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G^+ METHOD IN ACTION: NEW CLASSES OF NONNEGATIVE MATRICES, WITH RESULTS

Udrea Păun

"Gheorghe Mihoc-Caius Iacob" Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, Bucharest, Romania

upterra@gmail.com

In memory of Academician Marius Iosifescu (1936-2020)

Abstract The G^+ method is a new method, a powerful one, for the study of (homogeneous and nonhomogeneous) products of nonnegative matrices — for problems on the products of nonnegative matrices. To study such products, new classes of matrices are introduced: that of the sum-positive matrices, that of the $[\Delta]$-positive matrices on partitions (of the column index sets), that of the g_k^+-matrices... On the other hand, the g_k^+-matrices lead to necessary and sufficient conditions for the k-connected graphs. Using the G^+ method, we prove old and new results (Wielandt Theorem and a generalization of it, etc.) on the products of nonnegative matrices — mainly, sum-positive, $[\Delta]$-positive on partitions, irreducible, primitive, reducible, fully indecomposable, scrambling, or Sarymsakov matrices, in some cases the matrices being, moreover, g_k^+-matrices (not only irreducible).

Keywords: G^+ method, nonnegative matrix, sum-positive matrix, $[\Delta]$-positive matrix on partitions, g_k^+-matrix, k-connected graph, product of nonnegative matrices, positive matrix, row-allowable matrix, column-allowable matrix, irreducible matrix, primitive matrix, index of primitivity, reducible matrix, fully indecomposable matrix, Markov matrix, scrambling matrix, Sarymsakov matrix.
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1. A SHORT INTRODUCTION

The G method [17] led to good or very good results, see, *e.g.*, [17, Theorems 1.6 and 1.8; see also Theorems 2.2 and 2.3], [18], [19], and [20] — see also [16; see, *e.g.*, Examples 2.11, 2.19, and 2.22].

The G^+ method was suggested by the G method and Theorem 2.2, both from [17]. This method leads to good or very good results too; using this method, we prove old and new results (Wielandt Theorem and a generalization of it, etc.) on the (homogeneous or nonhomogeneous) products of nonnegative matrices — mainly, sum-positive, $[\Delta]$-positive on partitions (of the column index sets), irreducible, primitive, reducible, fully indecomposable, scrambling, or Sarymsakov matrices, in some cases the matrices being, moreover, g_k^+-matrices (not only irreducible).

The products of nonnegative matrices arise in several fields, such as,

1) (homogeneous and nonhomogeneous) Markov chains, see, *e.g.*, [10] and [21],

2) (fast or not too fast) exact sampling and other exactly solvable problems/things (the computation of normalization constants, etc.), all based on Markov chains (the methods used are not Monte Carlo), see, *e.g.*, [18] and [19], where products of reducible stochastic matrices are used (a surprise?),

3) Markov chain Monte Carlo, see, *e.g.*, [5], [11], [12], and [13, Chapter 9] — since the Markov chain Monte Carlo methods are not too good, it is very important to remove "Monte Carlo" from "Markov chain Monte Carlo" in as many as possible cases, *i.e.*, to obtain exact or approximate good/efficient methods based on Markov chains, methods which are not Monte Carlo (for exact methods, see 2) again), in as many as possible cases,

4) probabilistic automata, see, *e.g.*, [15],

5) fractals, see, *e.g.*, [7, Sections 4.3 and 4.5] and [8, Chapter 11],

6) economics and some related fields, see, *e.g.*, [8, Chapter 13] and [9, pp. 487−488],

7) consensus, see, *e.g.*, [21, pp. 153−158] and [23].

2. G^+ METHOD — DEFINITION AND BASIC RESULTS

In this section, we define the G^+ method and give certain basic results of it. To define the G^+ method, we first define the sum-positive matrices and $[\Delta]$-positive matrices on partitions — the Δ-positive matrices on partitions are also defined, and are also important.

Set

$$\langle m\rangle = \{1, 2, ..., m\} \ \ (m \in \mathbb{N},\ m \geq 1),$$

$$\langle\langle m\rangle\rangle = \{0, 1, ..., m\} \ \ (m \in \mathbb{N}),$$

$$N_{m,n} = \{P \mid P \text{ is a nonnegative } m \times n \text{ matrix}\},$$

$$S_{m,n} = \{P \mid P \text{ is a stochastic } m \times n \text{ matrix}\}$$

— here, a stochastic matrix is a row stochastic matrix —,

$$N_n = N_{n,n},$$

$$S_n = S_{n,n}.$$

The entry (i,j) of a matrix Z will be denoted Z_{ij} or, if confusion can arise, $Z_{i\to j}$.

Let $P = (P_{ij}) \in N_{m,n}$ ($i \in \langle m\rangle$ and $j \in \langle n\rangle$). $\langle m\rangle$ and $\langle n\rangle$ are called the *index sets of* P; moreover, $\langle m\rangle$ is called the *row index set of* P while $\langle n\rangle$ is called the *column index set of* P. If $P \in N_n$, $\langle n\rangle$ is called the *index set of* P.

Let $P = (P_{ij}) \in N_{m,n}$. Let $\emptyset \neq U \subseteq \langle m\rangle$ and $\emptyset \neq V \subseteq \langle n\rangle$. Set the matrices (these are submatrices of P)

$$P_U = (P_{ij})_{i\in U, j\in\langle n\rangle}\,,\ P^V = (P_{ij})_{i\in\langle m\rangle, j\in V}\,,\ \text{and } P_U^V = (P_{ij})_{i\in U, j\in V}\,.$$

Definition 2.1. (See, *e.g.*, [21, p. 80].) Let $P \in N_{m,n}$. We say that P is a *row-allowable matrix* if it has at least one positive entry in each row. We say that P is a *column-allowable matrix* if it has at least one positive entry in each column (equivalently, if the transpose of P is row-allowable).

Below we define a central notion of this article.

Definition 2.2. Let $P \in N_{m,n}$. Let $\emptyset \neq U \subseteq \langle m\rangle$ and $\emptyset \neq V \subseteq \langle n\rangle$. We say that P is *sum-positive on* $U \times V$ if

$$\sum_{j\in V} P_{ij} > 0,\ \forall i \in U.$$

Based on Definition 2.2, we will use the generic name "sum-positive matrix/matrices".

Remark 2.3. P is sum-positive on $U\times V$ if and only if P_U^V is a row-allowable matrix, where $P \in N_{m,n}$, $\emptyset \neq U \subseteq \langle m\rangle$, and $\emptyset \neq V \subseteq \langle n\rangle$.

Remark 2.4. Let $P \in N_{m,n}$.

(a) The column j of P is positive, *i.e.*, $P^{\{j\}} > 0 \Longleftrightarrow P$ is sum-positive on $\langle m\rangle \times \{j\}$.

(b) P is positive $\Longleftrightarrow P^{\{j\}} > 0, \forall j \in \langle n\rangle \Longleftrightarrow P$ is sum-positive on $\langle m\rangle\times\{j\}$, $\forall j \in \langle n\rangle$.

Remark 2.5. Let $P \in N_{m,n}$.

(a) If P is positive, then it is sum-positive on $\langle m\rangle \times \langle n\rangle$ — more generally, on $U \times V$, $\forall U, V$, $\emptyset \neq U \subseteq \langle m\rangle$ and $\emptyset \neq V \subseteq \langle n\rangle$. If P is sum-positive on $\langle m\rangle \times \langle n\rangle$, it does not follow that it is positive when $n \geq 2$. Due to these facts, the sum-positive matrices are generalizations of the positive matrices.

(b) $P \neq 0 \Longleftrightarrow \exists U \in \langle m\rangle$, $\exists V \in \langle n\rangle$ such that P is sum-positive on $U \times V$. Due to this fact, the class of all sum-positive matrices is equal to the class of all nonnegative matrices without the zero matrices.

Let $\emptyset \neq U \subseteq \langle m \rangle$ and $\emptyset \neq V \subseteq \langle n \rangle$. Set

$$G^{+}_{U,V} = G^{+}_{U,V}(m,n) = \{P \mid P \in S_{m,n} \text{ and } P \text{ is sum-positive on } U \times V\}$$

and

$$\overline{G}^{+}_{U,V} = \overline{G}^{+}_{U,V}(m,n) = \{P \mid P \in N_{m,n} \text{ and } P \text{ is sum-positive on } U \times V\}$$

— obviously,

$$G^{+}_{U,V} \subseteq S_{m,n},\ \overline{G}^{+}_{U,V} \subseteq N_{m,n},\ \text{and } G^{+}_{U,V} \subseteq \overline{G}^{+}_{U,V}.$$

Below we give a basic result on the sum-positivity of matrices.

THEOREM 2.6. (i) *Let* $P_1 \in \overline{G}^{+}_{U_1,U_2} \subseteq N_{m_1,m_2}$ *and* $P_2 \in \overline{G}^{+}_{U_2,U_3} \subseteq N_{m_2,m_3}$. *Then*

$$P_1P_2 \in \overline{G}^{+}_{U_1,U_3} \subseteq N_{m_1,m_3}.$$

(ii) (*a generalization of* (i)) *Let* $P_1 \in \overline{G}^{+}_{U_1,U_2} \subseteq N_{m_1,m_2}$, $P_2 \in \overline{G}^{+}_{U_2,U_3} \subseteq N_{m_2,m_3}$, ..., $P_n \in \overline{G}^{+}_{U_n,U_{n+1}} \subseteq N_{m_n,m_{n+1}}$. *Then*

$$P_1P_2...P_n \in \overline{G}^{+}_{U_1,U_{n+1}} \subseteq N_{m_1,m_{n+1}}.$$

Proof. (i) We have

$$(P_1P_2)^{U_3}_{U_1} = (P_1)^{\langle m_2 \rangle}_{U_1} (P_2)^{U_3}_{\langle m_2 \rangle} \geq (P_1)^{U_2}_{U_1} (P_2)^{U_3}_{U_2}.$$

The matrices $(P_1)^{U_2}_{U_1}$ and $(P_2)^{U_3}_{U_2}$ are row-allowable, so, $(P_1)^{U_2}_{U_1} (P_2)^{U_3}_{U_2}$ is row-allowable, and, further, $(P_1P_2)^{U_3}_{U_1}$ is row-allowable. By Remark 2.3 we have

$$P_1P_2 \in \overline{G}^{+}_{U_1,U_3}.$$

(ii) Induction. ∎

Remark 2.7. Since $S_{m,n} \subseteq N_{m,n}$, $G^{+}_{U,V} \subseteq \overline{G}^{+}_{U,V}$, and a product of stochastic matrices is a stochastic matrix, Theorem 2.6 holds, in particular, for stochastic matrices. From, *e.g.*, Theorem 2.6(i), we obtain

If $P_1 \in G^{+}_{U_1,U_2} \subseteq S_{m_1,m_2}$ *and* $P_2 \in G^{+}_{U_2,U_3} \subseteq S_{m_2,m_3}$, *then*

$$P_1P_2 \in G^{+}_{U_1,U_3} \subseteq S_{m_1,m_3}.$$

For other results from this article, we can proceed similarly — we will omit to specify this fact further.

The next result is simple, beautiful, and important — using it, it is proved, *e.g.*, the celebrated theorem of Wielandt for the index of primitivity of a primitive matrix (for this theorem, see, *e.g.*, [9, p. 520] or, here, Theorem 4.24).

THEOREM 2.8. *Let* $P_1 \in \overline{G}^+_{U_1,U_2} \subseteq N_{m_1,m_2}$, $P_2 \in \overline{G}^+_{U_2,U_3} \subseteq N_{m_2,m_3}$, ..., $P_n \in \overline{G}^+_{U_n,U_{n+1}} \subseteq N_{m_n,m_{n+1}}$. *Let* $j \in \langle m_{n+1} \rangle$. *Suppose that* $U_1 = \langle m_1 \rangle$ *and* $U_{n+1} = \{j\}$. *Then*

$$(P_1 P_2 ... P_n)^{\{j\}} > 0$$

(*i.e., the column* j *of* $P_1 P_2 ... P_n$ *is positive*).

Proof. Remark 2.4(a) and Theorem 2.6(ii). ∎

To give an example for the above result, we consider

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \end{pmatrix} \text{ and } P_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix}.$$

Since $P_1 \in \overline{G}^+_{\langle 4 \rangle, \langle 3 \rangle} \subseteq N_4$ ($(P_1)^{\langle 3 \rangle}_{\langle 4 \rangle}$ is row-allowable) and $P_2 \in \overline{G}^+_{\langle 3 \rangle, \{3\}} \subseteq N_4$, we have $(P_1 P_2)^{\{3\}} > 0$ (*i.e.*, the column 3 of $P_1 P_2$ is positive) — this thing can also be obtained by direct computation.

Let $P \in N_{m,n}$. Let $\emptyset \neq U \subseteq \langle m \rangle$ and $\emptyset \neq V \subseteq \langle n \rangle$. Set

$$U \to V \text{ or, if confusion can arise, } U \overset{P}{\to} V$$

if $\forall i \in U$, $\exists j \in V$ such that $P_{ij} > 0$. Set

$$V \leftarrow U \text{ or, if confusion can arise, } V \overset{P}{\leftarrow} U$$

if $U \to V$. Obviously, if $U \to V$ (or if $V \leftarrow U$), then P is sum-positive on $U \times V$.

Remark 2.9. Let $P_1 \in N_{m_1,m_2}$, $P_2 \in N_{m_2,m_3}$, ..., $P_n \in N_{m_n,m_{n+1}}$. Let $j \in \langle m_{n+1} \rangle$. If

$$U_1 = \langle m_1 \rangle \to U_2 \to ... \to U_n \to U_{n+1} = \{j\}$$

(equivalently, if

$$U_{n+1} = \{j\} \leftarrow U_n \leftarrow ... \leftarrow U_2 \leftarrow U_1 = \langle m_1 \rangle \text{)},$$

where, obviously, $\emptyset \neq U_2 \subseteq \langle m_2 \rangle$, ..., $\emptyset \neq U_n \subseteq \langle m_n \rangle$, then, by Theorem 2.8,

$$(P_1 P_2 ... P_n)^{\{j\}} > 0.$$

(We used
$$U_1 = \langle m_1 \rangle \to U_2 \to ... \to U_n \to U_{n+1} = \{j\},$$
not
$$U_1 = \langle m_1 \rangle \overset{P_1}{\to} U_2 \overset{P_2}{\to} ... \overset{P_{n-1}}{\to} U_n \overset{P_n}{\to} U_{n+1} = \{j\},$$
because no confusion can arise.)

Set
$$\mathrm{Par}\,(E) = \{\Delta \mid \Delta \text{ is a partition of } E\},$$
where E is a nonempty set. We will agree that the partitions do not contain the empty set. (E) is the improper (degenerate) partition of E.

Definition 2.10. Let $\Delta_1, \Delta_2 \in \mathrm{Par}(E)$. We say that Δ_1 *is finer than* Δ_2 if $\forall V \in \Delta_1$, $\exists W \in \Delta_2$ such that $V \subseteq W$.

Write $\Delta_1 \preceq \Delta_2$ when Δ_1 is finer than Δ_2.

Let W be a nonempty finite set. Suppose that $W = \{s_1, s_2, ..., s_t\}$. Set
$$(\{i\})_{i\in W} \in \mathrm{Par}\,(W), \ \ (\{i\})_{i\in W} = (\{s_1\}, \{s_2\}, ..., \{s_t\}).$$
E.g.,
$$(\{i\})_{i\in\langle n\rangle} = (\{1\}, \{2\}, ..., \{n\}).$$

Other generalizations of the positive matrices are given below, in Definitions 2.11 and 2.12.

Definition 2.11. Let $P \in N_{m,n}$. Let $\Delta \in \mathrm{Par}(\langle m\rangle)$ and $\Sigma \in \mathrm{Par}(\langle n\rangle)$. We say that P is a $[\Delta]$-*positive matrix on* Σ if $\forall V \in \Sigma$, $\exists U \in \Delta$ such that P is sum-positive on $U \times V$ (equivalently, if $\forall V \in \Sigma$, $\exists U \in \Delta$ such that P_U^V is a row-allowable matrix — see Remark 2.3). A $[\Delta]$-positive matrix on $(\{j\})_{j\in\langle n\rangle}$ is called $[\Delta]$-*positive* for short.

Definition 2.12. Let $P \in N_{m,n}$. Let $\Delta \in \mathrm{Par}(\langle m\rangle)$ and $\Sigma \in \mathrm{Par}(\langle n\rangle)$. We say that P is a Δ-*positive matrix on* Σ if Δ is the least fine partition for which P is a $[\Delta]$-positive matrix on Σ. A Δ-positive matrix on $(\{j\})_{j\in\langle n\rangle}$ is called Δ-*positive* while a $(\langle m\rangle)$-positive matrix on Σ is called *positive on* Σ for short. A positive matrix on $(\{j\})_{j\in\langle n\rangle}$ is called *positive* for short — in this case, $P \in \overline{G}^{+}_{(\langle m\rangle),(\{j\})_{j\in\langle n\rangle}}$, so, $P > 0$, and, therefore, "*positive*" is justified; we have more, namely, $P > 0$ if and only if $P \in \overline{G}^{+}_{(\langle m\rangle),(\{j\})_{j\in\langle n\rangle}}$.

Based on Definition 2.11, we will use the generic names (warning!) "$[\Delta]$-positive matrix/matrices on partitions (of the column index sets)" — this corresponds to the general case from Definition 2.11 — and "$[\Delta]$-positive matrix/matrices" — this corresponds to the special case from Definition 2.11. For generic names based on Definition 2.12, we proceed similarly.

The $[\Delta]$-positive matrices on partitions are both generalizations of the positive matrices — see Definition 2.12 — and of the column-allowable ones — see the next result (the collection of column-allowable matrices and that of $[\Delta]$-positive ones are equal).

THEOREM 2.13. *Let $P \in N_{m,n}$ $(m, n \geq 1)$. Then P is column-allowable if and only if $\exists \Delta \in$Par$(\langle m \rangle)$ such that P is $[\Delta]$-positive.*

Proof. "$\Longrightarrow$" Definition 2.11, taking $\Delta = (\{i\})_{i \in \langle m \rangle}$.
"$\Longleftarrow$" Obvious (see Definition 2.11). ∎

Let $\Delta \in$Par$(\langle m \rangle)$ and $\Sigma \in$Par$(\langle n \rangle)$. Set

$$G^+_{\Delta,\Sigma} = G^+_{\Delta,\Sigma}(m,n) = \{P \mid P \in S_{m,n} \text{ and } P \text{ is a } [\Delta]\text{-positive matrix on } \Sigma\}$$

and

$$\overline{G}^+_{\Delta,\Sigma} = \overline{G}^+_{\Delta,\Sigma}(m,n) = \{P \mid P \in N_{m,n} \text{ and } P \text{ is a } [\Delta]\text{-positive matrix on } \Sigma\}$$

— obviously, $G^+_{\Delta,\Sigma} \subseteq S_{m,n}$, $\overline{G}^+_{\Delta,\Sigma} \subseteq N_{m,n}$, and $G^+_{\Delta,\Sigma} \subseteq \overline{G}^+_{\Delta,\Sigma}$.

When we study or even when we construct products of nonnegative matrices (in particular, products of stochastic matrices) using $G^+_{U,V}$, $\overline{G}^+_{U,V}$, $G^+_{\Delta,\Sigma}$, or $\overline{G}^+_{\Delta,\Sigma}$, we will refer this as the G^+ *method*. G^+ comes from the verb *to group* and its derivatives and the adjective *positive*.

Remark 2.14. The G method from [17] can be renamed the G^s *method*, s coming from the adjective *stable*. But, for simplification, we do not make this thing — on the other hand, it is not strictly necessary.

The next result is a basic one, and is somehow similar to Theorem 2.6.

THEOREM 2.15. (i) *Let $P_1 \in \overline{G}^+_{\Delta_1,\Delta_2} \subseteq N_{m_1,m_2}$ and $P_2 \in \overline{G}^+_{\Delta_2,\Delta_3} \subseteq N_{m_2,m_3}$. Then*

$$P_1P_2 \in \overline{G}^+_{\Delta_1,\Delta_3} \subseteq N_{m_1,m_3}.$$

(ii) (*a generalization of* (i)) *Let $P_1 \in \overline{G}^+_{\Delta_1,\Delta_2} \subseteq N_{m_1,m_2}$, $P_2 \in \overline{G}^+_{\Delta_2,\Delta_3} \subseteq N_{m_2,m_3}$, ..., $P_n \in \overline{G}^+_{\Delta_n,\Delta_{n+1}} \subseteq N_{m_n,m_{n+1}}$. Then*

$$P_1P_2...P_n \in \overline{G}^+_{\Delta_1,\Delta_{n+1}} \subseteq N_{m_1,m_{n+1}}.$$

Proof. (i) Let $W \in \Delta_3$. Since $P_2 \in \overline{G}^+_{\Delta_2,\Delta_3}$, $\exists V \in \Delta_2$ such that P_2 is sum-positive on $V \times W$ — equivalently, $(P_2)^W_V$ is row-allowable. Since $P_1 \in \overline{G}^+_{\Delta_1,\Delta_2}$, $\exists U \in \Delta_1$ such that P_1 is sum-positive on $U \times V$ — equivalently, $(P_1)^V_U$ is row-allowable. By Theorem 2.6(i), P_1P_2 is sum-positive on $U \times W$.

We conclude that

$$P_1P_2 \in \overline{G}^+_{\Delta_1,\Delta_3}.$$

(ii) Induction. ∎

Remark 2.16. For Theorem 2.15, we have, obviously, a similar remark to Remark 2.7 — this is left to the reader.

The next result is somehow similar to Theorem 2.8 and, from [17], to Theorem 1.6(i) — it is also simple, beautiful, and important.

THEOREM 2.17. *Let* $P_1 \in \overline{G}^+_{\Delta_1,\Delta_2} \subseteq N_{m_1,m_2}$, $P_2 \in \overline{G}^+_{\Delta_2,\Delta_3} \subseteq N_{m_2,m_3}$, ..., $P_n \in \overline{G}^+_{\Delta_n,\Delta_{n+1}} \subseteq N_{m_n,m_{n+1}}$. *Suppose that* $\Delta_1 = (\langle m_1\rangle)$ *and* $\Delta_{n+1} = (\{j\})_{j\in\langle m_{n+1}\rangle}$. *Then*

$$P_1P_2...P_n > 0.$$

Proof. Definition 2.12 and Theorem 2.15(ii). ∎

3. g_k^+-MATRICES AND k-CONNECTED GRAPHS

In this section, we define the g_k^+-matrices. These matrices were suggested by the application of G^+ method (when this method was applied), and for them we give certain basic results and some examples. Further, the g_k^+-matrices led to k-connected graphs — necessary and sufficient conditions are given for these graphs.

Definition 3.1. Let $n \geq 2$ ($n \in \mathbb{N}$). Let $P \in N_n$. Let $k \in \langle n-1\rangle$. We say that P is a g_k^+*-matrix* if $\forall F$, $\emptyset \neq F \subset \langle n\rangle$,

$$\exists E,\ \emptyset \neq E \subseteq F^c,\ \ |E| \geq \min\left(k, |F^c|\right),\ \text{and}\ P \in \overline{G}^+_{E,F}$$

(F^c is the complement of F; $P \in \overline{G}^+_{E,F}$ (*i.e.*, P is sum-positive on $E \times F$) is equivalent to P_E^F is a row-allowable matrix (*i.e.*, to $\forall i \in E$, $\exists j \in F$ such that $P_{ij} > 0$)).

Concerning the notion of g_k^+-matrix, g and $+$ come from the G^+ method because the application of this method suggested the consideration of g_k^+-matrices.

If $P \in N_1$ and $P \neq 0$ (equivalently, $P > 0$), then, by definition, P is a g_1^+-matrix.

Let $n \geq 2$. Let $P \in N_n$. Let $\emptyset \neq F \subset \langle n\rangle$. Set

$$D_F = D_F\left(P\right) = \left\{i \mid i \in F^c \text{ and } \exists j \in F \text{ such that } P_{ij} > 0\right\}.$$

THEOREM 3.2. *Let* $n \geq 2$. *Let* $P \in N_n$. *Let* $k \in \langle n-1 \rangle$. *Then* P *is a* g_k^+*-matrix if and only if*

$$|D_F| \geq \min(k, |F^c|), \ \forall F, \ \emptyset \neq F \subset \langle n \rangle .$$

Proof. "$\Longrightarrow$" Let $\emptyset \neq F \subset \langle n \rangle$. By Definition 3.1, $\exists E$, $\emptyset \neq E \subseteq F^c$, $|E| \geq \min(k, |F^c|)$, and $\forall i \in E$, $\exists j \in F$ such that $P_{ij} > 0$. Obviously, $E \subseteq D_F$ ($D_F \subseteq F^c$); further, we have $|D_F| \geq |E|$. So, $|D_F| \geq \min(k, |F^c|)$.

"$\Longleftarrow$" Obvious (taking $E = D_F$, F fixed, $\emptyset \neq F \subset \langle n \rangle$). ∎

Set

$$G_{1,1}^+ = \left\{ P \mid P \in S_1 \text{ and } P \text{ is a } g_1^+\text{-matrix} \right\},$$

$$\overline{G}_{1,1}^+ = \left\{ P \mid P \in N_1 \text{ and } P \text{ is a } g_1^+\text{-matrix} \right\},$$

and, for any $n \geq 2$ and $k \in \langle n-1 \rangle$,

$$G_{n,k}^+ = \left\{ P \mid P \in S_n \text{ and } P \text{ is a } g_k^+\text{-matrix} \right\}$$

and

$$\overline{G}_{n,k}^+ = \left\{ P \mid P \in N_n \text{ and } P \text{ is a } g_k^+\text{-matrix} \right\}.$$

Remark 3.3. (a) Obviously, $G_{1,1}^+ = \{I\}$, I is the identity matrix, $I = (1)$, $\overline{G}_{1,1}^+ = \{P \mid P \in N_1 \text{ and } P > 0\}$, and $G_{1,1}^+ \subset \overline{G}_{1,1}^+$.

(b) Obviously, $G_{n,1}^+ \supseteq G_{n,2}^+ \supseteq ... \supseteq G_{n,n-1}^+$ and $\overline{G}_{n,1}^+ \supseteq \overline{G}_{n,2}^+ \supseteq ... \supseteq \overline{G}_{n,n-1}^+$, $\forall n \geq 2$. Moreover, "$\supseteq$" can be replaced with "$\supset$" in all places. For this, first, we consider the matrix $P \in N_n$ with $P^{\{j\}} > 0$, $\forall j \in \langle n-1 \rangle$ (*i.e.*, the columns $1, 2, ..., n-1$ of P are positive) and $P^{\{n\}}$ has the first k entries greater than 0 and the last $n-k$ entries equal to 0 (therefore, $P_{1n}, P_{2n}, ..., P_{kn} > 0$ and $P_{k+1 \to n} = P_{k+2 \to n} = ... = P_{n-1 \to n} = P_{nn} = 0$), where $k \in \langle n-2 \rangle$. Obviously, $P \in \overline{G}_{n,k}^+$ and $P \notin \overline{G}_{n,k+1}^+$, so, $\overline{G}_{n,k}^+ \supset \overline{G}_{n,k+1}^+$. For stochastic matrices, we can proceed similarly.

(c) Obviously, $G_{n,1}^+ \subset \overline{G}_{n,1}^+$, $G_{n,2}^+ \subset \overline{G}_{n,2}^+$, ..., $G_{n,n-1}^+ \subset \overline{G}_{n,n-1}^+$, $\forall n \geq 2$.

Definition 3.4. (See, *e.g.*, [9, p. 360].) Let $P \in N_n$. We say that P is *reducible* if either

(a) $n = 1$ and $P = 0$

or

(b) $n \geq 2$, $\exists Q \in N_n$, Q is a permutation matrix, and $\exists r \in \langle n-1 \rangle$ such that

$$^{\mathrm{t}}QPQ = \begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix}$$

(^tQ is the transpose of Q), where $X \in N_r$, 0 is a zero matrix, $0 \in N_{r,n-r}$, $Y \in N_{n-r,r}$, and $Z \in N_{n-r,n-r}$ (X, Y, and Z can be zero matrices).

Definition 3.5. (See, *e.g.*, [9, p. 361].) Let $P \in N_n$. We say that P is *irreducible* if it is not reducible.

Remark 3.6. Let $P \in N_n$. By Definitions 2.1 and 3.4, if P is not row-allowable, then it is reducible. By Definitions 2.1 and 3.5, if P is irreducible, then it is row-allowable.

The next result is simple, but useful — a basic result for the irreducible matrices.

THEOREM 3.7. *Let $n \geq 2$. Let $P \in N_n$. Suppose that P is irreducible. Then the following statements hold.*

(i) $\forall i \in \langle n \rangle$, $\exists j \in \langle n \rangle$, $j \neq i$, *such that* $P_{ij} > 0$.

(ii) $\forall i \in \langle n \rangle$, $\exists k \in \langle n \rangle$, $k \neq i$, *such that* $P_{ki} > 0$.

(iii) (*a generalization of* (i) *and* (ii)) $\forall A$, $\emptyset \neq A \subset \langle n \rangle$, $\exists i \in A$, $\exists j \in A^c$ *such that* $P_{ij} > 0$.

Proof. (iii) Let $\emptyset \neq A \subset \langle n \rangle$. If $\forall i \in A$, $\forall j \in A^c$, we have $P_{ij} = 0$, then, by Definition 3.4, P is reducible. Contradiction. ∎

Remark 3.8. Let $n \geq 2$. Let $P \in N_n$. If $\forall i \in \langle n \rangle$, $\exists j, k \in \langle n \rangle$, $j \neq i$, $k \neq i$ ($j = k$ or $j \neq k$) such that $P_{ij} > 0$ and $P_{ki} > 0$, it does not follow that P is irreducible. Indeed, letting

$$P \in N_4,\ P = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

P has the above property, but it is reducible. Note that for $i = 4$, we have $j = k = 3$.

In the next result we give a characterization of the irreducible matrices using the g_1^+-matrices — and thus we have examples of g_1^+-matrices.

THEOREM 3.9. *Let $P \in N_n$, $n \geq 1$. Then P is irreducible if and only if $P \in \overline{G}_{n,1}^{+}$.*

Proof. *Case* 1. $n = 1$. No problem.

Case 2. $n \geq 2$.

"$\Longrightarrow$" Let $\emptyset \neq F \subset \langle n \rangle$. Since P is irreducible and $\emptyset \neq F \subset \langle n \rangle$, using Theorem 3.7(iii), $\exists i \in F^c$, $\exists j \in F$ such that $P_{ij} > 0$. We take $E = \{i\}$, and have $|E| = 1$. So, P is a g_1^+-matrix.

"$\Longleftarrow$" Suppose that P is not irreducible. It follows from Definition 3.4 that $\exists Q \in N_n$, Q is a permutation matrix, and $\exists r \in \langle n-1 \rangle$ such that

$$^{t}QPQ = \begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix},$$

where $X \in N_r$, 0 is a zero matrix, $0 \in N_{r,n-r}$, $Y \in N_{n-r,r}$, and $Z \in N_{n-r,n-r}$. Suppose that the rows of X and of 0 are $i_1, i_2, , ..., i_r$ — the columns of X are $i_1, i_2, , ..., i_r$ — and that the columns of 0 are $i_{r+1}, i_{r+2}, , ..., i_n$. Take $F = \{i_{r+1}, i_{r+2}, , ..., i_n\}$. Then $\forall E$, $\emptyset \neq E \subseteq F^c = \{i_1, i_2, , ..., i_r\}$, $\forall i \in E$, $\forall j \in F$ we have $P_{ij} = 0$. So, $P \notin \overline{G}^{+}_{n,1}$. Contradiction. ∎

THEOREM 3.10. *Let* $P \in N_n$. *If* $P \in \overline{G}^{+}_{n,k}$ *for some* $k \in \langle n-1 \rangle$, *then* P *is row-allowable.*

Proof. Definition 2.1, Remarks 3.3(b) and 3.6, and Theorem 3.9 (any irreducible matrix is row-allowable). ∎

Remark 3.11. If a nonnegative $n \times n$ matrix is row-allowable, it does not follow that it is a g_k^+-matrix for some $k \in \langle n-1 \rangle$ if $n \geq 2$. Any nonnegative $n \times n$ matrix which is row-allowable and reducible is not a g_k^+-matrix, $\forall k \in \langle n-1 \rangle$. *E.g.*, let

$$P \in N_n, \ P = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ n \geq 2.$$

P is row-allowable (it is reducible), but $P \notin \overline{G}^{+}_{n,k}$, $\forall k \in \langle n-1 \rangle$.

THEOREM 3.12. *Let* $n \geq 2$ *and* $k \in \langle n-1 \rangle$. *Let* $P \in N_n$. *If* $P \in \overline{G}^{+}_{n,k}$, *then the matrix* $P^{\{j\}}_{\langle n \rangle - \{j\}}$ *has at least* k *positive entries,* $\forall j \in \langle n \rangle$ (*i.e., in each column,* P *has at least* k *positive entries which are not situated on the main diagonal*).

Proof. Let $j \in \langle n \rangle$. Take $F = \{j\}$. Since $P \in \overline{G}^{+}_{n,k}$, by Definition 3.1,

$$\exists E, \ \emptyset \neq E \subseteq \langle n \rangle - \{j\}, \ |E| \geq \min(k, |\langle n \rangle - \{j\}|), \text{ and } P_{ij} > 0, \ \forall i \in E.$$

We have

$$\min(k, |\langle n \rangle - \{j\}|) = k$$

because $n \geq 2$, $k \in \langle n-1 \rangle$, and $|\{j\}| = 1$. It follows that $|E| \geq k$. Since $P_{ij} > 0$, $\forall i \in E$, further, it follows that, in column j, P has at least k positive entries which are not situated on the main diagonal. ∎

Remark 3.13. Let $n \geq 2$ and $k \in \langle n-1 \rangle$. If a nonnegative $n \times n$ matrix has, in each column, at least k positive entries which are not situated on the main diagonal, it does not follow that it is a g_k^+-matrix. *E.g.*, let

$$P \in N_6,\ P = \begin{pmatrix} 0 & * & * & 0 & 0 & * \\ * & 0 & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & * & * & 0 \end{pmatrix},$$

where "$*$" stands for a positive entry. P has, in each column, at least 2 positive entries which are not situated on the main diagonal. But $P \notin \overline{G}_{6,2}^+$. Indeed, if we take $F = \{1,2,3\}$ — see Definition 3.1 —, E can be ($|E| \geq 2$) $\{4,5\}$, $\{4,6\}$, $\{5,6\}$, or $\{4,5,6\}$, but $P \notin \overline{G}_{E,F}^+$, $\forall E$, E being $\{4,5\}$, $\{4,6\}$, $\{5,6\}$, or $\{4,5,6\}$.

By Remark 3.3(b) and Theorem 3.9 we know that any g_k^+-matrix is irreducible; any irreducible matrix is a g_1^+-matrix — and thus we have examples of g_1^+-matrices. Further, we give other examples of g_k^+-matrices for $k = 1$ or, specially, for $k \geq 2$.

Example 3.14. Let $n \geq 2$. Let $P \in N_n$. If $P > 0$ or, more generally, if P is a matrix with $P_{ij} > 0$, $\forall i,j \in \langle n \rangle$, $i \neq j$, then it is a g_k^+-matrix, $\forall k \in \langle n-1 \rangle$. Moreover, P is a matrix with $P_{ij} > 0$, $\forall i,j \in \langle n \rangle$, $i \neq j$, if and only if it is a g_{n-1}^+-matrix.

The irreducible nonnegative matrices are (either) aperiodic or periodic, see, *e.g.*, [10, pp. 52−53], [21, Sections 1.2 and 1.3 (p. 18, etc.)], and, here, Definition 4.20.

Example 3.15. Consider the periodic irreducible matrix with period t ($t \geq 2$)

$$P = \begin{pmatrix} & Q_1 & & & \\ & & Q_2 & & \\ & & & \ddots & \\ & & & & Q_{t-1} \\ Q_t & & & & \end{pmatrix} \in N_n,$$

where $Q_1 \in N_{n_1,n_2}$, $Q_2 \in N_{n_2,n_3}$, ..., $Q_t \in N_{n_t,n_1}$ $n_1, n_2, ..., n_t \geq 1$ ($n_1 + n_2 + ... + n_t = n$). Suppose that Q_1, Q_2, ..., $Q_t > 0$. Let $1 \leq k \leq m = \min_{1 \leq l \leq t} n_l$ (obviously, in this case, $k \leq n-1$). Then $P \in \overline{G}_{n,k}^+$, $\forall k \in \langle m \rangle$ — it is easy to see this, it is also easy to see that $P \notin \overline{G}_{n,m+1}^+$ (but $P \in \overline{G}_{n,m}^+$).

Definition 3.16. (See, *e.g.*, [9, p. 356] and [10, p. 222].) Let $P \in N_{m,n}$. Set

$$\overline{P} = \left(\overline{P}_{ij}\right) \in N_{m,n},\ \overline{P}_{ij} = \begin{cases} 1 \text{ if } P_{ij} > 0, \\ 0 \text{ if } P_{ij} = 0, \end{cases}$$

$\forall i \in \langle m \rangle, \forall j \in \langle n \rangle$. We call $\overline{P}$ the *indicator* (or *incidence*) *matrix of* P.

Definition 3.17. (See, *e.g.*, [8, p. 61].) Let $P \in N_{m,n}$. Let $B \in N_{m,n}$ be a $(0,1)$-matrix, *i.e.*, B is a matrix with $B_{ij} \in \{0,1\}, \forall i \in \langle m \rangle, \forall j \in \langle n \rangle$. We say that P *has the pattern* B if $\overline{P} \geq B$.

Example 3.18. Consider an aperiodic irreducible matrix $Q \in N_n$ (in this case, Q is a primitive matrix — see, in Section 4, Theorem 4.21). Consider that Q has the pattern $\overline{P}$, where P is the matrix from Example 3.15. *E.g.*, $Q = P + A$ is an aperiodic irreducible matrix and has the pattern $\overline{P}$ if, *e.g.*, $A \in N_n$ and $A_{11} > 0$ or $A = I \in N_n$, I is identity matrix. Then — see Example 3.15 — $Q \in \overline{G}^{+}_{n,k}, \forall k \in \langle m \rangle$, and $Q \notin \overline{G}^{+}_{n,m+1}$.

Below we will characterize the k-connected graphs by means of the g_k^+-matrices — necessary and sufficient conditions are given for these graphs. To do this thing, we need certain notions from the graph theory. For the graph theory (basic notions, notation, etc.), see, *e.g.*, [2]-[4], [6], and [22] — all or some of these references (all are book references) could be available; Wikipedia could also be useful...

We work with nondirected finite graphs. Moreover, we work with graphs without multiple edges, but the loops (not the multiple loops) are allowed.

Definition 3.19. (See, *e.g.*, [9, p. 168].) Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a (nondirected finite) graph (without multiple edges, the loops are allowed), where $\mathcal{V} = \{V_1, V_2, ..., V_n\}$ is the vertex set ($n \geq 1$) and $\mathcal{E}$ is the edge set ($|\mathcal{E}| \geq 0$). Set

$$A = (A_{ij}) \in N_n,\ A_{ij} = \begin{cases} 1 & \text{if } [V_i, V_j] \in \mathcal{E}, \\ 0 & \text{if } [V_i, V_j] \notin \mathcal{E}, \end{cases}$$

$\forall i, j \in \langle n \rangle$. A is called the *adjacency matrix of* $\mathcal{G}$.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph. Let $\mathcal{W} \subset \mathcal{V}$. Consider the graph $\mathcal{G} - \mathcal{W}$. If $\mathcal{W} \neq \emptyset$, $\mathcal{G} - \mathcal{W}$ is the (sub)graph obtained from $\mathcal{G}$ by deleting the vertices in $\mathcal{W}$ together with, if any, their incident edges. $\mathcal{G} - \mathcal{W} = \mathcal{G}$ if $\mathcal{W} = \emptyset$. (See, *e.g.*, [3, p. 9].)

Definition 3.20. (See, *e.g.*, [3, p. 42].) Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with n vertices, $n \geq 1$. Let $\mathcal{C} \subset \mathcal{V}$ with $|\mathcal{C}| \geq 0$. We say that $\mathcal{C}$ is a *vertex cut of* $\mathcal{G}$ if $\mathcal{G} - \mathcal{C}$ is disconnected ($\mathcal{G} - \mathcal{C}$ is disconnected $\Longrightarrow n \geq 2$, $|\mathcal{V} - \mathcal{C}| \geq 2$, and $|\mathcal{C}| \leq n - 2$; when $n = 1$, the graph $\mathcal{G}$ has no vertex cuts ($\emptyset$ is not a vertex cut); when $n \geq 2$ and, moreover, the graph $\mathcal{G}$ is disconnected, $\emptyset$ is a vertex cut). We say that the vertex cut $\mathcal{C}$ is a *k-vertex cut* if $|\mathcal{C}| = k$ (obviously, now is very obviously, $0 \leq k \leq n-2$).

$\mathcal{K}_n$ is the complete graph with n vertices, $n \geq 1$. $\mathcal{K}_1$ has one vertex and no edge. $\cong$ is the isomorphism relation for graphs and $\ncong$ is its negation.

THEOREM 3.21. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with n vertices, $n \geq 1$. Let $\mathcal{G}'$ be the graph obtained from $\mathcal{G}$ deleting, if any, the loops. Then $\mathcal{G}' \ncong \mathcal{K}_n$ if and only if $\exists \mathcal{C}$, $\mathcal{C}$ is a vertex cut of $\mathcal{G}$.*

Proof. *Case* 1. $n = 1$. Nothing to prove.

Case 2. $n \geq 2$.

"$\Longrightarrow$" Since $\mathcal{G}' \ncong \mathcal{K}_n$, it follows that $\exists V, W \in \mathcal{G}$, $V \neq W$, such that V and W are not adjacent. Let

$$\mathcal{C} = \{X \mid X \in \mathcal{V} \text{ and } X \text{ is adjacent to } V\}.$$

We can have $\mathcal{C} = \emptyset$ or $\mathcal{C} \neq \emptyset$ — *e.g.*, $\mathcal{C} = \emptyset$ when $n = 2$. If $\mathcal{C} = \emptyset$, the graph $\mathcal{G}'$ is disconnected while, if $\mathcal{C} \neq \emptyset$, the graph $\mathcal{G}'$ is disconnected or connected. So, both when $\mathcal{C} = \emptyset$ and when $\mathcal{C} \neq \emptyset$, the graph $\mathcal{G}' - \mathcal{C}$ is disconnected. It follows that $\mathcal{G} - \mathcal{C}$ is disconnected. So, $\mathcal{C}$ is a vertex cut of $\mathcal{G}$.

"$\Longleftarrow$" Obvious (because $\mathcal{G} - \mathcal{C}$ is disconnected, so, $\exists Y, Z \in \mathcal{G}$, $Y \neq Z$, such that Y and Z are not adjacent). ∎

Due to Theorem 3.21, the next definition makes sense.

Definition 3.22. (See, *e.g.*, [3, p. 42].) Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with n vertices, $n \geq 1$. Let $\mathcal{G}'$ be the graph obtained from $\mathcal{G}$ deleting, if any, the loops. Set

$$\mathtt{k}(\mathcal{G}) = \begin{cases} \text{the minimum } k \text{ for which } \mathcal{G} \text{ has a } k\text{-vertex cut if } \mathcal{G}' \ncong \mathcal{K}_n, \\ |\mathcal{V}| - 1 \text{ (equivalently, } n-1) \text{ if } \mathcal{G}' \cong \mathcal{K}_n. \end{cases}$$

$\mathtt{k}(\mathcal{G})$ is called the *connectivity of* $\mathcal{G}$.

Definition 3.23. (See, *e.g.*, [3, p. 42].) Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with n vertices, $n \geq 1$. Let $k \geq 0$. We say that $\mathcal{G}$ is *k-connected* if $\mathtt{k}(\mathcal{G}) \geq k$ ($\mathtt{k}(\mathcal{G}) \geq k$ implies $k \leq n-1$, so, $k \in \langle\langle n-1 \rangle\rangle$).

Remark 3.24. (a) $0 \leq \mathtt{k}(\mathcal{G}) \leq n-1$ for any graph $\mathcal{G}$ with n vertices, $n \geq 1$; $\mathtt{k}(\mathcal{K}_1) = 0$, $\mathtt{k}(\mathcal{K}_n) = n-1$.

(b) By Definition 3.23 and (a) any graph (with n vertices, $n \geq 1$) is 0-connected, and, conversely, any 0-connected graph is a graph. Any graph with one vertex or which is disconnected has the connectivity equal to 0, so, it is only 0-connected.

(c) Any connected graph is 1-connected, and, conversely, any 1-connected graph is a connected graph.

(For 0-connected graphs, see, *e.g.*, [6, p. 11] (for k-connected graphs, see Section 1.4 and Chapter 3).)

The next result is a bridge between the k-connected graphs and g_k^+-matrices — and thus we have other examples of g_k^+-matrices.

THEOREM 3.25. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with n vertices, $n \geq 2$. Let $k \in \langle n-1 \rangle$ (this implies $n \geq k+1$). Then $\mathcal{G}$ is k-connected if and only if its adjacency matrix, A, is a g_k^+-matrix.*

Proof. Case 1. $k = 1$. Theorem 3.9, and the fact that $\mathcal{G}$ is connected/1-connected if and only if its adjacency matrix is irreducible.

Case 2. $k \neq 1$. Suppose that $\mathcal{V} = \{V_1, V_2, ..., V_n\}$. Taking into account Definition 3.19, we consider the bijective function $f : \langle n \rangle \longrightarrow \mathcal{V}$, $i \longmapsto f(i) = V_i$. Let $\emptyset \neq T \subset \langle n \rangle$. Set

$$\mathcal{V}_T = \{V_i \mid i \in T \text{ and } f(i) = V_i\}.$$

Obviously, $\emptyset \neq \mathcal{V}_T \subset \mathcal{V}$.

"$\Longrightarrow$" Suppose that A is not a g_k^+-matrix. By Theorem 3.2, $\exists F$, $\emptyset \neq F \subset \langle n \rangle$, such that $|D_F| < \min(k, |F^c|)$ (recall that $k \geq 2$). Since $\mathcal{G}$ is k-connected, it is 1-connected, so, A is irreducible. It follows that $|D_F| \geq 1$. From $|D_F| < \min(k, |F^c|)$, we have $|D_F| < |F^c|$. Therefore, we have $\emptyset \neq D_F \subset F^c$. By the definition of D_F and the fact that $F^c - D_F \neq \emptyset$, we have $A_{ij} = 0$, $\forall i \in F^c - D_F$, $\forall j \in F$ — equivalently, we have $[V_i, V_j] \notin \mathcal{E}$, $\forall i \in F^c - D_F$, $\forall j \in F$. It follows that $\mathcal{V}_{D_F}$ is a vertex cut of $\mathcal{G}$ ($\mathcal{G} - \mathcal{V}_{D_F}$ is disconnected). Since $\mathcal{G}$ is k-connected, $\mathtt{k}(\mathcal{G}) \geq k$. By $k > |D_F|$ and $|D_F| = |\mathcal{V}_{D_F}|$, we obtain $\mathtt{k}(\mathcal{G}) > |\mathcal{V}_{D_F}|$. Contradiction. So, A is a g_k^+-matrix.

"$\Longleftarrow$" Suppose that $\mathcal{G}$ is not k-connected. It follows that $\mathtt{k}(\mathcal{G}) < k$. By Remark 3.3(b), A is a g_1^+-matrix. By Theorem 3.9, A is irreducible. It follows that $\mathcal{G}$ is connected/1-connected. So, $\mathtt{k}(\mathcal{G}) \geq 1$. We now have $1 \leq \mathtt{k}(\mathcal{G}) < k$ ($k \geq 2$). It follows that $\exists \mathcal{C}$, $\mathcal{C}$ is a vertex cut of $\mathcal{G}$, with $1 \leq |\mathcal{C}| < k$. Further, it follows that $\mathcal{G} - \mathcal{C}$ is disconnected, and, as a result, $|\mathcal{V} - \mathcal{C}| \geq 2$ — further, $\exists \mathcal{W}_1, \mathcal{W}_2 \subset \mathcal{V} - \mathcal{C}$, $\mathcal{W}_1, \mathcal{W}_2 \neq \emptyset$, $\mathcal{W}_1 \cap \mathcal{W}_2 = \emptyset$, $\mathcal{W}_1 \cup \mathcal{W}_2 = \mathcal{V} - \mathcal{C}$, $1 \leq |\mathcal{W}_1| \leq |\mathcal{W}_2|$, and $[U, V] \notin \mathcal{E}$, $\forall U \in \mathcal{W}_1$, $\forall V \in \mathcal{W}_2$. Let

$$F = \{i \mid V_i \in \mathcal{W}_1\}.$$

$$(F = \{i \mid V_i \in \mathcal{W}_2\} \text{ is also good.})$$

Obviously, $\emptyset \neq F \subset \langle n \rangle$. Since $\mathcal{C} \neq \emptyset$ (because $|\mathcal{C}| \geq 1$) and $[U, V] \notin \mathcal{E}$, $\forall U \in \mathcal{W}_1$, $\forall V \in \mathcal{W}_2$, we have

$$D_F \subseteq \{i \mid V_i \in \mathcal{C}\}.$$

So, $|D_F| \leq |\mathcal{C}|$. Since $|\mathcal{C}| < k$, we have $|D_F| < k$. On the other hand, we have $|D_F| < |F^c|$ because

$$F^c = \{i \mid V_i \in \mathcal{C} \cup \mathcal{W}_2\} = \{i \mid V_i \in \mathcal{C}\} \cup \{i \mid V_i \in \mathcal{W}_2\} \supset$$

$$\supset \{i \mid V_i \in \mathcal{C}\} \supseteq D_F.$$

From $|D_F| < k$ and $|D_F| < |F^c|$, we have $|D_F| < \min(k, |F^c|)$. Contradiction (because A is a g_k^+-matrix). So, $\mathcal{G}$ is k-connected. ∎

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with n vertices, $n \geq 2$. Let $\emptyset \neq \mathcal{Y} \subset \mathcal{V}$. Set

$$D_{\mathcal{Y}} = \{V \mid V \in \mathcal{Y}^c \text{ and } \exists W \in \mathcal{Y} \text{ such that } [V, W] \in \mathcal{E}\}.$$

We arrived at an interesting result on the k-connected graphs.

THEOREM 3.26. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with n vertices, $n \geq 2$. Let $k \in \langle n-1 \rangle$. Then the following statements are equivalent.*

(i) *The graph $\mathcal{G}$ is k-connected.*

(ii) *The adjacency matrix of $\mathcal{G}$ is a g_k^+-matrix.*

(iii) $\forall \mathcal{Y}$, $\emptyset \neq \mathcal{Y} \subset \mathcal{V}$, $\exists \mathcal{X}$, $\emptyset \neq \mathcal{X} \subseteq \mathcal{Y}^c$, $|\mathcal{X}| \geq \min(k, |\mathcal{Y}^c|)$, *and* $\forall V \in \mathcal{X}$, $\exists W \in \mathcal{Y}$ *such that* $[V, W] \in \mathcal{E}$.

(iv) $|D_{\mathcal{Y}}| \geq \min(k, |\mathcal{Y}^c|)$, $\forall \mathcal{Y}$, $\emptyset \neq \mathcal{Y} \subset \mathcal{V}$.

Proof. Definition 3.1, Theorem 3.2, and Theorem 3.25 and its proof. ∎

We considered nondirected graphs. The directed graphs can also be considered, the strongly connected directed graphs can be considered, ... (For the directed graph (digraph) theory, see, *e.g.*, [1].)

4. APPLICATIONS, I

In this section, we give the first applications of G^+ method, more exactly, of Theorems 2.6 and 2.8. The results, old and new results, are mainly for the irreducible matrices and for the primitive ones, in some cases the matrices being, moreover, g_k^+-matrices (not only irreducible).

Recall that the row-allowable matrices and column-allowable ones were defined in Section 2 (see Definition 2.1). The next theorem refers to these matrices — it is simple, but very useful.

THEOREM 4.1. *Let $P_1 \in N_{n_1,n_2}$ and $P_2 \in N_{n_2,n_3}$, n_1, n_2, $n_3 \geq 1$.*

(i) *If P_1 is row-allowable and $(P_2)^{\{j\}} > 0$ (i.e., the column j of P_2 is positive), then $(P_1P_2)^{\{j\}} > 0$, where $j \in \langle n_3 \rangle$.*

(ii) *If* $(P_1)_{\{i\}} > 0$ (*i.e., the row* i *of* P_1 *is positive*) *and* P_2 *is column-allowable, then* $(P_1P_2)_{\{i\}} > 0$, *where* $i \in \langle n_1 \rangle$.

(iii) *If* $(P_1)^{\{j\}} > 0$, *where* $j \in \langle n_2 \rangle$, *and* P_2 *is row-allowable, then* $\exists k \in \langle n_3 \rangle$ *such that* $(P_1P_2)^{\{k\}} > 0$.

(iv) *If* P_1 *is column-allowable and* $(P_2)_{\{i\}} > 0$, *where* $i \in \langle n_2 \rangle$, *then* $\exists k \in \langle n_1 \rangle$ *such that* $(P_1P_2)_{\{k\}} > 0$.

(v) *If* P_1 *is row-allowable and* P_2 *is positive, then* P_1P_2 *is positive.*

(vi) *If* P_1 *is positive and* P_2 *is column-allowable, then* P_1P_2 *is positive.*

Proof. (i) Obvious. We can even apply Theorem 2.8.
Suppose that P_1 is row-allowable and $(P_2)^{\{j\}} > 0$. Then P_1 is sum-positive on $\langle n_1 \rangle \times \langle n_2 \rangle$ and P_2 is sum-positive on $\langle n_2 \rangle \times \{j\}$. It follows from Theorem 2.8 that $(P_1P_2)^{\{j\}} > 0$.

(ii) The transpose operation and (i).

(iii)-(vi) These are left to the reader — for (v)-(vi), see also Remark 2.4. ∎

Recall that the reducible matrices and irreducible ones were defined in Section 3 (see Definitions 3.4 and 3.5). Recall that the g_k^+-matrices (in particular, the irreducible ones) are row-allowable (see Theorem 3.10).

THEOREM 4.2. (See, *e.g.*, [9, p. 507].) *Let* $P \in N_n$, $n \geq 2$. *Then* P *is irreducible if and only if*

$$(I + P)^{n-1} > 0$$

(*recall that* I *is the identity matrix*).

Proof. "$\Longleftarrow$" Obviously, $I + P$ is irreducible ($I + P$ is reducible $\Longrightarrow (I + P)^{n-1} \not> 0$). So, P is irreducible.

"$\Longrightarrow$" Set $Q = I + P$. Q is irreducible because P is irreducible and $Q \geq P$. Moreover, we have $Q_{ii} > 0$, $\forall i \in \langle n \rangle$. We show that $Q^{n-1} > 0$, equivalently, that (see Remark 2.4)

$$\left(Q^{n-1}\right)^{\{j\}} > 0, \ \forall j \in \langle n \rangle .$$

Let $j \in \langle n \rangle$.

Case 1. $Q^{\{j\}} > 0$. Set $t = t(j) = 1$, and we have $\left(Q^t\right)^{\{j\}} > 0$.

Case 2. $Q^{\{j\}} \not> 0$. This case holds when $n \geq 3$ ($n = 2$, $Q_{ii} > 0$, $\forall i \in \langle n \rangle$, and Q is irreducible $\Longrightarrow Q > 0$). Set — a tail-to-head construction —

$$B_1 = \{j\}$$

and

$$B_{u+1} = B_u \cup \{i \mid i \in \langle n \rangle - B_u \text{ and } \exists k \in B_u \text{ such that } Q_{ik} > 0\},$$

$\forall u = u(j) \geq 1$ with $B_u \subset \langle n \rangle$. By the definition of sets B_1 and B_{u+1}, $u \geq 1$ with $B_u \subset \langle n \rangle$, and Theorem 3.7(iii) ($Q$ is irreducible...) we have

$$B_u \subset B_{u+1}, \ \forall u \geq 1 \text{ with } B_u \subset \langle n \rangle .$$

Since $Q^{\{j\}} \not> 0$, we have $B_2 \subset \langle n \rangle$. Since $B_u \subset B_{u+1} \subseteq \langle n \rangle$, $\forall u \geq 1$ with $B_u \subset \langle n \rangle$, it follows that $\exists u_0 = u_0(j) \in \langle n \rangle - \{1, 2\}$ such that $B_{u_0} = \langle n \rangle$ (recall that $n \geq 3$). Set $t = t(j) = u_0 - 1$. Obviously, $t \in \langle n-1 \rangle$. By the definition of sets B_1 and B_{u+1}, $u \geq 1$ with $B_u \subset \langle n \rangle$, and the fact that $Q_{ii} > 0$, $\forall i \in \langle n \rangle$, we have

$$B_1 = \{j\} \leftarrow B_2 \leftarrow ... \leftarrow B_t \leftarrow B_{t+1} = \langle n \rangle$$

($u_0 = t + 1$). By Theorem 2.8, see also Remark 2.9, we have $\left(Q^t\right)^{\{j\}} > 0$.

From Cases 1 and 2, we have $\left(Q^t\right)^{\{j\}} > 0$, where $t \in \langle n-1 \rangle$,

$$t = \begin{cases} 1 & \text{if } Q^{\{j\}} > 0, \\ u_0 - 1 & \text{if } Q^{\{j\}} \not> 0. \end{cases}$$

If $t = n-1$, no problem ($\left(Q^{n-1}\right)^{\{j\}} > 0$). If $1 \leq t < n-1$, by Theorem 4.1(i) we have

$$\left(Q^{n-1}\right)^{\{j\}} = \left(Q^{n-1-t}Q^t\right)^{\{j\}} = Q^{n-1-t}\left(Q^t\right)^{\{j\}} > 0.$$

∎

Let $x \in \mathbb{R}$. Set $\lfloor x \rfloor = \max\{k \mid k \in \mathbb{Z} \text{ and } k \leq x\}$.

The next result is a generalization of Theorem 4.2, "$\Longrightarrow$".

THEOREM 4.3. *Let* $P \in N_n$, $n \geq 2$. *Let* $k \in \langle n-1 \rangle$. *If* $P \in \overline{G}^+_{n,k}$, *then*

$$(I + P)^m > 0,$$

where $m = \left\lfloor \frac{n-2}{k} \right\rfloor + 1$.

Proof. Theorem 4.10 — a more general result. ∎

Remark 4.4. (a) $(I + P)^m > 0$ is only a necessary condition for a nonnegative $n \times n$ matrix P be a g_k^+-matrix, where $m = \left\lfloor \frac{n-2}{k} \right\rfloor + 1$. Indeed, considering

$$P = \begin{pmatrix} * & * & * & * \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ * & * & * & * \end{pmatrix},$$

where "$*$" stands for a positive entry, we have $(I+P)^2 > 0$, $2 = \left\lfloor \frac{4-2}{2} \right\rfloor + 1$, but P is not a g_2^+-matrix because (see Theorem 3.2) $D_{\{3,4\}} = \{1\}$, $\left|D_{\{3,4\}}\right| = 1 < \min(2,2) = 2$.

(b) By Theorem 4.3, $(I+P)^m \not> 0 \Longrightarrow P \notin \overline{G}^+_{n,k}$, where $P \in N_n$,... So, we have a method to show that a matrix $P \in N_n$ is not a g_k^+-matrix. In particular, due to Theorem 3.25, we have a method to show that a graph with n vertices, $n \geq 2$, is not k-connected.

Definition 4.5. (See, *e.g.*, [9, p. 516] and [14, p. 47].) Let $P \in N_n$, $n \geq 1$. We say that P is *primitive* if it is irreducible and has only one eigenvalue of maximum modulus.

THEOREM 4.6. (See, *e.g.*, [9, p. 516] and [14, p. 49].) *Let* $P \in N_n$, $n \geq 1$. *Then* P *is primitive if and only if* $\exists m \geq 1$ $(m \in \mathbb{N})$ *such that* $P^m > 0$.

Proof. See, *e.g.*, [9, pp. 516−517] and [14, pp. 49−50]. ∎

Let $P \in N_n$ be a primitive matrix. Set (see, *e.g.*, [9, p. 519] and [21, p. 56])

$$\gamma(P) = \text{the least } k \geq 1 \ (k \in \mathbb{N}) \text{ such that } P^k > 0.$$

$\gamma(P)$ is called the *index of primitivity of* P.

Remark 4.7. In general, it is not easy to compute $\gamma(P)$. In some cases (of interest or not) we can obtain good upper bounds for $\gamma(P)$ — the next result is for such a case.

THEOREM 4.8. (See, *e.g.*, [9, p. 517]). *Let* $P \in N_n$, $n \geq 2$, *be an irreducible matrix. If all the main diagonal entries of* P *are positive, then*

$$P^{n-1} > 0$$

(*and, therefore,* P *is primitive and* $\gamma(P) \leq n-1$).

Proof. Similar to the proof of Theorem 4.2, "$\Longrightarrow$". ∎

Remark 4.9. Theorem 4.8 can be generalized as follows. Let $P \in N_n$, $n \geq 2$, be an irreducible matrix. Suppose that all the main diagonal entries of P are positive. Let $P_1, P_2, ..., P_{n-1} \in N_n$. Suppose that $P_1, P_2, ..., P_{n-1}$ have the pattern P (see Definition 3.17). Then $P_1 P_2 ... P_{n-1} > 0$. The proof is left to the reader.

In this article, for simplification, the generalizations as that from Remark 4.9 will be omitted. Another generalization of Theorem 4.8 is given in the next result.

THEOREM 4.10. *Let* $P_1, P_2, ..., P_m \in \overline{G}^+_{n,k}$, *where* $n \geq 2$, $k \in \langle n-1 \rangle$, *and* $m = \left\lfloor \frac{n-2}{k} \right\rfloor + 1$. *Suppose that all the main diagonal entries of* $P_1, P_2, ..., P_m$ *are positive. Then*

$$P_1 P_2 ... P_m > 0.$$

In particular, if $P_1 = P_2 = ... = P_m := P$, *then* $P^m > 0$ *and, as a result,* P *is primitive and*

$$\gamma(P) \leq m = \left\lfloor \frac{n-2}{k} \right\rfloor + 1.$$

Proof. The proof is somehow similar to those of Theorems 4.2, "$\Longrightarrow$", and 4.8. We show that

$$(P_1 P_2 ... P_m)^{\{j\}} > 0, \ \forall j \in \langle n \rangle .$$

Let $j \in \langle n \rangle$.

Case 1. $(P_m)^{\{j\}} > 0$. Set $t = t(j) = 0$.

Case 2. $(P_m)^{\{j\}} \not> 0$. This case holds when $n \geq 3$ and $m \geq 2$ ($n = 2$, $(P_m)_{ii} > 0$, $\forall i \in \langle n \rangle$, and P_m is irreducible $\Longrightarrow P_m > 0$ (P_m is irreducible because $P_m \in \overline{G}^+_{n,k}$); $P_m \in \overline{G}^+_{n,k}$, $(P_m)_{jj} > 0$, and $k = n-1 \Longrightarrow (P_m)^{\{j\}} > 0$ (more generally, $P_m \in \overline{G}^+_{n,k}$, $(P_m)_{ii} > 0$, $\forall i \in \langle n \rangle$, and $k = n-1 \Longrightarrow P_m > 0$); so, $k \in \langle n-2 \rangle$, and, further, we obtain $m \geq 2$). Set — a tail-to-head construction too —

$$B_1 = \{j\}$$

and

$$B_{u+1} = B_u \cup C_u, \ \forall u = u(j) \geq 1 \text{ with } B_u \subset \langle n \rangle ,$$

where

$$C_u = \{ i \mid i \in \langle n \rangle - B_u \text{ and } \exists k \in B_u \text{ such that } (P_{m-u+1})_{ik} > 0 \} ,$$

$\forall u \geq 1$ with $B_u \subset \langle n \rangle$. Obviously, $B_u \cap C_u = \emptyset$, $\forall u \geq 1$ with $B_u \subset \langle n \rangle$. Since $P_l \in \overline{G}^+_{n,k}$, $\forall l \in \langle m \rangle$, on the one hand, using the definition of sets B_1 and B_{u+1}, $u \geq 1$ with $B_u \subset \langle n \rangle$, Remark 3.3(b), and Theorems 3.7(iii) and 3.9, we have

$$B_u \subset B_{u+1} = B_u \cup C_u, \ \forall u \geq 1 \text{ with } B_u \subset \langle n \rangle ,$$

and, on the other hand, we have

$$|B_{u+1}| = |B_u| + |C_u| \geq \begin{cases} |B_u| + k & \text{if } B_{u+1} \subset \langle n \rangle , \\ |B_u| + |\langle n \rangle - B_u| & \text{if } B_{u+1} = \langle n \rangle , \end{cases}$$

$$\geq \begin{cases} |B_{u-1}| + 2k & \text{if } B_{u+1} \subset \langle n \rangle , \\ |B_{u-1}| + k + |\langle n \rangle - B_u| & \text{if } B_{u+1} = \langle n \rangle , \end{cases}$$

$$\geq ... \geq \begin{cases} 1+uk & \text{if } B_{u+1} \subset \langle n \rangle, \\ 1+(u-1)k+|\langle n \rangle - B_u| & \text{if } B_{u+1} = \langle n \rangle, \end{cases}$$

$\forall u \geq 1$ with $B_u \subset \langle n \rangle$. Set

$$a = |\langle n \rangle - B_u| \text{ if } B_{u+1} = \langle n \rangle;$$

obviously, $1 \leq a \leq k$ (see the definition of g_k^+-matrices again). If $B_{u+1} = \langle n \rangle$, we have $|B_{u+1}| = n$, so, in this case,

$$n \geq 1 + (u-1)k + a.$$

Further, we obtain

$$u \leq \frac{n-1+k-a}{k} = \frac{n-1}{k} + \frac{k-a}{k} \leq$$

$$\leq \frac{n-1}{k} + \frac{k-1}{k} = \frac{n-2}{k} + 1.$$

Therefore,

$$u \leq \left\lfloor \frac{n-2}{k} \right\rfloor + 1 = m,$$

and, further,

$$u+1 \leq m+1,$$

and thus we justified that the number of matrices we need must be m (not $m+1$). Since $(P_m)^{\{j\}} \ngtr 0$, we have $B_2 \subset \langle n \rangle$. Since $B_u \subset B_{u+1} \subseteq \langle n \rangle$, $\forall u \geq 1$ with $B_u \subset \langle n \rangle$, it follows that $\exists u_0 = u_0(j) \in \langle m+1 \rangle - \{1,2\}$ such that $B_{u_0} = \langle n \rangle$ (see above ($n \geq 3$, $m \geq 2$, ...)). Set $t = t(j) = u_0 - 2$. Obviously, $t \in \langle m-1 \rangle$. By the definition of sets B_1 and B_{u+1}, $u \geq 1$ with $B_u \subset \langle n \rangle$, and the fact that $(P_l)_{ii} > 0$, $\forall l \in \langle m \rangle, \forall i \in \langle n \rangle$ we have

$$B_1 = \{j\} \leftarrow B_2 = B_1 \cup C_1 \leftarrow ... \leftarrow B_{t+1} = B_t \cup C_t \leftarrow B_{t+2} = \langle n \rangle$$

($u_0 = t+2$). By Theorem 2.8, see also Remark 2.9, we have

$$(P_{m-t}P_{m-t+1}...P_m)^{\{j\}} > 0$$

(recall that $t \in \langle m-1 \rangle$ — so, $t \geq 1$).

From Cases 1 and 2, we have $(P_{m-t}P_{m-t+1}...P_m)^{\{j\}} > 0$, where $t \in \langle\langle m-1 \rangle\rangle$,

$$t = \begin{cases} 0 & \text{if } (P_m)^{\{j\}} > 0, \\ u_0 - 2 & \text{if } (P_m)^{\{j\}} \ngtr 0. \end{cases}$$

If $t = m-1$, no problem $((P_1P_2...P_m)^{\{j\}} > 0)$. If $0 \le t < m-1$, by Theorem 4.1(i) we have

$$(P_1P_2...P_m)^{\{j\}} = ((P_1P_2...P_{m-t-1})(P_{m-t}P_{m-t+1}...P_m))^{\{j\}} =$$

$$= (P_1P_2...P_{m-t-1})(P_{m-t}P_{m-t+1}...P_m)^{\{j\}} > 0.$$

∎

Example 4.11. Let $P \in N_9$,

$$P = \begin{pmatrix} * & 0 & 0 & * & * & * & 0 & 0 & 0 \\ 0 & * & 0 & * & * & * & 0 & 0 & 0 \\ 0 & 0 & * & * & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * & * & * \\ * & * & * & 0 & 0 & 0 & * & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 & * & 0 \\ * & * & * & 0 & 0 & 0 & 0 & 0 & * \end{pmatrix},$$

where "$*$" stands for a positive entry. $P \in \overline{G}^+_{9,3}$. (See Examples 3.15 and 3.18 again.) By Theorem 4.8 we have $P^8 > 0$ while by Theorem 4.10 we have $P^3 > 0$. By direct computation, $P^2 \not> 0$ and $P^3 > 0$.

Remark 4.12. Theorem 4.10 can also be generalized. Let $P_1 \in \overline{G}^+_{n,k_1}$, $P_2 \in \overline{G}^+_{n,k_2}$, ..., $P_m \in \overline{G}^+_{n,k_m}$, where... — the study of this case is left to the reader.

We will give another generalization, a known one, of Theorem 4.8. To prove this generalization using our approach, we need the following result.

THEOREM 4.13. *Let $P \in N_n$, $n \ge 2$, be an irreducible matrix. Suppose that P has d positive main diagonal entries, $d \in \langle n \rangle$ — suppose that $P_{i_ui_u} > 0$, $\forall u \in \langle d \rangle$, where $i_1, i_2, ..., i_d \in \langle n \rangle$, $i_u \ne i_v$, $\forall u, v$, $u \ne v$. Let $W = \{i_1, i_2, ..., i_d\}$. Then*

$$\left(P^{n-d}\right)^W$$

is row-allowable (sum-positive on $\langle n \rangle \times W$) and

$$\left(P^{n-d}\right)_W$$

is column-allowable.

Proof. Case 1. $d = n$. Obvious.

Case 2. $1 \le d < n$. First, we show that $\left(P^{n-d}\right)^W$ is row-allowable. Since $P_{i_u i_u} > 0$, $\forall u \in \langle d \rangle$, it follows that $\left(P^{n-d}\right)_W^W$ is row-allowable. It remains to show that $\left(P^{n-d}\right)_{W^c}^W$ is row-allowable (W^c is the complement W).

Let $i \in W^c$. We show that $P_{ij}^{n-d} > 0$ for some $j \in W$. We can either have $P_{ij} > 0$ for some $j \in W$ or $P_{ij} = 0$, $\forall j \in W$. If $P_{ij} > 0$, we have $P \in \overline{G}^+_{\{i\},\{j\}}$. Obviously, $P^t \in \overline{G}^+_{\{k\},\{k\}}$, $\forall t \ge 0$, $\forall k \in W$. By Theorem 2.6(i), $P^{n-d} \in \overline{G}^+_{\{i\},\{j\}}$ ($P^{n-d} = PP^{n-d-1}$, ...), so, $P_{ij}^{n-d} > 0$. If $P_{ij} = 0$, $\forall j \in W$ — this subcase holds only when $|W^c| > 1$ because P is irreducible —, then $\exists l \in \langle n-d-1 \rangle$, $\exists j_1, j_2, ..., j_l \in W^c$, $j_u \ne j_v$, $\forall u, v \in \langle l \rangle$, $u \ne v$, and $j_u \ne i$, $\forall u \in \langle l \rangle$, $\exists j_{l+1} \in W$ such that $P_{ij_1}, P_{j_1 j_2}, ..., P_{j_l j_{l+1}} > 0$ (because P is irreducible and $|W^c - \{i\}| = n-d-1 > 0$). Further, we have $P \in \overline{G}^+_{\{i\},\{j_1\}} \cap \overline{G}^+_{\{j_1\},\{j_2\}} \cap ... \cap \overline{G}^+_{\{j_l\},\{j_{l+1}\}}$, so, by Theorem 2.6(ii), $P^{l+1} \in \overline{G}^+_{\{i\},\{j_{l+1}\}}$. Therefore, $P_{ij_{l+1}}^{l+1} > 0$.

If $l = n-d-1$, no problem — we have $P_{ij}^{n-d} > 0$ for some $j \in W$ ($j = j_{n-d}$).

If $1 \le l < n-d-1$, using Theorem 2.6(i) for $P^{l+1} \in \overline{G}^+_{\{i\},\{j_{l+1}\}}$ and $P^{n-d-l-1} \in \overline{G}^+_{\{j_{l+1}\},\{j_{l+1}\}}$ (recall that $P^t \in \overline{G}^+_{\{k\},\{k\}}$, $\forall t \ge 1$ (even $\forall t \ge 0$), $\forall k \in W$), we have $P^{n-d} \in \overline{G}^+_{\{i\},\{j_{l+1}\}}$, so, $P_{ij_{l+1}}^{n-d} > 0$.

We conclude that $\left(P^{n-d}\right)^W$ is row-allowable.

The above result leads to the fact that $\left(\left({}^{\mathrm{t}}P\right)^{n-d}\right)^W$ is row-allowable (${}^{\mathrm{t}}P$ is the transpose of P). So, $\left(P^{n-d}\right)_W$ is column-allowable. ∎

Now, we give the generalization of Theorem 4.8 we promised.

THEOREM 4.14. (See, *e.g.*, [9, pp. 520−521].) *Under the same conditions as in Theorem* 4.13 *we have*

$$P^{2n-d-1} > 0$$

(*so*, $\gamma(P) \le 2n-d-1$).

Proof. *Case* 1. $d = n$. Theorem 4.8.

Case 2. $d \in \langle n-1 \rangle$. Proceeding as in the proof of Theorem 4.2, "$\Longrightarrow$" (for the columns $i_1, i_2, ..., i_d$), we have

$$\left(P^{n-1}\right)^W > 0 \text{ and } \left(\left({}^{\mathrm{t}}P\right)^{n-1}\right)^W > 0.$$

The latter result leads to $\left(P^{n-1}\right)_W > 0$. By Theorem 4.13, $\left(P^{n-d}\right)^W$ is row-allowable and $\left(P^{n-d}\right)_W$ is column-allowable. Finally, by Theorem 4.1(v) we have

$$P^{2n-d-1} = P^{n-d}P^{n-1} \geq \left(P^{n-d}\right)^W \left(P^{n-1}\right)_W > 0.$$

(The inequality $P^{2n-d-1} > 0$ can also be obtained using Theorem 2.8 ($P^{n-d} \in \overline{G}^+_{\langle n \rangle, W}$, $P^{n-1} \in \overline{G}^+_{W,\{j\}}$, $\forall j \in \langle n \rangle$, ...) or the fact that $\left(P^{n-1}\right)^W > 0$ and $\left(P^{n-d}\right)_W$ is column-allowable — in the latter case, by Theorem 4.1(vi) we have

$$P^{2n-d-1} = P^{n-1}P^{n-d} \geq \left(P^{n-1}\right)^W \left(P^{n-d}\right)_W > 0.)$$

∎

Theorem 4.14 can be generalized — the next result is a generalization both of Theorem 4.14 and of Theorem 4.10.

THEOREM 4.15. *Let $P_1, P_2, ..., P_{m+1} \in N_n$ be irreducible matrices, where $m = \left\lfloor \frac{n-2}{k} \right\rfloor + 1$, $n \geq 2$, and $k \in \langle n-1 \rangle$. Let $W = \{i_1, i_2, ..., i_d\}$, $d \in \langle n \rangle$.*

(i) *If $(P_1)^W$ is row-allowable (irreducible or not), $(P_l)_{i_u i_u} > 0$, $\forall l \in \langle m+1 \rangle - \{1\}$, $\forall u \in \langle d \rangle$, and $P_2, P_3, ..., P_{m+1} \in \overline{G}^+_{n,k}$, then*

$$P_1 P_2 ... P_{m+1} > 0.$$

This result holds, in particular, for $P_2 = P_3 = ... = P_{m+1} := P$ and *$P_1 = P^{n-d}$, and we have*

$$P^{n+m-d} > 0$$

(so, $\gamma(P) \leq n + \left\lfloor \frac{n-2}{k} \right\rfloor + 1 - d$).

(ii) *If $(P_l)_{i_u i_u} > 0$, $\forall l \in \langle m \rangle$, $\forall u \in \langle d \rangle$, $P_1, P_2, ..., P_m \in \overline{G}^+_{n,k}$, and $(P_{m+1})_W$ is column-allowable (irreducible or not), then*

$$P_1 P_2 ... P_{m+1} > 0.$$

This result holds, in particular, for $P_1 = P_2 = ... = P_m := P$ and *$P_{m+1} = P^{n-d}$, and we have*

$$P^{n+m-d} > 0.$$

Proof. (i) *Case* 1. $d = n$. Theorems 4.1(v) and 4.10.

Case 2. $d \in \langle n-1 \rangle$. Proceeding as in the proof of Theorem 4.10 for the matrices ${}^{\mathrm{t}}P_{m+1}$, ${}^{\mathrm{t}}P_m$, ..., ${}^{\mathrm{t}}P_2$ and columns $i_1, i_2, ..., i_d$, we have

$$\left({}^{\mathrm{t}}P_{m+1} {}^{\mathrm{t}}P_m ... {}^{\mathrm{t}}P_2\right)^W > 0.$$

Further, we have $\left({}^{\mathrm{t}}\left(P_2 P_3 ... P_{m+1}\right)\right)^W > 0$. So, $\left(P_2 P_3 ... P_{m+1}\right)_W > 0$. Finally, by Theorem 4.1(v) we have

$$P_1 P_2 ... P_{m+1} = P_1 \left(P_2 P_3 ... P_{m+1}\right) \geq \left(P_1\right)^W \left(P_2 P_3 ... P_{m+1}\right)_W > 0.$$

(ii) Similar to (i) (using Theorem 4.1(vi) instead of Theorem 4.1(v)). ∎

Remark 4.16. Theorem 4.10 is a special case of Theorem 4.15 — it is the case when $d = n$ (equivalently, $W = \langle n \rangle$), $P_1 = I$ at (i), and $P_{m+1} = I$ at (ii).

Definition 4.17. (See, *e.g.*, [21, p. 11]; see, *e.g.*, also [9, p. 357].) Let $P \in N_n$, $n \geq 1$. Let $i, j \in \langle n \rangle$. Let $i_0, i_1, ..., i_u \in \langle n \rangle$, $u \geq 1$, $i_0 = i$, $i_u = j$. We say that $(i_0, i_1, ..., i_u)$ is a *path/chain* (*of* P) *from* i *to* j if $P_{i_0 i_1}, P_{i_1 i_2}, ..., P_{i_{u-1} i_u} > 0$. u is called the *length of path from* i *to* j. If $i = j$, the path is called a *cycle from* i *to* i (or *from* i *to itself*).

Definition 4.18. (See, *e.g.*, [21, p. 16].) Let $P \in N_n$, $n \geq 1$. Let $i \in \langle n \rangle$. Suppose that there exists a *cycle from* i *to* i. Set $d(i) =$ the greatest common divisor of those $k \geq 1$ for which $P^k_{ii} > 0$ ($P^k_{ii} = \left(P^k\right)_{ii}$). $d(i)$ is called *period of* i.

THEOREM 4.19. (See, *e.g.*, [21, pp. 17−18].) *Let* $P \in N_n$, $n \geq 1$, *be an irreducible matrix. Then* $\forall i, j \in \langle n \rangle$ *we have* $d(i) = d(j)$.

Proof. See, *e.g.*, [21, p. 17, Lemma 1.2, and p. 18, Definition 1.6]. ∎

Definition 4.20. (See, *e.g.*, [21, p. 18].) Let $P \in N_n$, $n \geq 1$, be an irreducible matrix. Set $d = d(1)$ (by Theorem 4.19, $d = d(1) = d(2) = ... = d(n)$). We say that P is *periodic/cyclic* (*with period* d) if $d > 1$. If $d = 1$, we say that P is *aperiodic/acyclic*.

THEOREM 4.21. (See, *e.g.*, [21, p. 21].) *Let* $P \in N_n$, $n \geq 1$. *Then* P *is primitive* (*see Definition* 4.5; *see also Theorem* 4.6) *if and only if it is aperiodic irreducible.*

Proof. See, *e.g.*, [21, p. 21]. ∎

THEOREM 4.22. (See, *e.g.*, [9, pp. 519−520].) *Let* $P \in N_n$, $n \geq 1$, *be a primitive matrix. Let* $s \geq 1$ *be the smallest* (*natural*) *number for which there exists a cycle* (*of* P) *with length* s. *Then*

$$P > 0 \ \text{ if } \ n = 1$$

and

$$P^{n+s(n-2)} > 0 \ (\text{so}, \ \gamma(P) \leq n + s(n-2)\,) \ \text{ if } \ n \geq 2.$$

Proof. No problem when $n = 1$. Further, we consider that $n \geq 2$, and show that

$$\left(P^{n+s(n-2)}\right)^{\{j\}} > 0, \ \forall j \in \langle n \rangle.$$

Fix $j \in \langle n \rangle$. Fix a cycle of P with length s. Suppose that the cycle is $(i_s, i_{s-1}, ..., i_0)$ with $i_s = i_0$, and, as a result, we have $P_{i_s i_{s-1}}, P_{i_{s-1} i_{s-2}}, ..., P_{i_1 i_0} > 0$.

Case 1. j belongs to the (fixed) cycle. Suppose that (no problem) $i_s = i_0 = j$. Set

$$U_0 = \{j\} \text{ and } U_{t+1} = \{i \mid i \in \langle n \rangle \text{ and } \exists k \in U_t \text{ such that } P_{ik} > 0\}, \forall t \geq 0.$$

Obviously, $i_0 \in U_{ks}$, $i_1 \in U_{ks+1}$, ..., $i_{s-1} \in U_{ks+s-1}$, $\forall k \geq 0$. Obviously, we have

$$U_0 = \{j\} \leftarrow U_1 \leftarrow ... \leftarrow U_{s-1} \leftarrow$$

$$\leftarrow U_s = U_0 \cup U_0^{(1)} \leftarrow U_{s+1} = U_1 \cup U_1^{(1)} \leftarrow ... \leftarrow U_{s+s-1} = U_{s-1} \cup U_{s-1}^{(1)} \leftarrow$$

$$\leftarrow U_{2s} = U_s \cup U_0^{(2)} \leftarrow U_{2s+1} = U_{s+1} \cup U_1^{(2)} \leftarrow ... \leftarrow U_{2s+s-1} = U_{s+s-1} \cup U_{s-1}^{(2)} \leftarrow ...,$$

where $U_0^{(1)} = U_s - U_0$, $U_1^{(1)} = U_{s+1} - U_1, ...$ It follows that

$$U_0 \subseteq U_s \subseteq U_{2s} \subseteq ...,$$

$$U_1 \subseteq U_{s+1} \subseteq U_{2s+1} \subseteq ...,$$

$$\vdots$$

$$U_{s-1} \subseteq U_{2s-1} \subseteq U_{3s-1} \subseteq ...$$

We cannot have

$$U_{ts} = U_{(t+1)s} \text{ if } U_{ts} \subset \langle n \rangle,$$

where $t \geq 0$, because P, being primitive, is not cyclic/periodic (see Theorem 4.21). More generally, we cannot have

$$U_{ts+w} = U_{(t+1)s+w} \text{ if } U_{ts+w} \subset \langle n \rangle,$$

where $t \geq 0$ and $w \in \langle\langle s-1 \rangle\rangle$.

U_0 ($U_0 = U_{0 \cdot s}$) has one element. Since P is aperiodic irreducible (not periodic irreducible), it follows that U_s ($U_s = U_{1 \cdot s}$) has at least 2 elements, U_{2s} has at least 3 elements, ..., $U_{(n-1)s}$ has at least n elements ($n-1+1 = n$), and, therefore, $U_{(n-1)s} = \langle n \rangle$. Using Theorem 2.8 for $U_0 = \{j\} \leftarrow U_1 \leftarrow ... \leftarrow U_{(n-1)s} = \langle n \rangle$, we have

$$\left(P^{(n-1)s}\right)^{\{j\}} > 0.$$

Case 2. j does not belong to the cycle. In this case, there exists a path from i_v for some $v \in \langle\langle s-1 \rangle\rangle$ to j with length at most $n - s$ ($s < n$ because

P is primitive, so, $n - s \geq 1$). Consider that this path is $(j_0, j_1, ..., j_z)$, where $j_0 = i_v$ and $j_z = j$, $z \in \langle n - s \rangle$. We have

$$\{j\} = \{j_z\} \leftarrow \{j_{z-1}\} \leftarrow ... \leftarrow \{j_1\} \leftarrow \{j_0\} = \{i_v\}.$$

Further, for i_v, we use Case 1, and keeping notation for sets, U_0, U_1, ..., but, here, $U_0 = \{i_v\}$, and, obviously, keeping the definitions for U_{t+1}, $t \geq 0$, we have $U_0 = \{i_v\} \leftarrow U_1 \leftarrow ... \leftarrow U_{(n-1)s} = \langle n \rangle$. Using Theorem 2.8 for

$$\{j\} = \{j_z\} \leftarrow \{j_{z-1}\} \leftarrow ... \leftarrow \{j_1\} \leftarrow U_0 = \{j_0\} = \{i_v\} \leftarrow ... \leftarrow U_{(n-1)s} = \langle n \rangle,$$

we have

$$\left(P^{z+(n-1)s}\right)^{\{j\}} > 0.$$

Both when $z = n - s$ and when $1 \leq z < n - s$, we have

$$\left(P^{n-s+(n-1)s}\right)^{\{j\}} > 0,$$

in the latter case using Theorem 4.1(i). Since $n - s + (n-1)s = n + s(n-2)$, we have

$$\left(P^{n+s(n-2)}\right)^{\{j\}} > 0.$$

From Cases 1 and 2, since $\max((n-1)s, n + s(n-2)) = n + s(n-2)$, by Theorem 4.1(i) we have

$$\left(P^{n+s(n-2)}\right)^{\{j\}} > 0.$$

∎

Theorem 4.22 can be generalized.

THEOREM 4.23. *Let* $P \in \overline{G}^+_{1,1} \cup \overline{G}^+_{n,k}$ *be a primitive matrix, where* $n \geq 2$ *and* $k \in \langle n-1 \rangle$. *Let* $s \geq 1$ *be the smallest number for which there exists a cycle with length* s. *Then*

$$P^g > 0,$$

where

$$g = \begin{cases} 1 & \text{if } P \in \overline{G}^+_{1,1}, \\ \left\lfloor \frac{n-s-2}{k} \right\rfloor + 2 + s\left[n - \max(2,k) + 1\right] & \text{if } P \in \overline{G}^+_{n,k}. \end{cases}$$

(*For* $n \geq 2$ *and* $k = 1$, *we have* $g = n + s(n-2)$ — *as in Theorem* 4.22.)

Proof. When $P \in \overline{G}^{+}_{1,1}$, we have $P \in N_1$, $P > 0$. So, no problem. Further, we consider that $P \in \overline{G}^{+}_{n,k}$, $n \geq 2$, $k \in \langle n-1 \rangle$, and show that

$$\left(P^g\right)^{\{j\}} > 0, \ \forall j \in \langle n \rangle .$$

Fix $j \in \langle n \rangle$. Fix a cycle of P with length s. Suppose that the cycle is $(i_s, i_{s-1}, ..., i_0)$ with $i_s = i_0$ — it follows that $P_{i_s i_{s-1}}$, $P_{i_{s-1} i_{s-2}}$, ..., $P_{i_1 i_0} > 0$.

Case 1. j belongs to the cycle. We proceed as in Case 1 from the proof of Theorem 4.22, and have a little difference due to the fact that $P \in \overline{G}^{+}_{n,k}$. Suppose that $i_s = i_0 = j$ as well, $U_0 = \{j\}$ as well, $U_{t+1} = ...$ — see there —, $\forall t \geq 0$, as well.

U_0 $(U_0 = U_{0 \cdot s})$ has one element. Since $P \in \overline{G}^{+}_{n,k}$ and is aperiodic irreducible, it follows that U_s $(U_s = U_{1 \cdot s})$ has at least b elements, where $b = \max(2, k)$ (not $b = k + 1$; $b = 2$ for $k = 1$ and $k = 2$; $b = k$ for $k \geq 3$), U_{2s} has at least $b + 1$ elements, U_{3s} has at least $b + 2$ elements, ..., $U_{(n-b+1)s}$ has at least n elements $(b + n - b + 1 - 1 = n)$, and, therefore, $U_{(n-b+1)s} = \langle n \rangle$. Using Theorem 2.8 for $U_0 = \{j\} \leftarrow U_1 \leftarrow ... \leftarrow U_{(n-b+1)s} = \langle n \rangle$, we have

$$\left(P^{(n-b+1)s}\right)^{\{j\}} > 0.$$

Case 2. j does not belong to the cycle. Set

$$B_1 = \{j\}$$

and

$$B_u = \left\{ i \;\middle|\; i \in \langle n \rangle - \bigcup_{t=1}^{u-1} B_t \text{ and } \exists k \in \bigcup_{t=1}^{u-1} B_t \text{ such that } P_{ik} > 0 \right\},$$

$\forall u \geq 2$ with $\left| \bigcup_{t=1}^{u-1} B_t \right| \leq n - s$. Let w be the smallest (natural) number such that $\left| \bigcup_{t=1}^{w} B_t \right| > n - s$. Obviously, $w \geq 2$. Since $P \in \overline{G}^{+}_{n,k}$, we have

$$|B_2| \geq 1$$

when $w = 2$ and

$$|B_2| \geq k, \ |B_3| \geq k, \ ..., \ |B_{w-1}| \geq k, \ |B_w| \geq 1 \ (\,|B_w| \geq k \text{ or } 1 \leq |B_w| < k)$$

when $w \geq 3$. Further, since the sets $B_1, B_2, ..., B_w$ are disjoint, we have

$$\left| \bigcup_{t=1}^{w} B_t \right| = |B_1| + |B_2| + ... + |B_w| \geq 2 + (w-2)\, k.$$

We must consider that

$$2 + (w-2)\,k > n - s.$$

Further, we have

$$w > \frac{n-s-2}{k} + 2,$$

so,

$$w = \left\lfloor \frac{n-s-2}{k} \right\rfloor + 3.$$

By the definition of w, $\exists l \in B_w$ such that l belongs to the cycle. It follows that $l = i_v$ for some $v \in \langle\langle s-1\rangle\rangle$. We have either $B_w = \{i_v\}$ or $B_w \supset \{i_v\}$. If $B_w \supset \{i_v\}$, we can work with $\{i_v\}$ instead of B_w because if a matrix is sum-positive on $(C_1 \cup C_2) \times D$, then it is sum-positive on $C_1 \times D$ and on $C_2 \times D$. So, we can work with $\{i_v\}$ in both cases. We work with $\{i_v\}$ in both cases, and, further, for i_v, we use Case 1, and keeping notation for sets, U_0, U_1, ..., but, here, $U_0 = \{i_v\}$, and, obviously, keeping the definition for U_{t+1}, $t \geq 0$, we have

$$U_0 = \{i_v\} \leftarrow U_1 \leftarrow \ldots \leftarrow U_{(n-b+1)s} = \langle n\rangle.$$

By the definition of sets $B_1, B_2, ..., B_w$ we have

$$\bigcup_{t=1}^{u-1} B_t \leftarrow B_u,\ \forall u,\ 2 \leq u \leq w.$$

But also we have

$$B_{u-1} \leftarrow B_u,\ \forall u,\ 2 \leq u \leq w$$

— we prove this statement. If $w = 2$ (recall that $w \geq 2$), we have only $B_1 \leftarrow B_2$, so, no problem. Further, we consider that $w \geq 3$. If $u = 2$, we have $B_1 \leftarrow B_2$, so, no problem. Further, we consider that $3 \leq u \leq w$. Let $i \in B_u$. Let $k \in \bigcup_{t=1}^{u-1} B_t$ such that $P_{ik} > 0$. If $k \in B_t$ for some $t \in \langle u-2\rangle$, then, using the definition of sets $B_1, B_2, ..., B_w$, we have $i \in B_{t+1}$. Contradiction (because $t+1 \leq u-1 < u$ and $B_u \cap B_{t+1} = \emptyset$). So, $k \in B_{u-1}$, and we have $B_{u-1} \leftarrow B_u$. Using Theorem 2.8 for

$$B_1 = \{j\} \leftarrow B_2 \leftarrow \ldots \leftarrow B_{w-1} \leftarrow U_0 = \{i_v\} \leftarrow \ldots \leftarrow U_{(n-b+1)s} = \langle n\rangle,$$

we have

$$\left(P^{w-1+(n-b+1)s}\right)^{\{j\}} > 0,$$

where

$$w = \left\lfloor \frac{n-s-2}{k} \right\rfloor + 3 \text{ and } b = \max(2, k).$$

From Cases 1 and 2, since $\max\left(\left(n-b+1\right)s, w-1+\left(n-b+1\right)s\right) = w-1+\left(n-b+1\right)s$, by Theorem 4.1(i) we have

$$\left(P^{g}\right)^{\{j\}} > 0,$$

where $g = w-1+\left(n-b+1\right)s = \left\lfloor \frac{n-s-2}{k} \right\rfloor + 2 + s\left[n - \max\left(2,k\right)+1\right]$. ∎

Now, we give Wielandt Theorem and a generalization of it — these are corollaries of Theorems 4.22 and 4.23, respectively.

THEOREM 4.24. *Let $P \in N_n$, $n \geq 1$. Then P is primitive if and only if*

$$P^{n^2-2n+2} > 0.$$

(For this part — this is Wielandt Theorem —, see, *e.g.*, [9, p. 520].)

More generally, if $P \in \overline{G}^{+}_{1,1} \cup \overline{G}^{+}_{n,k}$, $n \geq 2$, $k \in \langle n-1\rangle$, then P is primitive if and only if

$$P^{h} > 0,$$

where

$$h = \begin{cases} 1 & \text{if } P \in \overline{G}^{+}_{1,1}, \\ n^2-2n+2 & \text{if } P \in \overline{G}^{+}_{n,1}, \\ \left\lfloor \frac{n-m-3}{k} \right\rfloor + 2 + \left(m+1\right)\left(n-k+1\right) & \text{if } P \in \overline{G}^{+}_{n,k},\ k \neq 1, \end{cases}$$

$$m = \left\lfloor \frac{n-2}{k} \right\rfloor + 1 \text{ if } P \in \overline{G}^{+}_{n,k}.$$

Proof. We consider the more general case. If $P \in \overline{G}^{+}_{1,1}$, then $P \in N_1$, $P > 0$. So, no problem. Further, we consider that $P \in \overline{G}^{+}_{n,k}$, $n \geq 2$, $k \in \langle n-1\rangle$.

" $\Longrightarrow$ " Since $P \in \overline{G}^{+}_{n,k}$, we have $s \in \langle m+1\rangle$, s is that from Theorem 4.23 — we prove this statement. P being irreducible, any $j \in \langle n\rangle$ belongs to a cycle. Fix $j \in \langle n\rangle$. When $P_{jj} > 0$, we have $s = 1$, and, therefore, $s \in \langle m+1\rangle$. When $P_{jj} = 0$, set

$$C_0 = \{j\},$$

$$C_v = \left\{ i \;\middle|\; i \in \langle n\rangle - \bigcup_{t=0}^{v-1} C_t \text{ and } \exists k \in \bigcup_{t=0}^{v-1} C_t \text{ such that } P_{ik} > 0 \right\},$$

$\forall v \geq 1$ with $\bigcup_{t=0}^{v-1} C_t \subset \langle n\rangle$. C_0, C_1, ..., C_u are disjoint when $\bigcup_{t=0}^{u-1} C_t \subset \langle n\rangle$ and $\bigcup_{t=0}^{u} C_t = \langle n\rangle$ — this thing happens for some $u \geq 1$. It follows that

$$|C_0| + |C_1| + ... + |C_u| = n.$$

Since $P \in \overline{G}^+_{n,k}$, we have

$$|C_1| \geq k, \ \ |C_2| \geq k, \ \ ..., \ \ |C_{u-1}| \geq k, \ \ |C_u| \geq a,$$

where $a = n - \left| \bigcup_{t=0}^{u-1} C_t \right|$ — obviously, $1 \leq a \leq k$. So,

$$n \geq 1 + (u-1)\, k + a$$

Further, we proceed as in the proof of Theorem 4.10, and obtain that $u \leq m$. Further, we have — see the proof of Theorem 4.23, Case 2, for a similar situation —

$$C_0 = \{j\} \leftarrow C_1 \leftarrow ... \leftarrow C_u.$$

Further, since P is irreducible, $\exists q \in \langle u \rangle$ (q is unique or not; $q = u$ or $q \neq u$) such that

$$C_q \leftarrow \{j\}\,.$$

So, $1 \leq s \leq m+1$. Finally, both when $P_{jj} > 0$ and when $P_{jj} = 0$, we have $s \in \langle m+1 \rangle$ — the statement was proved.

Fix n and k; $n \geq 2$, $k \in \langle n-1 \rangle$. In this case, g from Theorem 4.23 depends only on s, and we consider the function

$$g\,(s) = g = \left\lfloor \frac{n-s-2}{k} \right\rfloor + 2 + s\,[n - \max{(2,k)} + 1]\,, \ \forall s \in \langle m+1 \rangle\,.$$

We show that g is increasing — moreover, it is strictly increasing if $n \geq 3$ ($m \geq 2 \Longrightarrow n \geq 3$; $n \geq 3 \not\Longrightarrow m \geq 2$). Fix $s \in \langle m \rangle$.

Case 1. $k = 1$. We have

$$g\,(s+1) - g\,(s) = n - 2 \begin{cases} = 0 & \text{if } n = 2, \\ > 0 & \text{if } n \geq 3. \end{cases}$$

Case 2. $2 \leq k \leq n-1$. We have

$$g\,(s+1) - g\,(s) = n - k + 1 + \left\lfloor \frac{n-(s+1)-2}{k} \right\rfloor - \left\lfloor \frac{n-s-2}{k} \right\rfloor.$$

Further, using the fact that $x - 1 < \lfloor x \rfloor \leq x$, $\forall x \in \mathbb{R}$, we have

$$g\,(s+1) - g\,(s) > n - k + 1 + \frac{n-(s+1)-2}{k} - 1 - \frac{n-s-2}{k} =$$

$$= n - k - \frac{1}{k} > n - k - 1 \geq 0$$

(because $2 \leq k \leq n-1$).

From Cases 1 and 2, a maximum value of g is $g(m+1)$;

$$g(m+1)=\left\lfloor\frac{n-m-3}{k}\right\rfloor+2+(m+1)\left[n-\max(2,k)+1\right].$$

If $s=m+1$, no problem — we have $P^{g(m+1)}>0$. If $1\leq s\leq m$, by Theorems 4.1(v) and 4.23 we have

$$P^{g(m+1)}=P^{g(m+1)-g(s)}P^{g(s)}>0$$

($g(s)=g$, g from Theorem 4.23, $P^{g(s)}>0$, ...).

In general, we cannot have $s\in\langle m\rangle$ — see/study for this fact the case when

$$P\in N_3,\ P=\begin{pmatrix}0&*&*\\ *&0&*\\ *&*&0\end{pmatrix},$$

where "$*$" stands for a positive entry. But when $k=1$, we have $s\in\langle m\rangle$, $m=n-1$ (if $s=n$, we obtain that P is periodic with period n — contradiction, see Theorem 4.21), and can obtain a better result. For $k=1$, we consider the function $g\,|_{\langle m\rangle}$, the restriction of g to $\langle m\rangle$. This function is also increasing, so, a maximum value of it is $g(m)$. Since $m=n-1$, we obtain

$$g(n-1)=n^2-2n+2.$$

If $s=n-1$, no problem — we have $P^{g(n-1)}>0$. If $1\leq s\leq n-2$, by Theorems 4.1(v) and 4.23 we have

$$P^{g(n-1)}=P^{g(n-1)-g(s)}P^{g(s)}>0.$$

"$\Longleftarrow$" Theorem 4.6. ∎

The first upper bound, n^2-2n+2, from Theorem 4.24 for the index of primitivity is sharp — for this, see the next example.

Example 4.25. Consider the Wielandt matrix,

$$P\in N_n,\ P=\begin{pmatrix}0&1&0&0&\cdots&0\\ 0&0&1&0&\cdots&0\\ \vdots&\vdots&\vdots&\ddots&\cdots&\vdots\\ 0&0&0&\cdots&\ddots&0\\ 0&0&0&0&\cdots&1\\ 1&1&0&0&\cdots&0\end{pmatrix},$$

see, *e.g.*, [9, Problem 4, p. 522]. We compute $\gamma(P)$ using the G^+ method, more precisely, Theorem 2.8. Since $\gamma(P)=\gamma\left({}^{\mathrm{t}}P\right)$, we compute $\gamma\left({}^{\mathrm{t}}P\right)$ — we have a reason, see below. Set $Q = {}^{\mathrm{t}}P$. We consider that $Q = (Q_{ij})_{i,j \in \langle\langle n-1 \rangle\rangle}$ (not $Q = (Q_{ij})_{i,j \in \langle n \rangle}$) — we made this thing to use the addition modulo n. Set

$$U_0 = \{0\}$$

and

$$U_t = \{i \mid i \in \langle\langle n-1 \rangle\rangle \text{ and } \exists j \in U_{t-1} \text{ such that } Q_{ij} > 0\}, \ \forall t \geq 1.$$

We have

$$Q \in N_n, \ Q = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

and

$$U_0 = \{0\} \leftarrow U_1 = \{1\} \leftarrow \ldots \leftarrow U_{n-1} = \{n-1\} \leftarrow \ldots$$

(here is the reason for the utilization of ${}^{\mathrm{t}}P$— to have $\{0\} \leftarrow \{1\} \leftarrow \ldots \leftarrow \{n-1\}$)

$$\leftarrow U_n = U_{n+0\cdot(n-1)} = \{0,1\} = \{n \bmod n, (n+1) \bmod n\} \leftarrow$$

$$\leftarrow U_{n+1} = \{1,2\} = \{(n+1) \bmod n, (n+2) \bmod n\} \leftarrow \ldots$$

$$\ldots \leftarrow U_{n+(n-2)} = \{n-2, n-1\} =$$

$$= \{(n+(n-2)) \bmod n, (n+(n-1)) \bmod n\} \leftarrow$$

$$\leftarrow U_{n+(n-1)} = U_{n+1\cdot(n-1)} = \{n-1, 0, 1\} =$$

$$= \{(n+(n-1)) \bmod n, (n+n) \bmod n, (n+(n+1)) \bmod n\} \leftarrow \ldots$$

$$\ldots \leftarrow U_{n+2\cdot(n-1)} \leftarrow \ldots \ldots \leftarrow U_{n+(n-2)(n-1)} = \langle\langle n-1 \rangle\rangle$$

— $U_{n+0\cdot(n-1)}$ has 2 elements, $U_{n+1\cdot(n-1)}$ has 3 elements, and proceeding in this way, we obtain that $U_{n+2(n-1)}$ has 4 elements, ..., $U_{n+(n-2)(n-1)}$ has n elements ($(n-2)+2=n$), so, $U_{n+(n-2)(n-1)} = \langle\langle n-1 \rangle\rangle$. Obviously,

$$|U_0| \leq |U_1| \leq \ldots \leq \left|U_{n+(n-2)(n-1)}\right|$$

and

$$|U_{n-1}| = 1 < |U_n| = \left|U_{n+0\cdot(n-1)}\right| = 2 \leq \ldots \leq \left|U_{n+(n-1)-1}\right| = 2 <$$

$$< \left|U_{n+(n-1)}\right| = \left|U_{n+1\cdot(n-1)}\right| = 3 \leq \dots \dots$$

$$\dots \dots \leq \left|U_{n+(n-2)(n-1)-1}\right| = n-1 < \left|U_{n+(n-2)(n-1)}\right| = n.$$

Since the number of sets U_t, $t \in \langle\langle n + (n-2)(n-1)\rangle\rangle$, is

$$n + (n-2)(n-1) + 1, \textit{i.e.}, \text{ is } n^2 - 2n + 3,$$

by Theorem 2.8 (see also Remark 2.9) we have

$$\left(Q^{n^2-2n+2}\right)^{\{0\}} > 0,$$

$$\left(Q^{n^2-2n+2-1}\right)^{\{1\}} = \left(Q^{n^2-2n+1}\right)^{\{1\}} > 0, \dots$$

$$\dots, \left(Q^{n^2-2n+2-(n-1)}\right)^{\{n-1\}} = \left(Q^{n^2-3(n-1)}\right)^{\{n-1\}} > 0.$$

We cannot have

$$\left(Q^k\right)^{\{0\}} > 0$$

for some $k \in \langle n^2 - 2n + 1\rangle$ because $U_t \subset \langle\langle n-1\rangle\rangle$, $\forall t \in \left\langle\left\langle n^2 - 2n + 2\right\rangle\right\rangle$. It follows that

$$Q^{n^2-2n+1} \not> 0 \text{ and } Q^{n^2-2n+2} > 0.$$

So, we have

$$\gamma(Q) = n^2 - 2n + 2.$$

Problem 4.26. (A good problem for the reader.) Is the second upper bound, h, from Theorem 4.24 for the index of primitivity sharp?

Remark 4.27. (See, *e.g.*, [9, p. 521].) To verify that a matrix, P, $P \in N_n$, is irreducible using Theorem 4.2, one way, a good one, is, when $P \not> 0$, to compute $(I+P)^2$, $(I+P)^4$, $(I+P)^8$, ..., $(I+P)^{2^t}$, t being the smallest integer for which $2^t \geq n-1$. We can proceed similarly, also when $P \not> 0$, to verify that a matrix, P, $P \in N_n$, is primitive using Theorem 4.24. For more information, see, *e.g.*, [9, p. 521]; see also Remark 4.4(b).

5. APPLICATIONS, II

In this section, we give other applications of the G^+ method — these are also applications of Theorems 2.6 and 2.8. The results are also old and new, and are for the irreducible matrices, for the reducible ones, for the fully indecomposable ones, for the scrambling ones, and for the Sarymsakov ones, in some cases the matrices being, moreover, g_k^+-matrices (not only irreducible). Finally, we mention an application of Theorem 2.17.

For the first part of the next result, see, *e.g.*, [14, p. 6] (in this book, this part is considered as being important — see, below, Remark 5.2).

THEOREM 5.1. *Let $y \in \mathbb{R}^n$, $n \geq 2$, be a nonnegative column vector with exactly h positive coordinates, $h \in \langle n-1 \rangle$. If $P \in N_n$, $n \geq 2$, is an irreducible matrix, then $(I+P)\,y$ has more than h positive coordinates. More generally, if $P \in \overline{G}^+_{n,k}$, $n \geq 2$, $k \in \langle n-1 \rangle$, then $(I+P)\,y$ has more than $h + \min(k, n-h) - 1$ positive coordinates.*

Proof. We prove the more general result. Let $Y \in N_{n,1}$ (therefore, Y is a nonnegative column matrix), $Y_{i1} = y_i$, $\forall i \in \langle n \rangle$. Due this definition, we can work with Y instead of y — we do this thing. Suppose that $Y_{i_1 1}, Y_{i_2 1}, ..., Y_{i_h 1} > 0$, where $i_1, i_2, ..., i_h \in \langle n \rangle$, $i_u \neq i_v$, $\forall u, v \in \langle h \rangle$, $u \neq v$. Set $V = \{i_1, i_2, ..., i_h\}$. It follows that $Y \in \overline{G}^+_{V,\{1\}}$.

Now, we consider $I + P$. $I + P \in \overline{G}^+_{n,k}$ because $P \in \overline{G}^+_{n,k}$. Further, we have, by Theorem 3.2, $\emptyset \neq D_V = D_V\,(I+P) \subseteq V^c$ and

$$|D_V| \geq \min(k, |V^c|) = \min(k, n-h)\,.$$

By Definition 3.1 (taking $E = D_V$), $I+P \in \overline{G}^+_{D_V,V}$. On the other hand, $I+P \in \overline{G}^+_{V,V}$. So, $I + P \in \overline{G}^+_{V \cup D_V, V}$. By Theorem 2.6(i), $(I+P)\,Y \in \overline{G}^+_{V \cup D_V,\{1\}}$. So, $(I+P)\,Y$ — this is a nonnegative column matrix — has $|V \cup D_V|$ positive entries. It follows that $(I+P)\,Y$ has at least

$$h + \min(k, n-h)$$

positive entries because

$$|V \cup D_V| = |V| + |D_V| \geq h + \min(k, n-h)$$

(recall that $\emptyset \neq D_V \subseteq V^c$; so, $D_V \cap V = \emptyset$). ∎

Remark 5.2. (a) The above result can be used to give another proof of Theorem 4.2, "$\Longrightarrow$", see, *e.g.*, [14, p. 6] — [14] contains other interesting applications of Theorem 5.1 (see Corollary 2.1, p. 6, Theorems 2.2 and 2.3, p. 7, Corollary 4.2, p. 16, ...).

(b) Theorem 5.1 and the hypotheses of Theorems 4.8 and 4.10 suggest the following result: if $y \in \mathbb{R}^n$, $n \geq 2$, is a column vector with exactly h positive coordinates, $h \in \langle n-1 \rangle$, and if $P \in N_n$, $n \geq 2$, is an irreducible matrix and all the main diagonal entries of P are positive, then Py has more than h positive coordinates. More generally, if ... — the completion and proof(s) are left to the reader.

Definition 5.3. (See, *e.g.*, [8, pp. 34−35] and, better, [14, p. 82].) Let $P \in N_n$, $n \geq 2$. Let $s \in \langle n-1 \rangle$. We say that P is *partly decomposable* if it contains an $s \times (n-s)$ zero submatrix.

Definition 5.4. (See, *e.g.*, [8, p. 35] and [14, pp. 82−83].) Let $P \in N_n$, $n \geq 2$. We say that P is *fully indecomposable* if it is not partly decomposable.

By definition, the 1×1 zero matrix is partly decomposable while a nonzero 1×1 matrix is fully indecomposable (see, *e.g.*, [14, p. 83]).

Remark 5.5. The partly decomposable matrices are either irreducible or reducible; they are generalizations of reducible matrices (any reducible matrix is partly decomposable). The fully indecomposable matrices are irreducible (not reducible), and, as a result, they are row-allowable.

Remark 5.6. (See, *e.g.*, [8, p. 35].) Let $P \in N_n$, $n \geq 2$. Then P is fully indecomposable if and only if whenever it contains a $p \times q$ zero submatrix, then $p + q \leq n - 1$.

THEOREM 5.7. (See, *e.g.*, [8, p. 37].) *Let $P_1, P_2, ..., P_{n-1} \in N_n$, $n \geq 2$, be fully indecomposable matrices. Then*

$$P_1 P_2 ... P_{n-1} > 0.$$

Proof. The proof is less formal than that of Theorem 4.2, "$\Longrightarrow$", or that of Theorem 4.10. We show that

$$(P_1 P_2 ... P_{n-1})^{\{j\}} > 0, \ \forall j \in \langle n \rangle.$$

Let $j \in \langle n \rangle$.

Case 1. $(P_{n-1})^{\{j\}} > 0$. When $n = 2$, we have only one matrix, P_1 — nothing to prove. ($n = 2$ and P_1 is fully indecomposable $\Longrightarrow P_1 > 0$.) When $n \geq 3$, by Theorem 4.1(i) (the fully indecomposable matrices are, by Remark 5.5 or Remark 5.6, row-allowable),

$$(P_1 P_2 ... P_{n-1})^{\{j\}} = (P_1 P_2 ... P_{n-2}) (P_{n-1})^{\{j\}} > 0.$$

Case 2. $(P_{n-1})^{\{j\}} \ngtr 0$. This case holds when $n \geq 3$ (see above for $n = 2$). Set

$$B_1 = \{j\} \text{ and } B_2 = \left\{ i \,\middle|\, i \in \langle n \rangle \text{ and } (P_{n-1})_{ij} > 0 \right\}.$$

$B_2 \neq \langle n \rangle$ because $(P_{n-1})^{\{j\}} \ngtr 0$. It follows that $(P_{n-1})^{B_1}_{B_2^c}$ is a $|B_2^c| \times |B_1|$ zero submatrix. By Remark 5.6 we have

$$|B_2^c| + |B_1| \leq n - 1.$$

Further, we have

$$n - |B_2| + |B_1| \leq n - 1,$$

and, therefore,

$$|B_2| \geq |B_1| + 1.$$

So,

$$|B_2| > |B_1|.$$

Moreover, $n > |B_2| > |B_1|$ (no contradiction, because $n > 2$). Set

$$B_3 = \{i \mid i \in \langle n \rangle \text{ and } (P_{n-2})_{ik} > 0 \text{ for some } k \in B_2\}.$$

If $B_3 = \langle n \rangle$, using Theorem 2.8 for $\{j\} = B_1 \leftarrow B_2 \leftarrow B_3 = \langle n \rangle$, we have

$$(P_{n-2}P_{n-1})^{\{j\}} > 0.$$

If $n = 3$, no problem. If $n > 3$, by Theorem 4.1(i) we have

$$(P_1P_2...P_{n-1})^{\{j\}} = (P_1P_2...P_{n-3})(P_{n-2}P_{n-1})^{\{j\}} > 0$$

(by Remark 5.5, P_1, P_2, ..., P_{n-3} are row-allowable; $P_1P_2...P_{n-3}$ is row-allowable...). If $B_3 \subset \langle n \rangle$, proceeding as above, $(P_{n-2})_{B_3^c}^{B_2}$ is a $|B_3^c| \times |B_2|$ zero submatrix... We proceed in this way until we obtain a set B_v with $B_v = \langle n \rangle$, where $v \geq 4$. We have $v \leq n$ because $1 = |B_1| < |B_2| < ... < |B_v| = n$. Using Theorem 2.8 for $\{j\} = B_1 \leftarrow B_2 \leftarrow ... \leftarrow B_v = \langle n \rangle$, we have

$$(P_{n-v+1}P_{n-v+2}...P_{n-1})^{\{j\}} > 0.$$

If $v = n$, no problem. If $v < n$, by Theorem 4.1(i) we have

$$(P_1P_2...P_{n-1})^{\{j\}} = (P_1P_2...P_{n-v})(P_{n-v+1}P_{n-v+2}...P_{n-1})^{\{j\}} > 0.$$

∎

The next result is a generalization of Theorem 5.7.

THEOREM 5.8. *Let* $P_1, P_2, ..., P_m \in \overline{G}^+_{n,k}$, *where*

$$m = \begin{cases} n-1 & \text{if } k = 1, \\ n-k+1 & \text{if } k \geq 2, \end{cases}$$

$n \geq 2$, *and* $k \in \langle n-1 \rangle$. *If* $P_1, P_2, ..., P_m$ *are fully indecomposable, then*

$$P_1P_2...P_m > 0.$$

Proof. For $k = 1$, see Theorem 5.7. Further, we suppose that $k \geq 2$. The proof is similar to that of Theorem 5.7. Proceeding as there, for $j \in \langle n \rangle$ fixed, we have 2 cases, when $(P_m)^{\{j\}} > 0$ and when $(P_m)^{\{j\}} \ngtr 0$. The first case is similar to Case 1 from the proof of Theorem 5.7. The second one is similar to Case 2 from the proof of Theorem 5.7 — we construct/have the sets $B_1, B_2, ..., B_v$, $B_1 = \{j\} \subset B_2 \subset ... \subset B_v = \langle n \rangle$, so, $1 = |B_1| < |B_2| < ... < |B_v| = n$. We have only a little difference: in the proof of Theorem 5.7, Case 2,

$$|B_2| \geq |B_1| + 1, \ \ |B_3| \geq |B_2| + 1, \ \ ..., \ \ |B_v| \geq |B_{v-1}| + 1,$$

while here, since $P_1, P_2, ..., P_m \in \overline{G}^+_{n,k}$, we have

$$|B_2| \geq \max(k, |B_1| + 1) = \max(k, 2) = k,$$

$$|B_3| \geq \max(k, |B_2| + 1) \geq \max(k, k + 1) = k + 1,$$

$$\vdots$$

$$|B_{n-k+2}| \geq k + (n - k) = n$$

— therefore, $v \leq n - k + 2$, and the number of matrices we need (the worst case) is $n - k + 1$, *i.e.*, is m (we justified the definition of m for $k \geq 2$). ∎

By Theorem 5.8, if $P \in \overline{G}^+_{n,k}$ and is fully indecomposable, then — it is interesting this thing —

$$\gamma(P) \leq \begin{cases} n - 1 & \text{if } k = 1, \\ n - k + 1 & \text{if } k \geq 2. \end{cases}$$

The nonnegative matrices which have at least one positive column are of interest, *e.g.*, in the finite Markov chain theory, and, in this case, the matrices are stochastic. If $P \in S_r$ (P can be considered as being the transition matrix of a Markov chain) and has at least one positive column, then $\mu(P) > 0$, where

$$\mu(P) = \max_{j \in \langle r \rangle} \left(\min_{i \in \langle r \rangle} P_{ij} \right),$$

$\mu(P)$ is the Markov ergodicity coefficient of P (see, *e.g.*, [10, pp. 56–57]), and, as a result — this is important —, $\lim_{n \to \infty} P^n$ exists. *E.g.*,

$$P = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

has a positive column (moreover, it is a primitive matrix, and a main diagonal entry of it is 0), and

$$\lim_{n \to \infty} P^n = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

Another example, when

$$P = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

(P has a positive column; moreover, it is reducible and all the main diagonal entries of it are positive), is left to the reader. For more information, see, *e.g.*, [10, Section 1.11 and Chapter 4]; see, *e.g.*, also [8, pp. 61–62].

Definition 5.9. (See, *e.g.*, [10, p. 57] and [21, p. 140].) Let $P \in N_{m,n}$, $m, n \geq 1$. We say that P is a *Markov matrix* if it has at least one positive column (equivalently, if P is sum-positive on $\langle m \rangle \times \{j\}$ for some $j \in \langle n \rangle$).

Theorem 2.8 is a general result for the products of nonnegative matrices, products which are Markov — it is also a general result to show that a product of nonnegative matrices is positive. Below we give a few results for the products of nonnegative matrices, products which have a positive column (therefore, are Markov or more).

THEOREM 5.10. (See, *e.g.*, [8, p. 62] — our result is better.) *Let $P \in N_n$, $n \geq 2$, be in the partitioned form*

$$P = \begin{pmatrix} Q & 0 \\ R & T \end{pmatrix},$$

where $Q \in N_m$ and is a primitive matrix ($1 \leq m < n$). If for any $i \in \{m+1, m+2, ..., n\}$ ($\{m+1, m+2, ..., n\} = \langle m \rangle^c$), there exists a path from i to some, say, j, $j \in \langle m \rangle$, $j = j(i)$ (see Definition 4.17), then

$$P^{\gamma(Q)(n-m+1)}$$

has the first m columns positive (equivalently,

$$\left(P^{\gamma(Q)(n-m+1)}\right)^{\langle m \rangle} > 0),$$

and, more generally,

$$\left(P^{t(n-m+1)}\right)^{\langle m \rangle} > 0, \ \forall t \geq \gamma(Q)$$

($\gamma(Q)$ is the index of primitivity of Q).

Proof. Since $Q^{\gamma(Q)} > 0$, we have

$$\left(P^{\gamma(Q)}\right)^{\langle m \rangle}_{\langle m \rangle} > 0.$$

Let $Z = P^{\gamma(Q)}$. Z also has the above property — a path property: $\forall i \in \langle m \rangle^c$, there exists a path from i to some, say, k, $k \in \langle m \rangle$, $k = k(i)$ (an elementary proof — use Definition 4.17 and the definition of P). We show that $\forall j \in \langle m \rangle$, $\exists l = l(j) \in \langle n-m+1 \rangle$ such that

$$\left(Z^l\right)^{\{j\}} > 0.$$

Let $j \in \langle m \rangle$. Set

$$D_1 = \{j\}.$$

If $Z^{\{j\}} > 0$, no problem, we take $l = 1$. If $Z^{\{j\}} \not> 0$, we consider a new set,

$$D_2 = D_1 \cup \{i \mid i \in \langle n \rangle \text{ and } Z_{ij} > 0\}.$$

Since $Z_{ij} > 0$, $\forall i \in \langle m \rangle$, we have $D_2 \supseteq \langle m \rangle$ (not $\supset \langle m \rangle$, because we can have $Z_{ij} = 0$, $\forall i \in \langle m \rangle^c$). Since $Z^{\{j\}} \not> 0$, we have $D_2 \subset \langle n \rangle$. So, $\langle n \rangle \supset D_2 \supseteq \langle m \rangle$. Further, we set

$$D_3 = D_2 \cup \{i \mid i \in \langle n \rangle \text{ and } \exists k \in D_2 \text{ such that } Z_{ik} > 0\}.$$

Obviously, $D_3 \supset D_2$ because $\langle n \rangle \supset D_2 \supseteq \langle m \rangle$ and Z has the above path property. If $D_3 = \langle n \rangle$, using Theorem 2.8, we have $\left(Z^2\right)^{\{j\}} > 0$, and we take $l = 2$. If $D_3 \subset \langle n \rangle$, we construct a new set,

$$D_4 = D_3 \cup \{i \mid i \in \langle n \rangle \text{ and } \exists k \in D_3 \text{ such that } Z_{ik} > 0\}.$$

Obviously, $D_4 \supset D_3$ because $\langle n \rangle \supset D_3 \supset D_2 \supseteq \langle m \rangle$ and Z has the above path property. If ... We proceed in this way until we obtain a set D_u, $u = u(j) \geq 1$, with $D_u = \langle n \rangle$. Obviously, $1 \leq u \leq 2 + n - m$ because $\langle m \rangle \subseteq D_2 \subset D_3 \subset \ldots \subset D_u = \langle n \rangle$. In this case (when we have $D_u = \langle n \rangle$), using Theorem 2.8 for $D_1 \leftarrow D_2 \leftarrow \ldots \leftarrow D_u$, we have $\left(Z^{u-1}\right)^{\{j\}} > 0$, and we take $l = u - 1$. It follows that $l \leq n - m + 1$.

From the above results, by Theorem 4.1(i) (P is row-allowable $\Longrightarrow \forall t \geq 1$, P^t is row-allowable $\Longrightarrow \forall t \geq 1$, Z^t is row-allowable),

$$\left(Z^{n-m+1}\right)^{\langle m \rangle} > 0.$$

So,

$$\left(P^{\gamma(Q)(n-m+1)}\right)^{\langle m \rangle} > 0,$$

and, more generally,

$$\left(P^{t(n-m+1)}\right)^{\langle m \rangle} > 0, \ \forall t \geq \gamma(Q).$$

∎

Definition 5.11. (See, *e.g.*, [8, p. 63], [10, p. 57], and [21, p. 143].) Let $P \in N_{m,n}$, $m \geq 2$, $n \geq 1$. We say that P is a *scrambling matrix* if $\forall i, j \in \langle m \rangle$, $i \neq j$, $\exists k \in \langle n \rangle$ such that $P_{ik}, P_{jk} > 0$ (equivalently, if $\forall i, j \in \langle m \rangle$, $i \neq j$, P is sum-positive on $\{i, j\} \times \{k\}$ for some $k \in \langle n \rangle$).

THEOREM 5.12. (See, e.g., [8, p. 63] — our result is better.) *Let $P \in N_n$, $n \geq 2$. If P is a scrambling matrix, then*

$$P^{n-1}$$

has a positive column, i.e., is a Markov matrix.

Proof. Case 1. P has a positive column. (Obviously, in this case, P is a scrambling matrix.) It follows that P^{n-1} has a positive column too because the scrambling matrices are row-allowable.

Case 2. P has no positive columns. (Obviously, there exist scrambling matrices which have no positive columns.) We do a head-to-tail construction for Theorem 2.8 (to apply Theorem 2.8). Set

$$U_1 = \langle n \rangle \text{ and } t_1 = |U_1| = n.$$

Let $j_1^{(1)}$, $j_2^{(1)}$, ..., $j_{n-1}^{(1)} \in \langle n \rangle$ such that

$$P_{1j_1^{(1)}},\ P_{nj_1^{(1)}} > 0,\ P_{2j_2^{(1)}},\ P_{nj_2^{(1)}} > 0,\ ...,\ P_{n-1 \to j_{n-1}^{(1)}},\ P_{nj_{n-1}^{(1)}} > 0$$

— this thing can be done because P is a scrambling matrix. Set

$$U_2 = \bigcup_{l=1}^{n-1} \left\{ j_l^{(1)} \right\} \text{ and } t_2 = |U_2|.$$

We have $t_2 > 1$ because P has no positive columns. Obviously, $U_2 \subset \langle n \rangle$. So, $1 < t_2 < t_1$. Consider that

$$U_2 = \left\{ i_1^{(2)}, i_2^{(2)}, ..., i_{t_2}^{(2)} \right\}.$$

Let $j_1^{(2)}$, $j_2^{(2)}$, ..., $j_{t_2-1}^{(2)} \in \langle n \rangle$ such that

$$P_{i_1^{(2)} j_1^{(2)}},\ P_{i_{t_2}^{(2)} j_1^{(2)}} > 0,\ P_{i_2^{(2)} j_2^{(2)}},\ P_{i_{t_2}^{(2)} j_2^{(2)}} > 0,\ ...,\ P_{i_{t_2}^{(2)} - 1 \to j_{t_2-1}^{(2)}},\ P_{i_{t_2}^{(2)}, j_{t_2-1}^{(2)}} > 0.$$

Set

$$U_3 = \bigcup_{l=1}^{t_2-1} \left\{ j_l^{(2)} \right\} \text{ and } t_3 = |U_3|.$$

Obviously, we have $t_2 > t_3 \geq 1$. Consider that

$$U_3 = \left\{ i_1^{(3)}, i_2^{(3)}, ..., i_{t_3}^{(3)} \right\}.$$

If $t_3 = 1$, since $\langle n \rangle = U_1 \to U_2 \to U_3$, by Theorem 2.8 we have

$$\left(P^2\right)^{\left\{i_1^{(3)}\right\}} > 0.$$

If $t_3 > 1$, let $j_1^{(3)}$, $j_2^{(3)}$, ..., $j_{t_3-1}^{(3)} \in \langle n \rangle$ such that... We proceed in this way until we find a set U_v with $|U_v| := t_v = 1$ — this thing happens because $n = t_1 > t_2 > ... > t_v$, and, obviously, $2 \leq v \leq n$. By Theorem 2.8, since $\langle n \rangle = U_1 \to U_2 \to ... \to U_v$ and $|U_v| = 1$, we have

$$\left(P^{v-1}\right)^{U_v} > 0.$$

If $v = n$, no problem. If $2 \leq v < n$, by Theorem 4.1(iii) ($P^{n-1} = P^{v-1}P^{n-v}$, P^{n-v} is row-allowable, ...), $\exists k \in \langle n \rangle$ such that

$$\left(P^{n-1}\right)^{\{k\}} > 0,$$

i.e., P^{n-1} is a Markov matrix. ∎

Theorem 5.12 can be generalized (see also [8, p. 64, Theorem 4.6] for a less good result).

THEOREM 5.13. *Let* $P_1, P_2, ..., P_{n-1} \in N_n$, $n \geq 2$. *If* $P_1, P_2, ..., P_{n-1}$ *are scrambling matrices, then*

$$P_1 P_2 ... P_{n-1}$$

is a Markov matrix. More generally, if $P_1 \in N_{n_1,n_2}$, $P_2 \in N_{n_2,n_3}$, ..., $P_t \in N_{n_t,n_{t+1}}$, $n_1, n_2, ..., n_t \geq 2$, $n_{t+1} \geq 1$, $t = n_1 - 1$, and $P_1, P_2, ..., P_t$ *are scrambling matrices, then*

$$P_1 P_2 ... P_t$$

is a Markov matrix — we even have a better result, namely,

$$P_1 P_2 ... P_z$$

is a Markov matrix, where $z = \min\limits_{0 \leq u \leq t-1} \left(u + n_{u+1} - 1\right)$.

Proof. Similar to the proof of Theorem 5.12. As to z, $z \leq t$ because $u + n_{u+1} - 1$ is equal to t if $u = 0$; we have u matrices before the matrix P_{u+1}, and when $u + n_{u+1} - 1 \leq t$, the number of matrices P_{u+1}, P_{u+2}, ..., $P_{u+n_{u+1}-1}$ is $n_{u+1} - 1$, so, when $u + n_{u+1} - 1 \leq t$, we have $u + n_{u+1} - 1$ matrices; the cases when $u + n_{u+1} - 1 > t$ do not count because $z \leq t$. ∎

Let $P \in N_{m,n}$, $m, n \geq 1$. Let $\emptyset \neq T \subseteq \langle m \rangle$. Set

$$F(T) = \{ j \mid j \in \langle n \rangle \text{ and for which } \exists i \in T \text{ such that } P_{ij} > 0 \}.$$

We call $F(T)$ the *set of consequent indices of* (*set of indices*) T (see, *e.g.*, [8, p. 64] and [21, p. 146]).

Definition 5.14. (See, *e.g.*, [8, p. 65], [21, p. 146], and [23].) Let $P \in N_{m,n}$, $m \geq 2$, $n \geq 1$. We say that P is a *Sarymsakov matrix* if $\forall I, J$, $\emptyset \neq I, J \subseteq \langle m \rangle$, $I \cap J = \emptyset$, either

(1) $F(I) \cap F(J) \neq \emptyset$

or

(2) $F(I) \cap F(J) = \emptyset$ and $|F(I) \cup F(J)| > |I \cup J|$.

Remark 5.15. (a) If P is a Sarymsakov matrix, then it is a row-allowable matrix (see, *e.g.*, [8, p. 65]).

(b) If P is a scrambling matrix, then it is a Sarymsakov matrix (see, *e.g.*, [21, p. 146]).

THEOREM 5.16. *Let $P \in N_{m,n}$ and $Q \in N_{n,p}$, $m \geq 2$, $n, p \geq 1$. If P is sum-positive on $\{i, j\} \times \{k\}$ ($\{i, j\} \subseteq \langle m \rangle$, $\{k\} \subseteq \langle n \rangle$) and Q is row-allowable or, more generally, $Q_{\{k\}}$ is row-allowable, then PQ is sum-positive on $\{i, j\} \times \{l\}$ for some $l \in \langle p \rangle$.*

Proof. Since P is sum-positive on $\{i, j\} \times \{k\}$ and Q is sum-positive on $\{k\} \times \{l\}$ for some $l \in \langle p \rangle$, then, by Theorem 2.6(i), PQ is sum-positive on $\{i, j\} \times \{l\}$ for some $l \in \langle p \rangle$. ∎

THEOREM 5.17. (See, *e.g.*, [8, p. 66] and [21, p. 146].) *Let P_1, P_2, ..., $P_{n-1} \in N_n$, $n \geq 2$. If P_1, P_2, ..., P_{n-1} are Sarymsakov matrices, then*

$$P_1 P_2 ... P_{n-1}$$

is a scrambling matrix.

Proof. See, *e.g.*, [8, p. 66] and [21, p. 146] — Remark 5.15(a) and Theorem 5.16 (not using our terminology) are used. ∎

Remark 5.18. Theorem 5.17 can be generalized considering $P_1 \in N_{m_1,m_2}$, $P_2 \in N_{m_2,m_3}$, ..., $P_{n-1} \in N_{m_{n-1},m_n}$, m_1, m_2, ..., $m_{n-1} \geq 2$, $m_n \geq 1$, m_1, m_2, ..., $m_{n-1} \leq n$, and P_1, P_2, ..., P_{n-1} be Sarymsakov matrices.

Recall that one of our aim is to obtain Markov matrices — these are of interest, *e.g.*, in the finite Markov chain theory. It is also of interest in the finite Markov chain theory to obtain scrambling matrices, see, *e.g.*, Theorem

5.17 for an example, because, if, *e.g.*, a stochastic $r \times r$ matrix, $r \geq 2$, say, P, is scrambling, then $\alpha(P) > 0$, where

$$\alpha(P) = \min_{1 \leq i,j \leq r} \sum_{k=1}^{r} \min(P_{ik}, P_{jk}),$$

$\alpha(P)$ is the Dobrushin ergodicity coefficient of P (see, *e.g.*, [10, pp. 56−57]), and, as a result — this is important —, $\lim_{n\to\infty} P^n$ exists.

THEOREM 5.19. *Let* $P_1, P_2, ..., P_{(n-1)^2} \in N_n$, $n \geq 2$. *If* $P_1, P_2, ..., P_{(n-1)^2}$ *are Sarymsakov matrices, then*

$$P_1 P_2 ... P_{(n-1)^2}$$

is a Markov matrix. In particular, if $P_1 = P_2 = ... = P_{(n-1)^2} := P$, *then*

$$P^{(n-1)^2}$$

is a Markov matrix.

Proof. Theorems 5.13 and 5.17. ∎

Recall that the G^+ method was suggested by the G method and Theorem 2.2, both from [17]. Theorem 2.2 from [17] can be proved — it is easy — using Theorem 2.17 from here. So, we have an application of Theorem 2.17, an important one because Theorem 2.2 from [17] was used to show that the transition matrix of our hybrid Metropolis-Hastings chain from [17] is positive, see Theorem 2.3 in [17], see also Theorem 1.2 in [20] — interestingly, this positive matrix is a product of stochastic matrices, all being reducible or, excepting the first one, the others are reducible. Since our Gibbs sampler in a generalized sense (see, *e.g.*, [20] for this chain) is a special case, an important one, of our hybrid Metropolis-Hastings chain, special cases for Theorem 2.3 from [17] can be found where we applied our Gibbs sampler in a generalized sense, see, *e.g.*, [18] and [19] — in each application from there, the transition matrix of our Gibbs sampler in a generalized sense is a product of reducible stochastic matrices; moreover, this transition matrix is stable (for stable matrices, see, *e.g.*, [19]). Another special case for Theorem 2.3 from [17] is at the cyclic Gibbs sampler in the finite case because this chain is a special case of our Gibbs sampler in a generalized sense (and, therefore, a special case of our hybrid Metropolis-Hastings chain).

Other applications, important applications, of Theorem 2.17 will be found — we believe this — in the future.

At present we have two G-type methods, the G method and G^+ method, and the results obtained using them are impressive.

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