain

$$f^i N_i = \vartheta(y)\left(f^3 - \frac{\partial r}{\partial y^a} f^a\right).$$

Consequently,

$$\mathcal{F}_f(S_r) = \iint\limits_D \vartheta(y)\left(f^3 - \frac{\partial r}{\partial y^a} f^a\right)\Bigg|_{y^3=r} \mathrm{d}y^1\mathrm{d}y^2. \tag{83}$$

Consider now a second order tensor Φ. The coordinate transformation (71) implies that the contravariant components of the tensor in the two coordinate system are related by

$$\Phi^{ij} = e^i_I e^j_J \Phi^{IJ}.$$

The main difficulty in this case is that the vectors of the basis depend on the variables (y^1, y^2, y^3) and there is no sense to find the components of the global vector quantity $\mathcal{F}_\Phi$ in the new system of coordinates. We proceed to find the Cartesian components of $\mathcal{F}_\Phi$, but calculated as functions of the contravariant components Φ^{IJ}.

On the surface Σ, one has

$$\Phi^{ij} N_j = e^i_I e^j_J \Phi^{IJ} N_j = \vartheta(y)\epsilon_{ab} w^a e^i_I \Phi^{Ib}$$

and the flux is given by

$$\mathcal{F}^i_\Phi(\Sigma) = \iint\limits_D \frac{\partial}{\partial y^a}\int\limits_u^w \vartheta(y) e^i_I \Phi^{Ia} \mathrm{d}y^3\mathrm{d}y^1\mathrm{d}y^2.$$

Using the relations (72) we get

$$\mathcal{F}^i_\Phi(\Sigma) = \iint\limits_D \frac{\partial}{\partial y^a}\left(\tau^i_c \int\limits_u^w q^c_b \vartheta(y)\Phi^{ba}\mathrm{d}y^3 + \nu^i \int\limits_u^w \vartheta(y)\Phi^{3a}\mathrm{d}y^3\right)\mathrm{d}y^1\mathrm{d}y^2.$$

Applying Weigartern formula, we can write

$$\begin{aligned}
\mathcal{F}^i_\Phi(\Sigma) &= \iint\limits_D \left(\tau^i_c \frac{\partial}{\partial y^a}\int\limits_u^w q^c_b \vartheta(y)\Phi^{ba}\mathrm{d}y^3 + \nu^i \frac{\partial}{\partial y^a}\int\limits_u^w \vartheta(y)\Phi^{3a}\mathrm{d}y^3\right)\mathrm{d}y^1\mathrm{d}y^2 \\
&+ \iint\limits_D \tau^i_c \left(\gamma^c_{ae}\int\limits_u^w q^e_b \vartheta(y)\Phi^{ba}\mathrm{d}y^3 - \kappa^c_a \int\limits_u^w \vartheta(y)\Phi^{3a}\mathrm{d}y^3\right)\mathrm{d}y^1\mathrm{d}y^2 \\
&+ \iint\limits_D \nu^i \kappa_{ea}\int\limits_u^w q^e_b \vartheta(y)\Phi^{ba}\mathrm{d}y^3\mathrm{d}y^1\mathrm{d}y^2.
\end{aligned}$$

Regrouping the terms, we obtain the result for $\mathcal{F}^i_\Phi(\Sigma)$.

Lemma 8. *Consider that the stress tensor of the fluid has the following form*

$$t^{ij} = -p\delta^{ij} + \tau^{ij}$$

and set

$$\mathcal{F}^i_{\text{stress}}(S_r) = \iint\limits_{S_r} t^{ij} n_j \mathrm{d}\sigma.$$

Then

$$\begin{aligned}\mathcal{F}^i_{\text{stress}}(S_r) = &\iint\limits_D \left[\tau^i_d q^d_a \left((p - \widetilde{\tau}^{33}) g^{ab} \frac{\partial r}{\partial y^b}\right.\right. \\ &\left.\left. + \widetilde{\tau}^{a3} \sqrt{1 + g^{bc} \frac{\partial r}{\partial y^b} \frac{\partial r}{\partial y^c}}\right) \vartheta(y)\right]\Bigg|_{y^3=r(y^1,y^2)} \mathrm{d}y^1 \mathrm{d}y^2 \\ &+ \iint\limits_D \left[\nu^i \left(-p + \widetilde{\tau}^{33} + \frac{\partial r}{\partial y^a} \widetilde{\tau}^{a3}\right.\right. \\ &\left.\left. \cdot \sqrt{1 + g^{bc} \frac{\partial r}{\partial y^b} \frac{\partial r}{\partial y^c}}\right) \vartheta(y)\right]\Bigg|_{y^3=r(y^1,y^2)} \mathrm{d}y^1 \mathrm{d}y^2.\end{aligned} \tag{84}$$

In this lemma, $\widetilde{\tau}^{IJ}$ denotes the contravariant components of the viscous stress tensor in the frame given by the tangent vectors to the surface $y^3 = r(y^1, y^2)$ and the unit normal to the tangent plan (which points to the same direction as the unit normal $\boldsymbol{\nu}$ to the support surface).

Proof. Let $r(y^1, y^2)$ be a parameterization of the surface S_r and let $\boldsymbol{\zeta}_1$, $\boldsymbol{\zeta}_2$ and $\boldsymbol{n}$ be the tangent vectors and the unit normal given by (81) and (82), respectively. One can write

$$t^{ij} n_j = -pn^i + \tau^{ij} n_j = -pn^i + \widetilde{\tau}^{a3} \zeta^i_a + \widetilde{\tau}^{33} n^i. \tag{85}$$

Using the basis $\{\boldsymbol{e}_I\}$, the unit normal has the form

$$\boldsymbol{n} = n^a \boldsymbol{e}_a + n^3 \boldsymbol{\nu}, \quad n^a = -g^{ab} \frac{\partial r}{\partial y^b} \frac{\vartheta(y)}{||\boldsymbol{N}||}, \quad n^3 = \frac{\vartheta(y)}{||\boldsymbol{N}||},$$
$$||\boldsymbol{N}|| = \vartheta(y) \sqrt{1 + g^{ab} \frac{\partial r}{\partial y^a} \frac{\partial r}{\partial y^b}}, \quad y^3 = r(y^1, y^2)$$

and the tangent vectors are expressed by

$$\boldsymbol{\zeta}_a = \boldsymbol{e}_a + \frac{\partial r}{\partial y^a} \boldsymbol{n} \boldsymbol{u}.$$

Since the area element is given by

$$\mathrm{d}\sigma = ||\boldsymbol{N}||\mathrm{d}y^1\mathrm{d}y^2,$$

then, we immediately obtain the conclusion of this lemma.