ROMAI J., v.22, no.1(2026), 89–127

G METHOD AND FINITE-TIME CONSENSUS

Udrea Păun

"Gheorghe Mihoc - Caius Iacob" Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, Bucharest, Romania

upterra@gmail.com

Abstract We give an extension of the G method, with results, the extension and results being partly suggested by the finite Markov chains and specially by the finite-time consensus problem for the DeGroot model and that for the DeGroot model on distributed systems. For the (homogeneous and nonhomogeneous) DeGroot model, using the G method, a result for reaching a partial or total consensus in a finite time is given. Further, we consider a special submodel/case of the DeGroot model, with examples and comments — a subset/subgroup property is discovered. For the DeGroot model on distributed systems, using the G method too, we have a result for reaching a partial or total (distributed) consensus in a finite time similar to that for the DeGroot model for reaching a partial or total consensus in a finite time. Then we show that for any connected graph having 2^m vertices, $m \geq 1$, and a spanning subgraph isomorphic to the m-cube graph, distributed averaging is performed in m steps — this result can be extended — research work — for any graph with $n_1 n_2 ... n_t$ vertices under certain conditions, where $t, n_1, n_2, ..., n_t \geq 2$, and, in this case, distributed averaging is performed in t steps.

Keywords: generalized stable matrix, G method, DeGroot model, consensus problem, finite-time consensus, finite-time partial consensus, subset/subgroup property, distributed system, DeGroot model on distributed systems, finite-time distributed consensus, finite-time partial distributed consensus, finite-time distributed averaging.

2020 MSC: 15B48, 15B51, 60J10, 68M14, 68W15, 91D15, 93D50.

1. AN EXTENSION OF *G* METHOD

In this section, we give an extension of the G method [18]-[19], see also [17] — a good thing both for the forward products of matrices, in particular, for the finite Markov chains, and for the backwards products of matrices, in particular, for the DeGroot model and for the DeGroot model on distributed systems.

Set

$$\operatorname{Par}(E) = \{\Delta \mid \Delta \text{ is a partition of } E\},$$

where E is a nonempty set. We shall agree that the partitions do not contain the empty set. (E) is the improper (degenerate) partition of E — we need this partition.

Definition 1.1. *Let $\Delta_1, \Delta_2 \in$Par(E). We say that Δ_1 is finer than Δ_2 if $\forall V \in \Delta_1$, $\exists W \in \Delta_2$ such that $V \subseteq W$.*

Write $\Delta_1 \preceq \Delta_2$ when Δ_1 is finer than Δ_2.

In this article, a vector is a row vector and a stochastic matrix is a row stochastic matrix.

Set

$$\langle m \rangle = \{1, 2, ..., m\} \ \ (m \geq 1).$$

Let Z be an $m \times n$ matrix. Z_{ij} or, if confusion can arise, $Z_{i \to j}$ denotes the entry (i, j) of matrix Z, $\forall i \in \langle m \rangle$, $\forall j \in \langle n \rangle$.

Set

$$C_{m,n} = \{P \mid P \text{ is a complex } m \times n \text{ matrix}\},$$

$$R_{m,n} = \{P \mid P \text{ is a real } m \times n \text{ matrix}\},$$

$$N_{m,n} = \{P \mid P \text{ is a nonnegative } m \times n \text{ matrix}\},$$

$$S_{m,n} = \{P \mid P \text{ is a stochastic } m \times n \text{ matrix}\}$$

(...the stochastic matrices are defined in Definition 1.3),

$$C_n = C_{n,n},$$

$$R_n = R_{n,n},$$

$$N_n = N_{n,n},$$

$$S_n = S_{n,n}.$$

Let $P = (P_{ij}) \in C_{m,n}$. Let $\emptyset \neq U \subseteq \langle m \rangle$ and $\emptyset \neq V \subseteq \langle n \rangle$. Set the matrices

$$P_U = (P_{ij})_{i \in U, j \in \langle n \rangle}, \ P^V = (P_{ij})_{i \in \langle m \rangle, j \in V}, \text{ and } P_U^V = (P_{ij})_{i \in U, j \in V}.$$

Remark 1.2. (Basic properties of the operators $(\cdot)_{(\cdot)}, (\cdot)^{(\cdot)}$, and $(\cdot)_{(\cdot)}^{(\cdot)}$ on the products of complex matrices.) Let $P_1 \in C_{n_1,n_2}$, $P_2 \in C_{n_2,n_3}$, ..., $P_t \in C_{n_t,n_{t+1}}$, where $t \geq 2$. Let $k \in \langle t-1 \rangle$. Let $\emptyset \neq U \subseteq \langle n_1 \rangle$ and $\emptyset \neq V \subseteq \langle n_{t+1} \rangle$. Obviously, we have

(a)

$$(P_1 P_2 ... P_t)_U = (P_1 P_2 ... P_k)_U P_{k+1} P_{k+2} ... P_t;$$

(b)

$$(P_1 P_2 ... P_t)^V = P_1 P_2 ... P_k (P_{k+1} P_{k+2} ... P_t)^V;$$

(c)

$$(P_1 P_2 ... P_t)_U^V = (P_1 P_2 ... P_k)_U (P_{k+1} P_{k+2} ... P_t)^V;$$

(d)

$$(P_1P_2...P_t)_U^V = \sum_{W\in\Delta} (P_1P_2...P_k)_U^W (P_{k+1}P_{k+2}...P_t)_W^V, \ \forall\Delta \in \mathrm{Par}(\langle n_{k+1}\rangle);$$

(e)

$$(P_1P_2...P_t)_U^V \geq (P_1P_2...P_k)_U^W (P_{k+1}P_{k+2}...P_t)_W^V, \ \forall W, \ \emptyset \neq W \subseteq \langle n_{k+1}\rangle.$$

((c)$\Longrightarrow$(d)$\Longrightarrow$(e))

Let W be a nonempty finite set. Suppose that $W = \{s_1, s_2, ..., s_t\}$. Set

$$(\{i\})_{i\in W} \in \mathrm{Par}(W), \ \ (\{i\})_{i\in W} = (\{s_1\}, \{s_2\}, ..., \{s_t\}).$$

E.g.,

$$(\{i\})_{i\in\langle n\rangle} = (\{1\}, \{2\}, ..., \{n\}).$$

$(\{i\})_{i\in W}$ is the finest partition of W while (W) is the coarsest partition of W. Let $L \subset W$. Let $K = L^c$, L^c is the complement of L — so, $\emptyset \neq K \subseteq W$. Suppose that $K = \{s_{j_1}, s_{j_2}, ..., s_{j_u}\}$ $(1 \leq u \leq t)$. Set

$$(\{i\}, L)_{i\in K} \in \mathrm{Par}(W), \ \ (\{i\}, L)_{i\in K} = \begin{cases} (\{i\})_{i\in W} \ \text{ if } |L| = 0, \\ (\{s_{j_1}\}, \{s_{j_2}\}, ..., \{s_{j_u}\}, L) \ \text{ if } |L| \geq 1 \end{cases}$$

($|\cdot|$ is the cardinality). Obviously, $(\{i\}, L)_{i\in K} = (\{i\})_{i\in W}$ if $|L| = 1$. So,

$$(\{i\}, L)_{i\in K} = (\{i\})_{i\in W} \ \text{ if } |L| \leq 1.$$

Further, we suppose that $t \geq 2$. Let $(L_1, L_2, ..., L_m) \in$Par(W), where $m \geq 2$ $(L_1, L_2, ..., L_m \neq \emptyset$; $m \geq 2 \Longrightarrow t \geq 2)$. Suppose that $L_1 = \{s_{l_1}, s_{l_2}, ..., s_{l_v}\}$ $(v \geq 1)$. Set

$$(\{i\}, L_2, ..., L_m)_{i\in L_1} \in \mathrm{Par}(W),$$

$$(\{i\}, L_2, ..., L_m)_{i\in L_1} = (\{s_{l_1}\}, \{s_{l_2}\}, ..., \{s_{l_v}\}, L_2, ..., L_m).$$

E.g.,

$$(\{i\}, \{4,5\}, \{6,7,8\})_{i\in\langle 3\rangle} = (\{1\}, \{2\}, \{3\}, \{4,5\}, \{6,7,8\}).$$

$(\{i\}, L)_{i\in K}$ is the finest partition of W when $|L| \leq 1$; $(\{i\}, L)_{i\in K}$ is the finest partition (of W) among the partitions of W which contain L when $|L| > 1$ while $(\{i\}, L_2, ..., L_m)_{i\in L_1}$ is the finest partition (of W) among the partitions of W which contain $L_2, ..., L_m$. Obviously, $(\{i\}, L_2, ..., L_m)_{i\in L_1} = (\{i\})_{i\in W}$ when $|L_2| = ... = |L_m| = 1$. Obviously, the partition $(\{i\}, L)_{i\in K}$ when $|L| \geq 1$ is a special one of $(\{i\}, L_2, ..., L_m)_{i\in L_1}$.

Definition 1.3. *Let $P \in C_{m,n}$. We say that P is a generalized stochastic (complex) matrix if $\exists a \in \mathbb{C}$ $(a = 0$ or $a \neq 0)$ such that*

$$\sum_{j \in \langle n \rangle} P_{ij} = a, \ \forall i \in \langle m \rangle .$$

(We have generalized stochastic complex matrices, generalized stochastic real matrices, generalized stochastic nonnegative matrices... If $P \in N_{m,n}$, then P is a generalized stochastic (nonnegative) matrix if and only if $\exists a \geq 0$, $\exists Q \in S_{m,n}$ such that $P = aQ$ — see, *e.g.*, [17, Definition 1.2].) The generalized stochastic matrices with row sums equal to 1 are called *stochastic complex* when the matrices are complex, *stochastic real* when the matrices are real, *stochastic nonnegative* or, for short, *stochastic* when the matrices are nonnegative...

Definition 1.4. (A generalization of Definition 1.3 from [17].)
*Let $P \in C_{m,n}$. Let $\Delta \in$*Par$(\langle m \rangle)$ *and $\Sigma \in$*Par$(\langle n \rangle)$. *We say that P is a $[\Delta]$-stable matrix on Σ if P_K^L is a generalized stochastic matrix, $\forall K \in \Delta$, $\forall L \in \Sigma$. In particular, a $[\Delta]$-stable matrix on $(\{j\})_{j \in \langle n \rangle}$ is called $[\Delta]$-stable for short.*

Definition 1.5.(A generalization of Definition 1.4 from [17].)
Let $P \in C_{m,n}$. Let $\Delta \in$Par$(\langle m \rangle)$ *and* $\Sigma \in$Par$(\langle n \rangle)$. *We say that P is a Δ-stable matrix on Σ if Δ is the least fine partition for which P is a $[\Delta]$-stable matrix on Σ. In particular, a Δ-stable matrix on $(\{j\})_{j \in \langle n \rangle}$ is called Δ-stable while a $(\langle m \rangle)$-stable matrix on Σ is called stable on Σ for short. A stable matrix on $(\{j\})_{j \in \langle n \rangle}$ is called stable for short.*

For the $[\Delta]$- and Δ-stable matrices on Σ, $\Delta \in$Par$(\langle m \rangle)$, $\Sigma \in$Par$(\langle n \rangle)$, we will use the generic name "generalized stable (complex) matrices" — we have generalized stable complex matrices, generalized stable real matrices, generalized stable nonnegative matrices, generalized stable stochastic matrices...

The stable matrices have a central role in the finite Markov chain theory, in the DeGroot model theory (see Section 2), in the theory of DeGroot model on distributed systems (see Section 3)... The stable matrices on Σ are important for the finite Markov chains (see, *e.g.*, [17, p. 389]), for the partial consensuses (see, in Section 2, Theorem 2.5; see also Example 2.8), for the partial distributed consensuses (see, in Section 3, Remark 3.2(b))...

Example 1.6. Let

$$P = \begin{pmatrix} 2 & 3 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 4 & 0 & \frac{1}{2} & \frac{1}{2} \\ 4 & 6 & 0 & 0 & 7 \\ 9 & 1 & 1 & 2 & 4 \end{pmatrix}, \ Q = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix},$$

$$T=\begin{pmatrix} 3 & -\frac{1}{4} & \frac{1}{4} & \frac{2}{4} \\ 3 & -\frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix}, \text{ and } Z=\begin{pmatrix} \frac{1}{4} & -\frac{3}{4} \\ \frac{1}{4} & -\frac{3}{4} \end{pmatrix}.$$

P is $[(\{1,2\},\{3,4\})]$-stable on $(\{1,2\},\{3,4,5\})$ because P_K^L is a generalized stochastic matrix, $\forall K\in(\{1,2\},\{3,4\})$, $\forall L\in(\{1,2\},\{3,4,5\})$ — indeed,

$$P_{\{1,2\}}^{\{1,2\}}=\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix},$$

and it is a generalized stochastic matrix because $2+3=1+4$,

$$P_{\{1,2\}}^{\{3,4,5\}}=\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

and it is a generalized stochastic matrix, etc. Moreover, P is $[\Delta]$-stable on $(\{1,2\},\{3,4,5\})$, $\forall\Delta\preceq(\{1,2\},\{3,4\})$. Moreover, P is $(\{1,2\},\{3,4\})$-stable on $(\{1,2\},\{3,4,5\})$. Q is $[(\{1,2\},\{3,4\})]$-stable

$$(Q_{\{1,2\}}^{\{1\}}=\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}=\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix},\ Q_{\{1,2\}}^{\{2\}}\equiv Q_{\{1,2\}}^{\{3\}}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}\cdots$$

— $\equiv$ is the identical sign). Moreover, it is $[\Delta]$-stable, $\forall\Delta\preceq(\{1,2\},\{3,4\})$. Moreover, it is $(\{1,2\},\{3,4\})$-stable. T is stable on $(\{1\},\{2\},\{3,4\})$ — it follows that $T^{\langle 2\rangle}$ is stable (the rows of $T^{\langle 2\rangle}$ are identical — any stable matrix has the rows identically). Z is stable (the rows of Z are identical).

Let $\Delta\in\mathrm{Par}(\langle m\rangle)$ and $\Sigma\in\mathrm{Par}(\langle n\rangle)$. Set (see [17] for $G_{\Delta,\Sigma}$ and [18] for $\overline{G}_{\Delta,\Sigma}$)

$$G_{\Delta,\Sigma}=G_{\Delta,\Sigma}(m,n)=\{P\mid P\in S_{m,n}\text{ and } P\text{ is a }[\Delta]\text{-stable matrix on }\Sigma\},$$

$$\overline{G}_{\Delta,\Sigma}=\overline{G}_{\Delta,\Sigma}(m,n)=\{P\mid P\in N_{m,n}\text{ and } P\text{ is a }[\Delta]\text{-stable matrix on }\Sigma\},$$

$$\widetilde{G}_{\Delta,\Sigma}=\widetilde{G}_{\Delta,\Sigma}(m,n)=\{P\mid P\in R_{m,n}\text{ and } P\text{ is a }[\Delta]\text{-stable matrix on }\Sigma\},$$

and

$$\widehat{G}_{\Delta,\Sigma}=\widehat{G}_{\Delta,\Sigma}(m,n)=\{P\mid P\in C_{m,n}\text{ and } P\text{ is a }[\Delta]\text{-stable matrix on }\Sigma\}$$

($G_{\Delta,\Sigma}$ is a set of generalized stable stochastic matrices, $\overline{G}_{\Delta,\Sigma}$ is a set of generalized stable nonnegative matrices...).

When we study or even when we construct products of complex matrices (in particular, products of stochastic matrices) using $G_{\Delta,\Sigma}$, $\overline{G}_{\Delta,\Sigma}$, $\widetilde{G}_{\Delta,\Sigma}$, or $\widehat{G}_{\Delta,\Sigma}$, we shall refer this as the *G method*. *G* comes from the verb *to group* and its derivatives.

The G method can more be extended considering other sets of matrices besides $S_{m,n}$, $N_{m,n}$, $R_{m,n}$, and $C_{m,n}$, then, if necessary, generalizing Definition 1.3...

The G method has serious applications, see, *e.g.*, [20]-[21].

Since $G_{\Delta,\Sigma} \subseteq \overline{G}_{\Delta,\Sigma} \subseteq \widetilde{G}_{\Delta,\Sigma} \subseteq \widehat{G}_{\Delta,\Sigma}$ ($\Delta \in$Par$(\langle m\rangle)$, $\Sigma \in$Par$(\langle n\rangle)$), any property which holds for $\widehat{G}_{\Delta,\Sigma}$ also holds for $G_{\Delta,\Sigma}$, $\overline{G}_{\Delta,\Sigma}$, and $\widetilde{G}_{\Delta,\Sigma}$, any property which holds for $\widetilde{G}_{\Delta,\Sigma}$ also holds for $G_{\Delta,\Sigma}$ and $\overline{G}_{\Delta,\Sigma}$... Several properties of $G_{\Delta,\Sigma}$ were given in Remark 2.1 from [17] — some of these properties can be extended for $\overline{G}_{\Delta,\Sigma}$, for $\widetilde{G}_{\Delta,\Sigma}$, or for $\widehat{G}_{\Delta,\Sigma}$. Below we give a few properties of $G_{\Delta,\Sigma}$, $\overline{G}_{\Delta,\Sigma}$, $\widetilde{G}_{\Delta,\Sigma}$, or $\widehat{G}_{\Delta,\Sigma}$.

Remark 1.7. (a)

$$S_{m,n} = G_{(\langle m\rangle),(\langle n\rangle)} = G_{(\{i\})_{i\in\langle m\rangle},\Sigma},\ \forall m,n\geq 1,\ \forall\Sigma\in \operatorname{Par}(\langle n\rangle)$$

($(\langle m\rangle)$ is the improper (degenerate) partition of $\langle m\rangle$...; Σ can be equal to $(\{j\})_{j\in\langle n\rangle}$ (so, $S_{m,n} = G_{(\{i\})_{i\in\langle m\rangle},(\{j\})_{j\in\langle n\rangle}}$), or, more generally, to $(\{j\}, K^c)_{j\in K}$, $\emptyset \neq K \subseteq \langle n\rangle$, or to...).

(b)

$$N_{m,n} \supset \overline{G}_{(\langle m\rangle),(\langle n\rangle)},\ R_{m,n} \supset \widetilde{G}_{(\langle m\rangle),(\langle n\rangle)},\ C_{m,n} \supset \widehat{G}_{(\langle m\rangle),(\langle n\rangle)},\ \forall m\geq 2,\ \forall n\geq 1,$$

and $\overline{G}_{(\langle m\rangle),(\langle n\rangle)}$ is equal to the set of generalized stochastic nonnegative $m\times n$ matrices, $\widetilde{G}_{(\langle m\rangle),(\langle n\rangle)}$ is equal to the set of generalized stochastic real $m\times n$ matrices, and $\widehat{G}_{(\langle m\rangle),(\langle n\rangle)}$ is equal to the set of generalized stochastic complex $m\times n$ matrices;

$$N_{m,n} = \overline{G}_{(\{i\})_{i\in\langle m\rangle},\Sigma},\ R_{m,n} = \widetilde{G}_{(\{i\})_{i\in\langle m\rangle},\Sigma},\ C_{m,n} = \widehat{G}_{(\{i\})_{i\in\langle m\rangle},\Sigma},$$

$\forall m\geq 1$, $\forall n\geq 1$, $\forall\Sigma \in$Par$(\langle n\rangle)$.

(c) The stable complex $m\times n$ matrices on Σ are generalized stochastic, $\forall\Sigma \in$Par$(\langle n\rangle)$ ($\overline{G}_{(\langle m\rangle),\Sigma}, \widetilde{G}_{(\langle m\rangle),\Sigma}$, and $\widehat{G}_{(\langle m\rangle),\Sigma}$ are included in the set of generalized stochastic nonnegative $m\times n$ matrices, that of generalized stochastic real $m\times n$ matrices, and that of generalized stochastic complex $m\times n$ matrices, respectively).

(d)

$$\overline{G}_{(\langle m\rangle),(\{j\},L_2,L_3,\ldots,L_v)_{j\in L_1}} \subseteq \overline{G}_{(\langle m\rangle),\left(\{j\},\bigcup_{b=2}^{v} L_b\right)_{j\in L_1}},$$

$\forall m, n\geq 1$, $\forall (L_1,L_2,\ldots,L_v) \in$Par$(\langle n\rangle)$, $v\geq 2$ (obviously, $(\{j\}, L_2, L_3, \ldots, L_v)_{j\in L_1} \in$Par$(\langle n\rangle)$). Similarly for $\widetilde{G}_{(\langle m\rangle),(\{j\},L_2,L_3,\ldots,L_v)_{j\in L_1}}$ and for $\widehat{G}_{(\langle m\rangle),(\{j\},L_2,L_3,\ldots,L_v)_{j\in L_1}}$.

Let $P \in \widehat{G}_{\Delta,\Sigma}$. Then (see Definition 1.4) $\forall K \in \Delta$, $\forall L \in \Sigma$, $\exists a_{K,L} = a_{K,L}(P) \in \mathbb{C}$ such that

$$\sum_{j\in L} P_{ij} = a_{K,L}, \forall i \in K.$$

Set the matrix (for the case when $P \in \overline{G}_{\Delta,\Sigma}$, see, *e.g.*, also [18])

$$P^{-+} = \left(P^{-+}_{KL}\right)_{K\in\Delta, L\in\Sigma}, \ P^{-+}_{KL} = a_{K,L}, \ \forall K \in \Delta, \ \forall L \in \Sigma.$$

We call the matrix P^{-+} the (Δ, Σ)-*grouped matrix* (*of* P), or, for short, if no confusion can arise, the *grouped matrix* (*of* P) — in [17] this matrix was differently called... The labels of rows and columns of P^{-+} are K, $K \in \Delta$, and L, $L \in \Sigma$, respectively; P^{-+}_{KL}, $K \in \Delta$, $L \in \Sigma$, are the entries of matrix P^{-+}; P^{-+}_{KL} is the entry (K, L) of matrix P^{-+}, $\forall K \in \Delta$, $\forall L \in \Sigma$. If confusion can arise, we write $P^{-+(\Delta,\Sigma)}$ instead of P^{-+}.

Example 1.8. Consider the matrices P, Q, T, and Z from Example 1.6. We have

$$P^{-+} = \begin{pmatrix} 5 & 1 \\ 10 & 7 \end{pmatrix} \ (\Delta = (\{1,2\}, \{3,4\}), \ \Sigma = (\{1,2\}, \{3,4,5\})),$$

$$Q^{-+} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 1 & 2 & 3 \end{pmatrix} \ (\Delta = (\{1,2\}, \{3,4\}), \ \Sigma = (\{j\})_{j\in\langle 3\rangle}),$$

$$T^{-+} = \begin{pmatrix} 3 & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}, \ Z^{-+} = \begin{pmatrix} \frac{1}{4} & -\frac{3}{4} \end{pmatrix}.$$

T^{-+} and Z^{-+} have just one row because T and Z are stable on $(\{1\}, \{2\}, \{3,4\})$ and stable, respectively. Moreover, the row of $\left(T^{\langle 2\rangle}\right)^{-+} = \left(T^{\langle 2\rangle}\right)^{-+((\langle 2\rangle),(\{1\},\{2\}))}$ is identical with any row of $T^{\langle 2\rangle}$ and the row of Z^{-+} is identical with any row of Z. Generally speaking, letting A and B, A, $B \in C_{m,n}$, be a stable matrix on $(\{j\}, K^c)_{j\in K}$, $\emptyset \neq K \subseteq \langle n\rangle$, $|K^c| > 1$, and a stable matrix, respectively, $\left(A^K\right)^{-+} = \left(A^K\right)^{-+\left((\langle m\rangle),(\{j\})_{j\in K}\right)}$ has just one row and this is identical with any row of A^K and $B^{-+} = B^{-+\left((\langle m\rangle),(\{j\})_{j\in\langle n\rangle}\right)}$ has just one row and this is identical with any row of B — do not forget! (see Theorem 1.10...)

Below we give a fundamental result.

Theorem 1.9. (i) (a generalization of Theorem 1.5 from [18]) *Let* $P_1 \in \widehat{G}_{\Delta_1,\Delta_2} \subseteq C_{n_1,n_2}$ *and* $P_2 \in \widehat{G}_{\Delta_2,\Delta_3} \subseteq C_{n_2,n_3}$. *Then*

$$P_1P_2 \in \widehat{G}_{\Delta_1,\Delta_3} \subseteq C_{n_1,n_3} \quad \text{and} \quad (P_1P_2)^{-+} = P_1^{-+} P_2^{-+}.$$

(ii) (a generalization of (i)) *Let* $P_1 \in \widehat{G}_{\Delta_1,\Delta_2} \subseteq C_{n_1,n_2}$, $P_2 \in \widehat{G}_{\Delta_2,\Delta_3} \subseteq C_{n_2,n_3}$, ..., $P_t \in \widehat{G}_{\Delta_t,\Delta_{t+1}} \subseteq C_{n_t,n_{t+1}}$, *where* $t \geq 2$. *Then*

$$P_1P_2...P_t \in \widehat{G}_{\Delta_1,\Delta_{t+1}} \subseteq C_{n_1,n_{t+1}} \quad \text{and} \quad (P_1P_2...P_t)^{-+} = P_1^{-+} P_2^{-+}...P_t^{-+}.$$

Proof. (i) Similar to the proof of Theorem 1.5 from [18]. Another proof, a simple one too, of the first part of (i) is as follows. Let $U \in \Delta_1$ and $V \in \Delta_3$. By Remark 1.2(d) we have

$$(P_1P_2)_U^V = \sum_{W\in\Delta_2} (P_1)_U^W (P_2)_W^V .$$

Since $(P_1)_U^W$ and $(P_2)_W^V$ are generalized stochastic matrices, $\forall W \in \Delta_2$, it follows that

$$(P_1)_U^W (P_2)_W^V \text{ and } \sum_{W\in\Delta_2} (P_1)_U^W (P_2)_W^V$$

are generalized stochastic matrices. So, $(P_1P_2)_U^V$ is a generalized stochastic matrix. Therefore, $P_1P_2 \in \widehat{G}_{\Delta_1,\Delta_3}$.

(ii) Induction. ∎

Recall that in this article, a vector is a row vector. Set $e = e(n) = (1, 1, ..., 1) \in \mathbb{R}^n$ ($e \in \mathbb{C}^n$ too), $\forall n \geq 1$; e' is its transpose.

Below we give the best result of this section.

Theorem 1.10. (An extension of Theorem 1.6 from [18].) *Let* $P_1 \in \widehat{G}_{\Delta_1,\Delta_2} \subseteq C_{n_1,n_2}$, $P_2 \in \widehat{G}_{\Delta_2,\Delta_3} \subseteq C_{n_2,n_3}$, ..., $P_t \in \widehat{G}_{\Delta_t,\Delta_{t+1}} \subseteq C_{n_t,n_{t+1}}$, *where* $t \geq 2$. *Suppose that*

$$\Delta_1 = (\langle n_1\rangle) \quad \textit{and } \Delta_{t+1} = (\{j\}, K^c)_{j\in K}$$

($\emptyset \neq K \subseteq \langle n_{t+1}\rangle$; $|K^c| \leq 1$ *or* $|K^c| > 1$). *Then*

$$(P_1P_2...P_t)^K$$

$$(\textit{equivalently, } P_1P_2...P_{t-1}P_t^K \textit{ (see Remark } 1.2\text{(b)}))$$

is a stable matrix (*i.e., a matrix with identical rows, see Definition* 1.5 *and Example* 1.6; $(P_1P_2...P_t)^K$ *is a submatrix of* $P_1P_2...P_t$, *see the definition of operator* $(\cdot)^{(\cdot)}$ *again*; *if* $K = \langle n_{t+1}\rangle$, *then* $(P_1P_2...P_t)^K = P_1P_2...P_t$, and $P_1P_2...P_t$ *is a stable matrix*),

$$(P_1P_2...P_t)_{\{i\}}^K = \left(P_1^{-+}P_2^{-+}...P_t^{-+}\right)^{\bigcup\limits_{j\in K}\{\{j\}\}} =$$

$$= P_1^{-+}P_2^{-+}...P_{t-1}^{-+}\left(P_t^K\right)^{-+}, \ \forall i \in \langle n_1\rangle,$$

where

$$\left(P_t^K\right)^{-+} = \left(P_t^K\right)^{-+\left(\Delta_t,(\{j\})_{j\in K}\right)}$$

$((P_1P_2...P_t)^K_{\{i\}}$ *is the row* i *of matrix* $(P_1P_2...P_t)^K$; $\left(P_1^{-+}P_2^{-+}...P_t^{-+}\right)^{\bigcup_{j\in K}\{\{j\}\}}$ *is a submatrix of* $P_1^{-+}P_2^{-+}...P_t^{-+}$ *(see the definition of operator* $(\cdot)^{(\cdot)}$*);* $P_b^{-+} = P_b^{-+(\Delta_b,\Delta_{b+1})}$, $\forall b \in \langle t\rangle$*; we used* $\{\{j\}\}$ *because the labels of columns of* $P_1^{-+}P_2^{-+}$ $...P_t^{-+}$ *are, here,* $\{j\}$, $j \in \langle n_{t+1}\rangle$, *when* $|K^c| \le 1$ *and* $\{j\}$, $j \in K$, *and* K^c *when* $|K^c| > 1$ *(see the definition of operator* $(\cdot)^{-+}$ *again)), and*

$$(P_1P_2...P_t)^K = e'\left(P_1^{-+}P_2^{-+}...P_t^{-+}\right)^{\bigcup_{j\in K}\{\{j\}\}} = e'P_1^{-+}P_2^{-+}...P_{t-1}^{-+}\left(P_t^K\right)^{-+},$$

where $e = e(n_1)$ *(if* $K = \langle n_{t+1}\rangle$, *then*

$$(P_1P_2...P_t)_{\{i\}} = P_1^{-+}P_2^{-+}...P_t^{-+},\ \forall i \in \langle n_1\rangle,$$

and

$$P_1P_2...P_t = e'P_1^{-+}P_2^{-+}...P_t^{-+}).$$

Proof. Since $\Delta_1 = (\langle n_1\rangle)$ and $\Delta_{t+1} = (\{j\}, K^c)_{j\in K}$, by Theorem 1.9(ii) and Definition 1.5, $P_1P_2...P_t$ is a stable matrix on $(\{j\}, K^c)_{j\in K}$ ($|K^c| \le 1$ or $|K^c| > 1$). It follows that $(P_1P_2...P_t)^K$ is a stable matrix and, consequently,

$$\left((P_1P_2...P_t)^K\right)^{-+\left((\langle n_1\rangle),(\{j\})_{j\in K}\right)}$$

has only one row — the label of this row is $\langle n_1\rangle$ (see the definition of operator $(\cdot)^{-+}$); further, we have

$$(P_1P_2...P_t)^K_{\{i\}} = \left(P_1P_2...P_{t-1}P_t^K\right)_{\{i\}} =$$

(due to the partitions $(\langle n_1\rangle)$ and $(\{j\})_{j\in K}$ ... — see Example 1.8 again)

$$= \left(P_1P_2...P_{t-1}P_t^K\right)^{-+\left((\langle n_1\rangle),(\{j\})_{j\in K}\right)} =$$

$$= P_1^{-+}P_2^{-+}...P_{t-1}^{-+}\left(P_t^K\right)^{-+},\ \forall i \in \langle n_1\rangle.$$

On the other hand, it is easy to see (see also Example 1.8) that

$$(P_1P_2...P_t)^K_{\{i\}} = \left(P_1^{-+}P_2^{-+}...P_t^{-+}\right)^{\bigcup_{j\in K}\{\{j\}\}},\ \forall i \in \langle n_1\rangle.$$

Finally, we have

$$(P_1P_2...P_t)^K = e'\left(P_1^{-+}P_2^{-+}...P_t^{-+}\right)^{\bigcup_{j\in K}\{\{j\}\}} = e'P_1^{-+}P_2^{-+}...P_{t-1}^{-+}\left(P_t^K\right)^{-+}.$$

∎

Remark 1.11. On Theorem 1.10, we consider 3 things.

(a) $\Delta_{t+1} = (\{j\})_{j\in\langle n_{t+1}\rangle}$ both when $|K^c| = 0$ and when $|K^c| = 1$ — practically speaking, we use "$|K^c| = 0$" because, in this case, $K = \langle n_{t+1}\rangle$.

(b) We have, in fact,

$$(P_1P_2...P_t)^K_{\{i\}} \equiv \left(P_1^{-+}P_2^{-+}...P_t^{-+}\right)^{\bigcup\limits_{j\in K}\{\{j\}\}}, \ \forall i \in \langle n_1\rangle,$$

because the row matrices

$$(P_1P_2...P_t)^K_{\{i\}} \text{ and } \left(P_1^{-+}P_2^{-+}...P_t^{-+}\right)^{\bigcup\limits_{j\in K}\{\{j\}\}}$$

have different labels for rows and, separately, for columns, $\forall i \in \langle n_1\rangle$, but abusively, for simplification, we used " $=$ " instead of " $\equiv$ " ($\equiv$ is the identical sign; since the one-to-one correspondence (the bijective function) is column $j \longmapsto$ column (with label) $\{j\}$, $\forall j \in K$, obviously, $\left(P_1^{-+}P_2^{-+}...P_t^{-+}\right)^{\bigcup\limits_{j\in K}\{\{j\}\}}$ must have the property: the column $\{j_1\}$ is before the column $\{j_2\}$ if $j_1 < j_2$, $\forall j_1, j_2 \in K$). Also,

$$(P_1P_2...P_t)^K_{\{i\}} \equiv P_1^{-+}P_2^{-+}...P_{t-1}^{-+}\left(P_t^K\right)^{-+}, \ \forall i \in \langle n_1\rangle,$$

but, due to the notion of equal matrices and notation used for the equal matrices,

$$\left(P_1^{-+}P_2^{-+}...P_t^{-+}\right)^{\bigcup\limits_{j\in K}\{\{j\}\}} = P_1^{-+}P_2^{-+}...P_{t-1}^{-+}\left(P_t^K\right)^{-+}.$$

Also,

$$(P_1P_2...P_t)^K \equiv e'\left(P_1^{-+}P_2^{-+}...P_t^{-+}\right)^{\bigcup\limits_{j\in K}\{\{j\}\}} ...$$

(c) Theorem 1.10 leads to another one if we replace $\Delta_{t+1} = (\{j\}, K^c)_{j\in K}$ with $\Delta_{t+1} = (\{j\}, L_2, L_3, ..., L_v)_{j\in L_1}$, where $(L_1, L_2, L_3, ..., L_v) \in$Par$(\langle n_{t+1}\rangle)$ and $v \geq 2$. But, by Remark 1.7(d), $P_1P_2...P_t$ is stable on $(\{j\}, L_2, L_3, ..., L_v)_{j\in L_1}$ $\Longrightarrow P_1P_2...P_t$ is stable on $\left(\{j\}, \bigcup\limits_{b=2}^{v} L_b\right)_{j\in L_1}$. So, nothing new.

Example 1.12. Let

$$P_1 = \begin{pmatrix} \frac{2}{4} & 0 & \frac{2}{4} & 0 \\ \frac{1}{4} & \frac{2}{4} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{2}{4} & \frac{2}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{2}{4} \end{pmatrix}, \ P_2 = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 0 & \frac{2}{4} \\ \frac{2}{4} & 0 & 0 & \frac{2}{4} \\ \frac{2}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{2}{4} & \frac{2}{4} & 0 \end{pmatrix},$$

$$P_3=\begin{pmatrix}\frac14&0&\frac24&\frac14\\\frac14&0&\frac24&\frac14\\0&\frac14&\frac34&0\\0&\frac14&\frac34&0\end{pmatrix},\text{ and } P_3'=\begin{pmatrix}\frac14&0&\frac24&\frac14\\\frac14&0&\frac14&\frac24\\0&\frac14&\frac24&\frac14\\0&\frac14&\frac14&\frac24\end{pmatrix}.$$

We have $P_1 \in G_{(\langle 4\rangle),(\langle 4\rangle)}$, $P_2 \in G_{(\langle 4\rangle),(\{1,2\},\{3,4\})}$, $P_3 \in G_{(\{1,2\},\{3,4\}),(\{j\})_{j\in\langle 4\rangle}}$, and $P_3' \in G_{(\{1,2\},\{3,4\}),(\{1\},\{2\},\{3,4\})}$. ($P_1 \notin G_{(\langle 4\rangle),\Sigma}$, $\forall\Sigma \in$Par$(\langle 4\rangle)$, $\Sigma \neq (\langle 4\rangle)$ — P_1 was deliberately chosen with this property.) So, by Theorem 1.10, $P_1P_2P_3$ is a stable matrix and

$$P_1P_2P_3=e'P_1^{-+}P_2^{-+}P_3^{-+}=\begin{pmatrix}1\\1\\1\\1\end{pmatrix}(1)\begin{pmatrix}\frac24&\frac24\end{pmatrix}\begin{pmatrix}\frac14&0&\frac24&\frac14\\0&\frac14&\frac34&0\end{pmatrix}=$$

$$=\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\begin{pmatrix}\frac{2}{16}&\frac{2}{16}&\frac{10}{16}&\frac{2}{16}\end{pmatrix}=\begin{pmatrix}\frac18&\frac18&\frac58&\frac18\\\frac18&\frac18&\frac58&\frac18\\\frac18&\frac18&\frac58&\frac18\\\frac18&\frac18&\frac58&\frac18\end{pmatrix}$$

while $(P_1P_2P_3')^{\{1,2\}}$ is a stable matrix and

$$\left(P_1P_2P_3'\right)^{\{1,2\}}=e'\left(P_1^{-+}P_2^{-+}P_3'^{-+}\right)^{\{\{1\},\{2\}\}}=e'P_1^{-+}P_2^{-+}\left(P_3'^{-+}\right)^{\{\{1\},\{2\}\}}=$$

$$=\begin{pmatrix}1\\1\\1\\1\end{pmatrix}(1)\begin{pmatrix}\frac24&\frac24\end{pmatrix}\begin{pmatrix}\frac14&0\\0&\frac14\end{pmatrix}=\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\begin{pmatrix}\frac{2}{16}&\frac{2}{16}\end{pmatrix}=\begin{pmatrix}\frac18&\frac18\\\frac18&\frac18\\\frac18&\frac18\\\frac18&\frac18\end{pmatrix}.$$

These results can also be obtained by direct computation. P_2P_3 and $(P_2P_3')^{\{1,2\}}$ are also stable matrices.

To easy recognize when Theorem 1.10 could or cannot be used/applied, see (i) and (ii) in the next theorem.

Theorem 1.13. *Suppose that the conditions of Theorem* 1.10 *hold.*

(i) *If* $n_2, n_3, ..., n_{t+1} \geq 2$ ($n_1 \geq 1$), *then* $\exists u \in \langle t\rangle$ *such that* P_u *is stable on* Δ_{u+1}, *and* $\Delta_{u+1} \neq (\langle n_{u+1}\rangle)$ (P_u *can be "broken" into at least two generalized stochastic matrices with* n_u *rows*).

(ii) *If* $n_1, n_2, ..., n_t \geq 2$ ($n_{t+1} \geq 1$), *then* $\exists w \in \langle t-1\rangle$ ($t \geq 2$) *such that* $\Delta_w \neq (\{i\})_{i\in\langle n_w\rangle}$ *and* $(P_w)_U$ *is a matrix with identical rows,* $\forall U \in \Delta_w$, $|U| \geq 2$ (*as a result,* P_w *has at least two identical rows*) *or* $(P_t)_U^K$ *is a matrix with* *identical rows,* $\forall U \in \Delta_t$, $|U| \geq 2$ *(as a result,* P_t^K *has at least two identical rows* ($P_t^K = P_t$ *when* $K = \langle n_{t+1} \rangle$*)).*

(iii) P_1, P_1P_2, ..., $P_1P_2...P_t$ *are stable on* Δ_2, Δ_3, ..., Δ_{t+1} ($\Delta_{t+1} = (\{j\}, K^c)_{j \in K}$), *respectively, and, as a result, they are generalized stochastic matrices.*

(iv) P_t, $P_{t-1}P_t$, ..., $P_1P_2...P_t$ *are* $[\Delta_t]$-, $[\Delta_{t-1}]$-, ..., $[\Delta_1]$*-stable on* $(\{j\}, K^c)_{j \in K}$, *respectively* ($\Delta_1 = (\langle n_1 \rangle)$).

Proof. (i) Since $n_2, n_3, ..., n_{t+1} \geq 2$, $\Delta_1 = (\langle n_1 \rangle)$, and $\Delta_{t+1} = (\{j\}, K^c)_{j \in K}$, it follows that $\exists u \in \langle t \rangle$ such that $\Delta_1 = (\langle n_1 \rangle)$, $\Delta_2 = (\langle n_2 \rangle)$, ..., $\Delta_u = (\langle n_u \rangle)$, and $\Delta_{u+1} \neq (\langle n_{u+1} \rangle)$ ("$u = t$" leads to "$\Delta_{t+1} \neq (\langle n_{t+1} \rangle)$"; $n_{t+1} \geq 2$ and $\emptyset \neq K \subseteq \langle n_{t+1} \rangle \Longrightarrow |\Delta_{t+1}| \geq 2$ (equivalently, $\Delta_{t+1} \neq (\langle n_{t+1} \rangle)$); so, no problem). It follows that P_u is stable on Δ_{u+1}, and $\Delta_{u+1} \neq (\langle n_{u+1} \rangle)$.

(ii) *Case* 1. $\Delta_t \neq (\{i\})_{i \in \langle n_t \rangle}$ — this case is possible because $n_t \geq 2$. In this case, $\exists V \in \Delta_t$ with $|V| \geq 2$. Since $P_t \in \widehat{G}_{\Delta_t, \Delta_{t+1}}$ and $\Delta_{t+1} = (\{j\}, K^c)_{j \in K}$, it follows that $(P_t)_U^K$ is a (complex) matrix with identical rows, $\forall U \in \Delta_t$, $|U| \geq 2$.

Case 2. $\Delta_t = (\{i\})_{i \in \langle n_t \rangle}$. Since $n_1, n_2, ..., n_{t-1} \geq 2$ (here, n_{t-1}, not n_t), $\Delta_1 = (\langle n_1 \rangle)$, and $\Delta_t = (\{i\})_{i \in \langle n_t \rangle}$, it follows that $\exists w \in \langle t-1 \rangle$ ($t \geq 2$) such that $\Delta_t = (\{i\})_{i \in \langle n_t \rangle}$, $\Delta_{t-1} = (\{i\})_{i \in \langle n_{t-1} \rangle}$, ..., $\Delta_{w+1} = (\{i\})_{i \in \langle n_{w+1} \rangle}$, and $\Delta_w \neq (\{i\})_{i \in \langle n_w \rangle}$ ($n_w \geq 2$, so, "$\Delta_w \neq (\{i\})_{i \in \langle n_w \rangle}$" is possible). Since $P_w \in \widehat{G}_{\Delta_w, \Delta_{w+1}}$, it follows that $(P_w)_U$ is a matrix with identical rows, $\forall U \in \Delta_w$, $|U| \geq 2$ ($\Delta_w \neq (\{i\})_{i \in \langle n_w \rangle} \Longrightarrow \exists V \in \Delta_w$ with $|V| \geq 2$).

(iii) and (iv) Theorem 1.9. ∎

Theorem 1.10 was illustrated in Example 1.12. To illustrate Theorem 1.13(i)-(ii), considering Example 1.12, the matrix P_1 cannot be "broken" into at least two generalized stochastic matrices with 4 rows, the matrix P_2 can be "broken" into two generalized stochastic (nonnegative) matrices with 4 rows, namely, $P_2^{\langle 2 \rangle}$ and $P_2^{\{3,4\}}$,

$$P_2^{\langle 2 \rangle} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{2}{4} & 0 \\ \frac{2}{4} & 0 \\ 0 & \frac{2}{4} \end{pmatrix} \text{ and } P_2^{\{3,4\}} = \begin{pmatrix} 0 & \frac{2}{4} \\ 0 & \frac{2}{4} \\ \frac{1}{4} & \frac{1}{4} \\ \frac{2}{4} & 0 \end{pmatrix},$$

the matrix P_3 is $[(\langle 2 \rangle, \{3,4\})]$-stable, so, $(P_3)_{\langle 2 \rangle}$ and $(P_3)_{\{3,4\}}$ are stable matrices

$$(\,(P_3)_{\langle 2 \rangle} = \begin{pmatrix} \frac{1}{4} & 0 & \frac{2}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{2}{4} & \frac{1}{4} \end{pmatrix} ...),$$

and the matrix $P_3^{\langle 2\rangle}$ is $[(\langle 2\rangle, \{3,4\})]$-stable, so, $(P_3)_{\langle 2\rangle}^{\langle 2\rangle}$ and $(P_3)_{\{3,4\}}^{\langle 2\rangle}$ are stable matrices.

Below we consider a similarity relation, and give a result about it — the relation and result lead, under certain conditions, to a subset/subgroup property of the DeGroot submodel from Section 2, see Example 2.11.

Definition 1.14. (A generalization of Definition 2.1 from [19].)
Let $P, Q \in \widehat{G}_{\Delta,\Sigma} \subseteq C_{m,n}$. We say that P is similar to Q (with respect to Δ and Σ) if

$$P^{-+} = Q^{-+}.$$

Set $P \backsim Q$ when P is similar to Q. Obviously, $\backsim$ is an equivalence relation on $\widehat{G}_{\Delta,\Sigma}$.

Example 1.15. (For another example, see Example 1.1 in [21].) Let

$$P = \begin{pmatrix} \frac{1}{4} & 0 & \frac{2}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}, \ Q = \begin{pmatrix} \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ \frac{1}{4} & 0 & 0 & \frac{3}{4} \\ \frac{2}{4} & 0 & \frac{2}{4} & 0 \\ 0 & \frac{2}{4} & \frac{2}{4} & 0 \end{pmatrix}, \text{ and } T = \begin{pmatrix} \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ \frac{1}{4} & 0 & \frac{2}{4} & \frac{1}{4} \\ 0 & \frac{2}{4} & \frac{2}{4} & 0 \\ \frac{2}{4} & 0 & \frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

We have $P, Q, R \in G_{\Delta,\Sigma} \subseteq S_4$, where $\Delta = \Sigma = (\{1,2\}, \{3,4\})$, and

$$P^{-+} = Q^{-+} = T^{-+} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{2}{4} & \frac{2}{4} \end{pmatrix}.$$

So, $P \backsim Q \backsim T$. Note that: 1) any matrix which is similar to P with respect to Δ and Σ has at least 8 positive entries; 2) Q has the smallest possible number of positive entries, *i.e.*, 8, while T has not — the matrices with a small number of positive entries are important for algorithms (for sampling...).

The next result says, among other things, that under certain conditions a product of matrices can be replaced with another product of matrices.

Theorem 1.16. (A generalization of Theorem 2.3 from [19].)
Let $P_1, U_1 \in \widehat{G}_{\Delta_1,\Delta_2} \subseteq C_{n_1,n_2}$, $P_2, U_2 \in \widehat{G}_{\Delta_2,\Delta_3} \subseteq C_{n_2,n_3}$, ..., $P_t, U_t \in \widehat{G}_{\Delta_t,\Delta_{t+1}} \subseteq C_{n_t,n_{t+1}}$. Suppose that

$$P_1 \backsim U_1, \ P_2 \backsim U_2, \ ..., P_t \backsim U_t.$$

Then

$$P_1P_2...P_t \backsim U_1U_2...U_t.$$

If, moreover, $\Delta_1 = (\langle n_1\rangle)$ and $\Delta_{t+1} = (\{j\}, K^c)_{j\in K}$, where $\emptyset \neq K \subseteq \langle n_{t+1}\rangle$, then

$$(P_1P_2...P_t)^K = (U_1U_2...U_t)^K$$

$$\text{— equivalently, } P_1P_2...P_{t-1}P_t^K = U_1U_2...U_{t-1}U_t^K$$

(*therefore, when* $\Delta_1 = (\langle n_1 \rangle)$ *and* $\Delta_{t+1} = (\{j\}, K^c)_{j\in K}$, *a product of* n *representatives, the first of an equivalence class included in* $\widehat{G}_{\Delta_1,\Delta_2}$, *the second of an equivalence class included in* $\widehat{G}_{\Delta_2,\Delta_3}$, ..., *the* tth *of an equivalence class included in* $\widehat{G}_{\Delta_t,(\{j\})_{j\in K}}$, *does not depend on the choice of representatives — we can work* (*this is important*) *with* $U_1U_2...U_{t-1}U_t^K$ *instead of* $P_1P_2...P_{t-1}P_t^K$ *and, conversely, with* $P_1P_2...P_{t-1}P_t^K$ *instead of* $U_1U_2...U_{t-1}U_t^K$; *due to Theorem* 1.10, $P_1P_2...P_{t-1}P_t^K$ *and* $U_1U_2...U_{t-1}U_t^K$ *are stable matrices*; $P_1P_2...P_{t-1}P_t^K = P_1P_2...P_t$ *and* $U_1U_2...U_{t-1}U_t^K = U_1U_2...U_t$ *when* $K = \langle n_{t+1} \rangle$).

Proof. The first part is similar to the proof of first part of Theorem 2.3 from [19]. For the second part, we consider two cases.

Case 1. $|K^c| \leq 1$. The proof is similar to the proof of second part of Theorem 2.3 from [19].

Case 2. $|K^c| > 1$. We have (see Remark 1.2)

$$(P_1P_2...P_t)^K = P_1P_2...P_{t-1}P_t^K$$

and

$$(U_1U_2...U_t)^K = U_1U_2...U_{t-1}U_t^K.$$

Since $P_t, U_t \in \widehat{G}_{\Delta_t,(\{j\},K^c)_{j\in K}}$, we have $P_t^K, U_t^K \in \widehat{G}_{\Delta_t,(\{j\})_{j\in K}}$. To finish the proof, we apply Case 1 to the matrices P_1, P_2, ..., P_{t-1}, P_t^K and U_1, U_2, ..., U_{t-1}, U_t^K, and obtain

$$P_1P_2...P_{t-1}P_t^K = U_1U_2...U_{t-1}U_t^K.$$

∎

For examples for Theorem 1.16, see Example 1.2 in [21] and, in this article, Example 2.11 (in Section 2).

Remark 1.17. The equivalence relation $\backsim$ is, in particular, an equivalence relation on the set of generalized stochastic complex $m \times n$ matrices. Indeed, by Remark 1.7(b) we have that $\widehat{G}_{(\langle m \rangle),(\langle n \rangle)}$ is equal to the set of generalized stochastic complex $m \times n$ matrices. Since $\backsim$ is an equivalence relation (with respect to $(\langle m \rangle)$ and $(\langle n \rangle)$) on $\widehat{G}_{(\langle m \rangle),(\langle n \rangle)}$, it follows that it is an equivalence relation (with respect to $(\langle m \rangle)$ and $(\langle n \rangle)$ — see Definition 1.14 again) on the set of generalized stochastic complex $m \times n$ matrices. Similarly, $\backsim$ is an equivalence relation on the set of generalized stochastic real $m \times n$ matrices and on that of generalized stochastic nonnegative $m \times n$ matrices.

2. FINITE-TIME CONSENSUS FOR DEGROOT MODEL

In this section, we consider the (homogeneous and nonhomogeneous) DeGroot model and the finite-time consensus problem for this model — using the G method, a result for reaching a partial or total consensus in finite time is given. Then we consider a special submodel/case of the DeGroot model, and for it we give two examples, and for the latter example (Example 2.11) we give certain comments — a subset/subgroup property of the DeGroot submodel is illustrated in this latter example.

Consider a group (society, team, committee, etc.) of r individuals (experts, agents, etc.) — formally/mathematically we consider a set with r elements, $r \geq 2$; in this article, we consider the set $\langle r \rangle$ ($\langle r \rangle = \{1, 2, ..., r\}$, $r \geq 2$), and we call it the *individual set.* Each individual has an estimation/opinion, an initial one, on a subject (the estimation of a parameter, or of a probability, or...). The initial estimations/opinions of individuals are represented by a (row) vector $p_0 = (p_{0i})_{i \in \langle r \rangle} = (p_{01}, p_{02}, ..., p_{0r})$, p_{0i} is the initial estimation/opinion of individual i, $\forall i \in \langle r \rangle$, and p_0 is a probability (row) vector, or a nonnegative vector, or a real vector, or a complex vector, or... When the individuals are made aware of each others' opinions, they may modify their own opinion by taking into account the opinion of others, and the modifications, if any, are done at discrete times $1, 2, ...$ — formally, we consider a sequence $(p_n)_{n \geq 0}$ of (row) vectors and a sequence $(P_n)_{n \geq 1}$ of stochastic matrices, $p_n = (p_{ni})_{i \in \langle r \rangle} = (p_{n1}, p_{n2}, ..., p_{nr})$, $\forall n \geq 0$, and $P_n = \left((P_n)_{ij} \right) \in S_r$, $\forall n \geq 1$, where p_{ni} is the estimation/opinion of individual i at time n, $\forall n \geq 0$, $\forall i \in \langle r \rangle$, and $(P_n)_{ij}$ is the probability which the individual i assigns to the estimation/opinion p_{n-1j} (warning! not p_{0j}) of individual j at time n (not $n-1$), $\forall n \geq 1$, $\forall i, j \in \langle r \rangle$. We call p_n the *vector of estimations/opinions* (*of individuals*) *at time* n *or the value vector at time* n, $\forall n \geq 0$, and P_n the *trust* or *weight matrix at time* n, $\forall n \geq 1$. We also call p_0 the *initial vector of estimations/opinions* (*of individuals*) *or the initial value vector.* p_n is a probability (row) vector, or a nonnegative vector, or a real vector, or a complex vector, or..., $\forall n \geq 0$. The above considerations lead to

$$p'_n = P_n p'_{n-1}, \ \forall n \geq 1$$

(p'_n and p'_{n-1} are the transposes of p_n and p_{n-1}, respectively, $\forall n \geq 1$), and, as a result, to

$$p'_n = P_n P_{n-1} ... P_1 p'_0, \ \forall n \geq 1.$$

The above model [7]... is determined by the set $\langle r \rangle$, vector p_0, and stochastic matrices P_n, $n \geq 1$. Due this fact, we call this model the *DeGroot model on* (*the triple*) $\left(\langle r\rangle, p_0, (P_n)_{n\geq 1}\right)$, and will use the generic name "(the) DeGroot model". We say that DeGroot model on $\left(\langle r\rangle, p_0, (P_n)_{n\geq 1}\right)$ is *homogeneous* if $P_1 = P_2 = P_3 = ...$ and *nonhomogeneous* if $\exists u, v \geq 1$, $u \neq v$, such that $P_u \neq P_v$, and will use the generic names "(the) homogeneous DeGroot model" and "(the) nonhomogeneous DeGroot model", respectively.

Definition 2.1. ([7]; see, *e.g.*, also [5].) *We say that the individual set reaches a consensus or that a consensus is reached* (*for the individual set*) *if all r components of p_n converge to the same limit as $n \longrightarrow \infty$, and we call this limit the consensus.*

Reaching a consensus for the DeGroot model — this is the *consensus problem* for/of the DeGroot model. When a consensus is reached, it is reached in a finite or infinite time — obviously, the first case is the best.

The DeGroot model and its consensus problem and the consensus problem in computer science have some things in common with the finite Markov chains — for the first three, see, *e.g.*, [1, Chapter 5, pp. 91−124], [2]-[3], [5]-[7], [9], [12, Chapter 8], [15], [22], [26]-[28], and [30], and for the last, see, *e.g.*, [10]-[11], [13], [16], and [24]-[26] (all or some of these references could be available; arXiv and Wikipedia could also be useful... — expressions for search: DeGroot model, DeGroot learning, DeGroot learning model, consensus (computer science), distributed consensus, distributed averaging, Markov chain, homogeneous Markov chain, inhomogeneous Markov chain, nonhomogeneous Markov chain...).

Theorem 2.2. (A simple, important, and known result, see [7], [2]...) *Consider the DeGroot model on $\left(\langle r\rangle, p_0, (P_n)_{n\geq 1}\right)$. If $\lim_{n\to\infty} P_n P_{n-1} ... P_1$ exists and is a stable matrix, then a consensus is reached.*

Proof. (A known proof, but we give it — the best thing is to give it.) Suppose that $\lim_{n\to\infty} P_n P_{n-1} ... P_1$ exists and is a stable matrix. In this case, $\exists \pi$, π is a probability (row) vector with r components, such that

$$\lim_{n\to\infty} P_n P_{n-1} ... P_1 = e'\pi$$

(recall that $e = e(r) = (1, 1, ..., 1)$). Further, we have

$$\lim_{n\to\infty} p'_n = \lim_{n\to\infty} P_n P_{n-1} ... P_1 p'_0 = e'\pi p'_0 = e' \sum_{i=1}^{r} \pi_i p_{0i} = \begin{pmatrix} \sum_{i=1}^{r} \pi_i p_{0i} \\ \sum_{i=1}^{r} \pi_i p_{0i} \\ \vdots \\ \sum_{i=1}^{r} \pi_i p_{0i} \end{pmatrix}.$$

Therefore, a consensus is reached; the consensus is

$$\sum_{i=1}^{r} \pi_i p_{0i}.$$

∎

Remark 2.3. The condition from the above theorem is only a sufficient condition for reaching a consensus. Indeed, if

$$p_0 = (2,4,3) \text{ and } P_n = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \forall n \geq 1$$

(a homogeneous case), then a consensus is reached at time 1 (a finite-time consensus), the consensus is 3, but

$$\lim_{n\to\infty} P_n P_{n-1} ... P_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is not a stable matrix. Another example, a nonhomogeneous case:

$$p_0 = \left(2, 4, \frac{10}{3}\right), \ P_1 = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } P_n = \begin{pmatrix} \begin{matrix} & \\ & Q_n & \\ & \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix}, \forall n \geq 2,$$

where Q_n is a stochastic 2×2 matrix, $\forall n \geq 2$ — a consensus is reached at time 1, and this is $\frac{10}{3}$. Note that in the above cases/examples the matrices are reducible — therefore, a consensus, even a finite-time consensus, can be reached even when the matrices are reducible.

The above remark is both for Theorem 2.2 and for an incorrect statement in [7, Sec. 4, p. 119] discovered by R.L. Berger [2].

Remark 2.4. (a) If $P \in S_{m,n}$ and $Q \in S_{n,p}$, and P is a stable matrix, then PQ is a stable matrix. More generally, if $P \in C_{m,n}$ and $Q \in C_{n,p}$, and P is a stable (complex) matrix, then PQ is a stable matrix.

(b) If $P \in S_{m,n}$ and $Q \in C_{n,p}$, and Q is a stable matrix, then PQ is a stable matrix, and, moreover,

$$PQ = Q.$$

More generally, if $P \in C_{m,n}$ and $Q \in C_{n,p}$, P is a generalized stochastic complex matrix (see Definition 1.3) and Q is a stable (complex) matrix, then PQ is a stable matrix, and, moreover,

$$PQ = Q,$$

but only when P is a stochastic complex matrix.

(c) (simple and important proprieties of the (homogeneous and nonhomogeneous) DeGroot model) Consider the DeGroot model on $\left(\langle r\rangle, p_0, (P_n)_{n\geq 1}\right)$.

(c1) If $P_t P_{t-1} ... P_m$ is a stable matrix for some t and m, $t \geq m \geq 1$, then (see (a) and (b))

$$P_n P_{n-1} ... P_1$$

is a stable matrix, $\forall n \geq t$, and, moreover,

$$P_n P_{n-1} ... P_1 = P_t P_{t-1} ... P_1, \ \forall n \geq t.$$

This fact leads to other ones: 1) the sequence of matrices $(P_n)_{n\geq t+1}$ does not count, it can be replaced with any other infinite sequence of stochastic $r \times r$ matrices; 2) a consensus is reached in a finite time, and it is

$$\sum_{i=1}^{r} \pi_i p_{0i},$$

where $\pi = (\pi_1, \pi_2, ..., \pi_r)$, $\pi = (P_t P_{t-1} ... P_1)_{\{i\}}$, $\forall i \in \langle r\rangle$ ($P_t P_{t-1} ... P_1$ is a stable matrix, *i.e.*, a matrix with identical rows; $(P_t P_{t-1} ... P_1)_{\{i\}}$ is the row i of $P_t P_{t-1} ... P_1$, $\forall i \in \langle r\rangle$).

(c2) If $(P_t P_{t-1} ... P_1)^K$ is a stable matrix for some t and K, $t \geq 1$ and $\emptyset \neq K \subseteq \langle r\rangle$, then (see Remark 1.2(b) and (b))

$$(P_n P_{n-1} ... P_1)^K$$

is a stable matrix, $\forall n \geq t$, and, moreover,

$$(P_n P_{n-1} ... P_1)^K = (P_t P_{t-1} ... P_1)^K, \ \forall n \geq t$$

— $\forall n \geq t$, $P_n P_{n-1} ... P_1 = P_t P_{t-1} ... P_1$ and $P_n P_{n-1} ... P_1$ is a stable matrix when $|K^c| = 0$; $\forall n \geq t$, $(P_n P_{n-1} ... P_1)^K = (P_t P_{t-1} ... P_1)^K$ and $(P_n P_{n-1} ... P_1)^K$ is a stable matrix $\Longrightarrow P_n P_{n-1} ... P_1 = P_t P_{t-1} ... P_1$ and $P_n P_{n-1} ... P_1$ is a stable matrix when $|K^c| = 1$ because the matrices P_b, $b \geq 1$, are stochastic; so, $\forall n \geq t$, $P_n P_{n-1} ... P_1 = P_t P_{t-1} ... P_1$ and $P_n P_{n-1} ... P_1$ is a stable matrix when $|K^c| \leq 1$.

Below we give a fundamental result concerning the finite-time partial and total consensuses (for reaching a partial or total consensus in a finite time).

Theorem 2.5. *Consider the DeGroot model on* $\left(\langle r\rangle, p_0, (P_n)_{n\geq 1}\right)$. *If* $P_1 \in G_{\Delta_2,\Delta_1} \subseteq S_r$, $P_2 \in G_{\Delta_3,\Delta_2} \subseteq S_r$ ..., $P_t \in G_{\Delta_{t+1},\Delta_t} \subseteq S_r$, *and* $\Delta_{t+1} = (\langle r\rangle)$ *for some* $t \geq 1$ *and*

$$\Delta_1 = (\{j\}, K^c)_{j\in K} \ (\emptyset \neq K \subseteq \langle r\rangle),$$

then

$$(P_tP_{t-1}...P_1)^K$$

(*equivalently,* $P_tP_{t-1}...P_2P_1^K$) *is a stable matrix,*

$$(P_tP_{t-1}...P_1)^K_{\{i\}} = \left(P_t^{-+}P_{t-1}^{-+}...P_1^{-+}\right)^{\bigcup_{j\in K}\{\{j\}\}} = P_t^{-+}P_{t-1}^{-+}...P_2^{-+}\left(P_1^K\right)^{-+}, \forall i\in\langle r\rangle,$$

where

$$\left(P_1^K\right)^{-+} = \left(P_1^K\right)^{-+\left(\Delta_2,(\{j\})_{j\in K}\right)}$$

($(P_tP_{t-1}...P_1)^K_{\{i\}}$ *is the row* i *of matrix* $(P_tP_{t-1}...P_1)^K$; $\left(P_t^{-+}P_{t-1}^{-+}...P_1^{-+}\right)^{\bigcup_{j\in K}\{\{j\}\}}$ *is a submatrix of* $P_t^{-+}P_{t-1}^{-+}...P_1^{-+}$ (*see the definition of operator* $(\cdot)^{(\cdot)}$ *again*); $P_b^{-+} = P_b^{-+(\Delta_{b+1},\Delta_b)}$, $\forall b \in \langle t\rangle$; *we used* $\{\{j\}\}$ *because the labels of columns of* $P_t^{-+}P_{t-1}^{-+}...P_1^{-+}$ *are, here,* $\{j\}$, $j \in \langle r\rangle$, *when* $|K^c| \leq 1$ *and* $\{j\}$, $j \in K$, *and* K^c *when* $|K^c| > 1$ (*see the definition of operator* $(\cdot)^{-+}$ *again*)), *and*

$$(P_tP_{t-1}...P_1)^K = e'\left(P_t^{-+}P_{t-1}^{-+}...P_1^{-+}\right)^{\bigcup_{j\in K}\{\{j\}\}} =$$

$$= e'P_t^{-+}P_{t-1}^{-+}...P_2^{-+}\left(P_1^K\right)^{-+},$$

where $e = e(r)$. *Further, since*

$$p_n' = P_nP_{n-1}...P_1p_0'$$

and (*see Remark* 2.4(c2))

$$(P_nP_{n-1}...P_1)^K = (P_tP_{t-1}...P_1)^K, \ \forall n \geq t, \ \textit{ if } \ |K^c| > 1$$

and

$$P_nP_{n-1}...P_1 = P_tP_{t-1}...P_1 \ \textit{ and...}, \forall n \geq t, \ \textit{ if } \ |K^c| \leq 1,$$

we have

$$p_{nl} = \begin{cases} \sum_{j\in\langle r\rangle}\left(P_t^{-+}P_{t-1}^{-+}...P_1^{-+}\right)_{\langle r\rangle\{j\}}p_{0j} \\ \qquad\qquad \textit{if } \ |K^c| \leq 1, \\ \sum_{j\in K}\left(P_t^{-+}P_{t-1}^{-+}...P_1^{-+}\right)_{\langle r\rangle\{j\}}p_{0j} + \sum_{u\in K^c}(P_nP_{n-1}...P_1)_{lu}\,p_{0u} \\ \qquad\qquad \textit{if } \ |K^c| > 1, \end{cases}$$

$\forall n \geq t$, $\forall l \in \langle r\rangle$ (*when* $|K^c| > 1$, *we can replace* $\left(P_t^{-+}P_{t-1}^{-+}...P_1^{-+}\right)_{\langle r\rangle\{j\}}$ *with* $\left(P_t^{-+}P_{t-1}^{-+}...P_2^{-+}\left(P_1^K\right)^{-+}\right)_{\langle r\rangle\{j\}}$; $P_t^{-+}P_{t-1}^{-+}...P_1^{-+}$ *has just one row, and the label of this row is* $\langle r\rangle$) — *when* $\emptyset \neq K \subseteq \langle r\rangle$ ($K = \langle r\rangle$ *or* $K \neq \langle r\rangle$), *the linear* *combination of* p_{0j}, $j \in K$, *namely,* $\sum_{j\in K} \left(P_t^{-+} P_{t-1}^{-+} ... P_1^{-+}\right)_{\langle r\rangle \{j\}} p_{0j}$, *does not depend on* n *and* l *when* $n \geq t$, *and, therefore, a partial or total consensus is reached at time* t (*the linear combination of* p_{0j}, $j \in K$, *at time/step* t *is kept at any time* $n > t$ — *the utilization of phrases "partial consensus" and "total consensus" are thus justified*); *when* $|K^c| \leq 1$, *a* (*total*) *consensus is reached at time* t *and this is*

$$\sum_{j\in \langle r\rangle} \left(P_t^{-+} P_{t-1}^{-+} ... P_1^{-+}\right)_{\langle r\rangle \{j\}} p_{0j};$$

when $|K^c| > 1$,

$$\sum_{j\in K} \left(P_t^{-+} P_{t-1}^{-+} ... P_1^{-+}\right)_{\langle r\rangle \{j\}} p_{0j}$$

is a partial consensus reached at time t *for the initial opinions,* p_{0j}, $j \in K$, *of individuals from* K *and the linear combination of* p_{0u}, $u \in K^c$, *namely,* $\sum_{u\in K^c} (P_n P_{n-1} ... P_1)_{lu} \, p_{0u}$, *depends or not on* n *and* l. (See also Remark 1.11.)

Proof. Theorems 1.10 and 2.2 and Remarks 1.2(b) and 2.4, (b) or (c). (Using Remark 1.2(b), we have

$$(P_n P_{n-1} ... P_1)^K = P_n P_{n-1} ... P_{t+1} (P_t P_{t-1} ... P_1)^K = (P_t P_{t-1} ... P_1)^K, \ \forall n \geq t$$

— the latter equation follows from the fact that $(P_t P_{t-1} ... P_1)^K$ is a stable (stochastic) matrix ($P_n P_{n-1} ... P_{t+1}$ vanishes when $n = t$).) ∎

Remark 2.6. (a) $P \in G_{(\{i\})_{i\in\langle n\rangle}, (\{i\})_{i\in\langle n\rangle}}$, $\forall P \in S_n$, where $n \geq 1$ (by Remark 1.7(a)).

(b) A case, an interesting one, of Theorem 2.5 is when $\Delta_1 = \Delta_2 = ... = \Delta_u = (\{i\})_{i\in\langle r\rangle}$ and $\Delta_{u+1} \neq (\{i\})_{i\in\langle r\rangle}$ for some u, $2 \leq u \leq t$ (due to (a), "$P_1 \in G_{\Delta_2,\Delta_1}$, $P_2 \in G_{\Delta_3,\Delta_2}$, ..., $P_{u-1} \in G_{\Delta_u,\Delta_{u-1}}$" holds without problems). In this case, a (total) consensus is reached.

(c) On Theorem 2.5, our interest is to have a set K as large as possible and a natural number $t \geq 1$ as small as possible — interesting optimization problems.

Below we give two examples to illustrate Theorem 2.5; Remark 2.6(b) is illustrated in the next example.

Example 2.7. Consider the DeGroot model on $\left(\langle r\rangle, p_0, (P_n)_{n\geq 1}\right)$, where $r = 4$, $p_0 = (2, 4, 1, 1)$, and the first four matrices are

$$P_1 = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} & \frac{2}{4} \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{8} & \frac{1}{8} & \frac{2}{4} & \frac{1}{4} \\ \frac{3}{16} & \frac{1}{16} & \frac{2}{4} & \frac{1}{4} \end{pmatrix}, \ P_2 = \begin{pmatrix} \frac{1}{4} & 0 & 0 & \frac{3}{4} \\ \frac{1}{4} & 0 & 0 & \frac{3}{4} \\ \frac{3}{4} & 0 & \frac{1}{4} & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} & 0 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} \frac{1}{4} & 0 & \frac{2}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{2}{4} \\ \frac{1}{4} & 0 & 0 & \frac{3}{4} \\ \frac{1}{8} & \frac{1}{8} & \frac{3}{4} & 0 \end{pmatrix}, \text{ and } P_4 = \begin{pmatrix} \frac{1}{8} & \frac{2}{8} & \frac{3}{8} & \frac{2}{8} \\ \frac{2}{8} & \frac{3}{8} & \frac{2}{8} & \frac{1}{8} \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

We have $P_1 \in G_{\Delta_2,\Delta_1}$, $P_2 \in G_{\Delta_3,\Delta_2}$, and $P_3 \in G_{\Delta_4,\Delta_3}$, where $\Delta_1 = \Delta_2 = (\{j\})_{j\in\langle 4\rangle}$ (see now Remark 2.6(b)), $\Delta_3 = (\langle 2\rangle, \{3,4\})$, and $\Delta_4 = (\langle 4\rangle)$ ($P_1 \in G_{\Delta_4,\Delta_3}$, but this thing does not count here). By Theorem 2.5 (or by Theorem 1.10), $P_3P_2P_1$ is a stable matrix (P_3P_2 is also a stable matrix). Further, we have

$$P_3^{-+}P_2^{-+}P_1^{-+} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} \frac{1}{4} & 0 & 0 & \frac{3}{4} \\ \frac{3}{4} & 0 & \frac{1}{4} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} & \frac{2}{4} \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{8} & \frac{1}{8} & \frac{2}{4} & \frac{1}{4} \\ \frac{3}{16} & \frac{1}{16} & \frac{2}{4} & \frac{1}{4} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{55}{256} & \frac{9}{256} & \frac{88}{256} & \frac{104}{256} \end{pmatrix},$$

so, the consensus is

$$P_3^{-+}P_2^{-+}P_1^{-+}p_0' = \frac{110+36+88+104}{256} = \frac{338}{256}$$

(it is reached at time 3). The sequence of matrices $(P_n)_{n\geq 4}$ does not count — this can be replaced with any other infinite sequence of stochastic 4×4 matrices (see Remark 2.4(c1)).

Example 2.8. Consider the DeGroot model on $\left(\langle r\rangle, p_0, (P_n)_{n\geq 1}\right)$, where $r = 4$, $p_0 = (10, 20, 2, 4)$, and the first two matrices are

$$P_1 = \begin{pmatrix} \frac{2}{10} & \frac{1}{10} & \frac{7}{10} & 0 \\ \frac{2}{10} & \frac{1}{10} & \frac{1}{10} & \frac{6}{10} \\ 0 & \frac{3}{10} & 0 & \frac{7}{10} \\ 0 & \frac{3}{10} & \frac{2}{10} & \frac{5}{10} \end{pmatrix} \text{ and } P_2 = \begin{pmatrix} \frac{4}{10} & 0 & 0 & \frac{6}{10} \\ \frac{1}{10} & \frac{3}{10} & \frac{4}{10} & \frac{2}{10} \\ \frac{1}{10} & \frac{3}{10} & \frac{1}{10} & \frac{5}{10} \\ \frac{2}{10} & \frac{2}{10} & \frac{1}{10} & \frac{5}{10} \end{pmatrix}.$$

We have $P_1 \in G_{\Delta_2,\Delta_1}$ and $P_2 \in G_{\Delta_3,\Delta_2}$, where $\Delta_1 = (\{1\}, \{2\}, \{3,4\})$, $\Delta_2 = (\{1,2\}, \{3,4\})$, and $\Delta_3 = (\langle 4\rangle)$. So, by Theorem 2.5 (or by Theorem 1.10),

$$(P_2P_1)^{\{1,2\}}$$

is a stable matrix. Further, we have

$$P_2^{-+}P_1^{-+} = \begin{pmatrix} \frac{4}{10} & \frac{6}{10} \end{pmatrix} \begin{pmatrix} \frac{2}{10} & \frac{1}{10} & \frac{7}{10} \\ 0 & \frac{3}{10} & \frac{7}{10} \end{pmatrix} = \begin{pmatrix} \frac{8}{100} & \frac{22}{100} & \frac{70}{100} \end{pmatrix}.$$

So (see Theorem 2.5),

$$p_{nl} = \frac{8}{100} \cdot 10 + \frac{22}{100} \cdot 20 + (P_nP_{n-1}...P_1)_{l3} \cdot 2 + (P_nP_{n-1}...P_1)_{l4} \cdot 4 =$$
$$= 5.2 + 2\,(P_nP_{n-1}...P_1)_{l3} + 4\,(P_nP_{n-1}...P_1)_{l4}\,,\ \forall n \geq 2,\ \forall l \in \langle 4\rangle$$

— the linear combination,

$$\frac{8}{100} \cdot 10 + \frac{22}{100} \cdot 20,$$

for the initial opinions 10 and 20 (of individuals 1 and 2) holds at any time $n \geq 2$; the consensus, this is a partial consensus, for the initial opinions 10 and 20 is 5.2, and is reached at time 2.

Below we consider a possible case of the DeGroot model on $\left(\langle r\rangle, p_0, (P_n)_{n\geq 1}\right)$ — a submodel of the DeGroot model which works from small subsets (subgroups of individuals) of $\langle r\rangle$ to larger and larger subsets of $\langle r\rangle$ until the whole set $\langle r\rangle$.

Let $\Delta_1, \Delta_2, ..., \Delta_{t+1} \in \mathrm{Par}(\langle r\rangle)$, $r \geq 2$, $t \geq 1$. Assume that $\Delta_1 \preceq \Delta_2 \preceq ... \preceq \Delta_{t+1}$, $\Delta_1 = (\{j\})_{j\in\langle r\rangle}$, and $\Delta_{t+1} = (\langle r\rangle)$. Consider the DeGroot model on $\left(\langle r\rangle, p_0, (P_n)_{n\geq 1}\right)$, where the matrices $P_1, P_2, ..., P_{t-1}$ (not $P_1, P_2, ..., P_t$) when $t \geq 2$ have the property

$$(P_l)_K^L = 0,\ \forall l \in \langle t-1\rangle, \forall K, L \in \Delta_{l+1},\ K \neq L$$

(this assumption implies that P_l is a block diagonal matrix, eventually by permutation of rows and columns, and Δ_{l+1}-stable matrix on Δ_{l+1}, $\forall l \in \langle t-1\rangle$).

Remark 2.9. Theorem 2.5 holds, in particular, for the above DeGroot submodel when $P_1 \in G_{\Delta_2,\Delta_1}$, $P_2 \in G_{\Delta_3,\Delta_2}$, ..., $P_t \in G_{\Delta_{t+1},\Delta_t}$.

To illustrate the DeGroot submodel, we give two examples.

Example 2.10. Consider the DeGroot model on $\left(\langle r\rangle, p_0, (P_n)_{n\geq 1}\right)$, where

$$r = 6,\ p_0 = \left(\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{1}{10}, \frac{1}{10}, \frac{2}{10}\right),$$

and the first three matrices are

$$P_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & & & & \\ \frac{2}{3} & \frac{1}{3} & & & & \\ & & \frac{1}{4} & \frac{3}{4} & & \\ & & \frac{2}{4} & \frac{2}{4} & & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix}, \ P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & & \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & & \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & & \\ 0 & 1 & 0 & 0 & & \\ & & & & 1 & 0 \\ & & & & \frac{1}{10} & \frac{9}{10} \end{pmatrix},$$

and

$$P_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}.$$

Taking $\Delta_1 = (\{j\})_{j\in\langle 6\rangle}$, $\Delta_2 = (\langle 2\rangle, \{3,4\}, \{5,6\})$, $\Delta_3 = (\langle 4\rangle, \{5,6\})$, and $\Delta_4 = (\langle 6\rangle)$, we have $\Delta_1 \preceq \Delta_2 \preceq \Delta_3 \preceq \Delta_4$ and

$$(P_1)_{\langle 2\rangle}^{\{3,4\}} \equiv (P_1)_{\langle 2\rangle}^{\{5,6\}} = 0, \ (P_1)_{\{3,4\}}^{\langle 2\rangle} \equiv (P_1)_{\{3,4\}}^{\{5,6\}} = 0,$$

$$(P_1)_{\{5,6\}}^{\langle 2\rangle} \equiv (P_1)_{\{5,6\}}^{\{3,4\}} = 0, \ (P_2)_{\langle 4\rangle}^{\{5,6\}} = 0, \text{ and } (P_2)_{\{5,6\}}^{\langle 4\rangle} = 0$$

— the conditions of the DeGroot submodel are satisfied, and thus we have an example for this submodel. Learning begins with the small subsets (subgroups of individuals) $\langle 2\rangle$, $\{3,4\}$, and $\{5,6\}$ of $\langle 6\rangle$, the larger subsets $\langle 4\rangle$ and $\{5,6\}$ of $\langle 6\rangle$ are then used, then the whole set, $\langle 6\rangle$, is used. Theorem 2.5 cannot be applied for Δ_1, Δ_2, Δ_3, and Δ_4 because $P_1 \notin G_{\Delta_2,\Delta_1}$ (moreover, $P_2 \notin G_{\Delta_3,\Delta_2}$).

In the next example we show that our DeGroot submodel on $\left(\langle r\rangle, p_0, (P_n)_{n\geq 1}\right)$ has a subset/subgroup property when $P_1 \in G_{\Delta_2,\Delta_1}$, $P_2 \in G_{\Delta_3,\Delta_2}$, ..., $P_t \in G_{\Delta_{t+1},\Delta_t}$.

Example 2.11. (The subset/subgroup property illustrated.) Consider the DeGroot model on $\left(\langle r\rangle, p_0, (P_n)_{n\geq 1}\right)$, where

$$r = 6, \ p_0 = \left(\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{1}{10}, \frac{1}{10}, \frac{2}{10}\right),$$

and the first four matrices are

$$P_1=\begin{pmatrix} \frac{4}{10} & \frac{6}{10} & & & & \\ \frac{4}{10} & \frac{6}{10} & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \frac{1}{10} & \frac{9}{10} \\ & & & & \frac{1}{10} & \frac{9}{10} \end{pmatrix},\ P_2=\begin{pmatrix} \frac{3}{10} & \frac{4}{10} & \frac{3}{10} & & & \\ \frac{2}{10} & \frac{5}{10} & \frac{3}{10} & & & \\ \frac{1}{10} & \frac{6}{10} & \frac{3}{10} & & & \\ & & & 1 & & \\ & & & & \frac{2}{10} & \frac{8}{10} \\ & & & & \frac{4}{10} & \frac{6}{10} \end{pmatrix},$$

$$P_3=\begin{pmatrix} \frac{3}{10} & \frac{2}{10} & \frac{5}{10} & & & \\ \frac{2}{10} & \frac{4}{10} & \frac{4}{10} & & & \\ \frac{1}{10} & \frac{6}{10} & \frac{3}{10} & & & \\ & & & \frac{1}{10} & \frac{1}{10} & \frac{8}{10} \\ & & & \frac{1}{10} & \frac{2}{10} & \frac{7}{10} \\ & & & \frac{1}{10} & \frac{1}{10} & \frac{8}{10} \end{pmatrix},\ P_4=\begin{pmatrix} \frac{2}{10} & \frac{2}{10} & \frac{2}{10} & \frac{1}{10} & \frac{1}{10} & \frac{2}{10} \\ \frac{1}{10} & \frac{3}{10} & \frac{2}{10} & \frac{1}{10} & \frac{2}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{2}{10} & \frac{3}{10} & \frac{2}{10} & \frac{1}{10} & \frac{1}{10} \\ \frac{3}{10} & \frac{2}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{2}{10} \\ \frac{1}{10} & \frac{4}{10} & \frac{1}{10} & \frac{2}{10} & \frac{1}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{4}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{2}{10} \end{pmatrix}.$$

It is easy to see that this model satisfies the conditions of DeGroot submodel for $\Delta_1 = (\{j\})_{j\in\langle 6\rangle}$, $\Delta_2 = (\langle 2\rangle, \{3\}, \{4\}, \{5,6\})$, $\Delta_3 = (\langle 3\rangle, \{4\}, \{5,6\})$, $\Delta_4 = (\langle 3\rangle, \{4,5,6\})$, and $\Delta_5 = (\langle 6\rangle)$. We have $P_1 \in G_{\Delta_2,\Delta_1}$, $P_2 \in G_{\Delta_3,\Delta_2}$, $P_3 \in G_{\Delta_4,\Delta_3}$, and $P_4 \in G_{\Delta_5,\Delta_4}$. By Theorem 2.5, $P_4P_3P_2P_1$ is a stable matrix, so, a consensus is reached at time 4... — the continuation, using Theorem 2.5, is left to the reader (see also Example 2.7). Further, we comment on the matrices P_1, P_2, P_3, and P_4, and use Theorem 1.16. An interpretation of P_1 (at time 1 — see the DeGroot model again): the individuals 1 and 2 negotiate — negotiation, a good idea —, and agree the same probability, $\frac{4}{10}$, for the initial opinion/estimation of individual 1, *i.e.*, for p_{01}, and, separately, the same probability, $\frac{6}{10}$, for the initial opinion of individual 2, *i.e.*, for p_{02}; the individuals 5 and 6 proceed similarly; the individuals 3 and 4 do not negotiate... An interpretation of P_2 (at time 2): the individuals 1, 2, and 3 negotiate, and agree the same probability, $\frac{7}{10}$ ($\frac{7}{10} = \frac{3}{10}+\frac{4}{10} = \frac{2}{10}+\frac{5}{10} = \frac{1}{10}+\frac{6}{10}$), for the opinions of individuals 1 and 2 at time 1 considered together, *i.e.*, for p_{11} and p_{12} considered together, and, separately, the same probability, $\frac{3}{10}$, for the opinion of individual 3 at time 1, *i.e.*, for p_{13}; the individuals 5 and 6 renegotiate, and agree the same probability, 1 ($1 = \frac{2}{10} + \frac{8}{10} = \frac{4}{10} + \frac{6}{10}$), for their opinions at time 1 considered together, *i.e.*, for p_{15} and p_{16} considered together; the individual 4 does not negotiate... An interpretation of P_3 (at time 3): the individuals 1, 2, and 3 renegotiate, and agree the same probability, 1 ($1 = \frac{3}{10} + \frac{2}{10} + \frac{5}{10} = \frac{2}{10} + \frac{4}{10} + \frac{4}{10} = \frac{1}{10} + \frac{6}{10} + \frac{3}{10}$), for their opinions at time 2 considered together, *i.e.*, for p_{21}, p_{22}, and p_{23} considered together; the individuals 4, 5, and 6 negotiate, and agree the same probability, $\frac{1}{10}$, for the opinion of individual 4 at time 2, *i.e.*, for p_{24}, and, separately, the same probability, $\frac{9}{10}$ ($\frac{9}{10} = \frac{1}{10}+\frac{8}{10} = \frac{2}{10}+\frac{7}{10} = \frac{1}{10}+\frac{8}{10}$), for the opinions of individuals 5 and 6 at time 2 considered together, *i.e.*, for p_{25} and p_{26} considered together. An interpretation of P_4 (at time 4): all the individuals negotiate, and agree the same probability, $\frac{6}{10}$ ($\frac{6}{10} = \frac{2}{10} + \frac{2}{10} + \frac{2}{10} = \frac{1}{10} + \frac{3}{10} + \frac{2}{10} = ...$), for the opinions of individuals 1, 2, and 3 at time 3 considered together, *i.e.*, for p_{31}, p_{32}, and p_{33} considered together, and, separately, the same probability, $\frac{4}{10}$ ($\frac{4}{10} = \frac{1}{10} + \frac{1}{10} + \frac{2}{10} = \frac{1}{10} + \frac{2}{10} + \frac{1}{10} = ...$), for the opinions of individuals 4, 5, and 6 at time 3 considered together, *i.e.*, for p_{34}, p_{35}, and p_{36} considered together. The negotiations and renegotiations took place from small subsets (of $\langle 6\rangle$)/subgroups to larger and larger subsets/subgroups until to the whole set/group. By Theorem 1.16, P_2 can be replaced with any matrix which is similar to it, such as,

$$\begin{pmatrix} \frac{3}{10} & \frac{4}{10} & \frac{3}{10} & & & \\ \frac{3}{10} & \frac{4}{10} & \frac{3}{10} & & & \\ \frac{3}{10} & \frac{4}{10} & \frac{3}{10} & & & \\ & & & 1 & & \\ & & & & 1 & 0 \\ & & & & 0 & 1 \end{pmatrix} := P_2',$$

and

$$\begin{pmatrix} \frac{7}{10} & 0 & \frac{3}{10} & & & \\ 0 & \frac{7}{10} & \frac{3}{10} & & & \\ \frac{7}{10} & 0 & \frac{3}{10} & & & \\ & & & 1 & & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix} := P_2''.$$

The matrices $(P_2)_{\langle 3\rangle}^{\langle 2\rangle}$,

$$(P_2)_{\langle 3\rangle}^{\langle 2\rangle} = \begin{pmatrix} \frac{3}{10} & \frac{4}{10} \\ \frac{2}{10} & \frac{5}{10} \\ \frac{1}{10} & \frac{6}{10} \end{pmatrix},$$

$(P_2')_{\langle 3\rangle}^{\langle 2\rangle}$,

$$(P_2')_{\langle 3\rangle}^{\langle 2\rangle} = \begin{pmatrix} \frac{3}{10} & \frac{4}{10} \\ \frac{3}{10} & \frac{4}{10} \\ \frac{3}{10} & \frac{4}{10} \end{pmatrix},$$

and $(P_2'')^{\langle 2\rangle}_{\langle 3\rangle}$,

$$\left(P_2''\right)^{\langle 2\rangle}_{\langle 3\rangle} = \begin{pmatrix} \frac{7}{10} & 0 \\ 0 & \frac{7}{10} \\ \frac{7}{10} & 0 \end{pmatrix}$$

(each of them has 6 entries,

$$\begin{pmatrix} \star & \star \\ \star & \star \\ \star & \star \end{pmatrix},$$

"$\star$" stands for a positive or zero entry), have the property, a subset/subgroup property: the row sums are equal to $\frac{7}{10}$ — only the row sums count, not the positions of positive entries, but at least one entry must be positive in each row (because $\frac{7}{10} > 0$), only $\frac{7}{10}$ counts, $\frac{7}{10}$ is the probability... (see above), and this probability is for the subset $\langle 2\rangle$ of $\langle 6\rangle$. Among P_2, P_2', and P_2'', P_2'' is the best because it has the smallest number of positive entries — these are the positive probabilities for the subsets involved here: $\frac{7}{10}$ for (the subset) $\langle 2\rangle$ ($\frac{7}{10}$ = the probability assigned by the individual i for the opinions of individuals 1 and 2 at time 1 considered together, $\forall i \in \langle 3\rangle$), $\frac{3}{10}$ for $\{3\}$, 1 for $\{4\}$, and 1 for $\{5, 6\}$. Similarly, P_3 and P_4 can be replaced with any matrices, Q_3 and Q_4, which are similar to P_3 and P_4, respectively. By Theorem 1.16 we have

$$P_4P_3P_2P_1 = P_4P_3P_2'P_1 = P_4P_3P_2''P_1 = P_4Q_3P_2P_1 = ...$$

(we can work with $P_4P_3P_2P_1$, or with $P_4P_3P_2'P_1$, or with...). Note that P_1, P_2, and P_3 are reducible while P_4 is irreducible (being positive), but the last matrix can be replaced with any reducible matrix which is similar to it, such as,

$$\begin{pmatrix} \frac{6}{10} & 0 & 0 & \frac{4}{10} & 0 & 0 \\ \frac{6}{10} & 0 & 0 & \frac{4}{10} & 0 & 0 \\ \frac{6}{10} & 0 & 0 & \frac{4}{10} & 0 & 0 \\ \frac{6}{10} & 0 & 0 & \frac{4}{10} & 0 & 0 \\ \frac{6}{10} & 0 & 0 & \frac{4}{10} & 0 & 0 \\ \frac{6}{10} & 0 & 0 & \frac{4}{10} & 0 & 0 \end{pmatrix} := U_4$$

— P_1, P_2, P_3, and U_4 are reducible, but, interestingly, $U_4P_3P_2P_1$ is irreducible (because it is positive — it is also stable).

For other examples for the DeGroot submodel, see, in the next section, Remark 3.7.

3. FIINITE-TIME CONSENSUS FOR DEGROOT MODEL ON DISTRIBUTED SYSTEMS

In this section, we consider the finite-time consensus problem for the DeGroot model on distributed systems (for these systems, see, *e.g.*, [6], see, *e.g.*, also [4]) — this section can also be useful for the study of other systems, such as, the multi-agent systems (for the multi-agent systems, see, *e.g.*, [14], [23], and [29]). For the DeGroot model on distributed systems, using the G method too, we have a result for reaching a partial or total (distributed) consensus in a finite time similar to that for the DeGroot model for reaching a partial or total consensus in a finite time (see Remark 3.2(b)). We show that for any connected graph having 2^m vertices, $m \geq 1$, and a spanning subgraph isomorphic to the m-cube graph, distributed averaging is performed in m steps — this result can be extended — research work — for any graph with $n_1 n_2 ... n_t$ vertices under certain conditions, where t, $n_1, n_2, ..., n_t \geq 2$, and, in this case, distributed averaging is performed in t steps.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a (nondirected simple finite) connected graph, where $\mathcal{V}$ is the vertex/node set and $\mathcal{E}$ is the edge set. (A *simple graph* is a graph without multiple edges and loops.) Suppose that $\mathcal{V} = \langle r \rangle$, and that $r \geq 2$. The edge set $\mathcal{E}$ is a set of 2-element subsets of $\langle r \rangle$, so,

$$\mathcal{E} \subseteq \{\{i,j\} \mid i,j \in \langle r \rangle \, , \; i \neq j\} \;\; (\mathcal{E} \neq \emptyset)$$

— *e.g.*, $\mathcal{E} = \{\{1,2\}, \{1,3\}, ..., \{1,r\}\}$ ($r \geq 2$). This graph, on the distributed systems or not, can be a mathematical model for a network of computers, processors, or sensors, for a team, when the graphs are involved, of robots, aircraft, or spacecraft... Set

$$\mathcal{N}_i = \{j \mid j \in \langle r \rangle \text{ and } \{i,j\} \in \mathcal{E}\}, \; \forall i \in \langle r \rangle .$$

$\mathcal{N}_i$ is called the *set of neighbors of* i, $\forall i \in \langle r \rangle$. $i \notin \mathcal{N}_i$, $\forall i \in \langle r \rangle$. Suppose that each vertex/node $i \in \langle r \rangle$ holds an initial value, say, q_{0i}, $q_{0i} \in \mathbb{R}$, and we call the vector

$$q_0 = (q_{0i})_{i \in \langle r \rangle} = (q_{01}, q_{02}, ..., q_{0r})$$

the *initial value vector*. We use homogeneous and nonhomogeneous linear iterations: at each time $n \geq 1$ ($n \in \mathbb{N}^*$), each vertex i updates its value as (see, *e.g.*, [27] and [30] — only the homogeneous linear iterations are considered in these articles)

$$q_{ni} = \sum_{j \in \{i\} \cup \mathcal{N}_i} (W_n)_{ij} \, q_{n-1j},$$

where $(W_n)_{ij} \in \mathbb{R}$, $\forall j \in \{i\} \cup \mathcal{N}_i$, $(W_n)_{ij}$ is the weight on q_{n-1j} at vertex i at time n, $\forall j \in \{i\} \cup \mathcal{N}_i$. Set the (row) vector

$$q_n = (q_{ni})_{i \in \langle r \rangle} = (q_{n1}, q_{n2}, ..., q_{nr}) \, , \; \forall n \geq 0$$

— using q_0, we obtain q_1 (see above), using q_1, we obtain q_2... We call the vector q_n the *value vector at time/step* n, $\forall n \geq 0$ — the value vector at time 0 is also (see above) called the initial value vector. Set the real $r \times r$ matrix

$$W_n = \left((W_n)_{ij} \right)_{i,j \in \langle r \rangle},$$

$(W_n)_{ij}$ was defined above, $\forall i \in \langle r \rangle$, $\forall j \in \{i\} \cup \mathcal{N}_i$, and

$$(W_n)_{ij} = 0 \text{ if } j \notin \{i\} \cup \mathcal{N}_i, \ \forall i \in \langle r \rangle$$

(we can have $(W_n)_{ij} = 0$ even when $j \in \{i\} \cup \mathcal{N}_i$, where $i \in \langle r \rangle$ and $n \geq 1$), $\forall n \geq 1$. We call the matrix W_n the *weight matrix at time/step* n, $\forall n \geq 1$. W_n, n fixed, is symmetric or not (each (nondirected/undirected) edge can be considered as two directed edges). Obviously, we have

$$q_n' = W_n q_{n-1}' = W_n W_{n-1} ... W_1 q_0', \ \forall n \geq 1$$

(q_n' is the transpose of q_n...).

The above model is a DeGroot-type model — compare this model with the DeGroot model from Section 2. It is determined by the graph $\mathcal{G}$, real vector q_0, and real matrices W_n, $n \geq 1$. Due these facts, we call it the *DeGroot model on* (*the triple*) $\left(\mathcal{G}, q_0, (W_n)_{n\geq 1}\right)$, and will use the generic name "(the) DeGroot model on distributed systems". We say that the DeGroot model on $\left(\mathcal{G}, q_0, (W_n)_{n\geq 1}\right)$ is *homogeneous* if $W_1 = W_2 = W_3 = ...$ and *nonhomogeneous* if $\exists u, v \geq 1$, $u \neq v$, such that $W_u \neq W_v$, and will use the generic names "(the) homogeneous DeGroot model on distributed systems" and "(the) nonhomogeneous DeGroot model on distributed systems", respectively.

Note that the DeGroot model on $\left(\langle r \rangle, p_0, (P_n)_{n\geq 1}\right)$ (see Section 2) is trivially identical with the DeGroot model on $\left(\mathcal{K}_r, p_0, (P_n)_{n\geq 1}\right)$, $\forall r \geq 2$, $\forall p_0$, p_0 is a real vector here, $\forall (P_n)_{n\geq 1}$, $(P_n)_{n\geq 1}$ is a sequence of stochastic matrices here, where $\mathcal{K}_r$ is the complete graph with r vertices, $\forall r \geq 2$ — moreover, it is identical with the DeGroot model on $\left(\mathcal{G}_1, p_0, (P_n)_{n\geq 1}\right)$, $\forall r \geq 2$, $\forall p_0$, p_0 is a real vector here, $\forall (P_n)_{n\geq 1}$, $(P_n)_{n\geq 1}$ is a sequence of stochastic matrices here, where $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$, $\mathcal{V}_1 = \langle r \rangle$ and
$\mathcal{E}_1 = \left\{ \{i,j\} \,\middle|\, i,j \in \langle r \rangle,\ i \neq j, \text{ and } \exists n \geq 1 \text{ such that } (P_n)_{ij} > 0 \text{ or } (P_n)_{ji} > 0 \right\}.$

Definition 2.1 (from Section 2) and the first part of the next definition are similar.

Definition 3.1. (See also [27] and [30].) *Consider the DeGroot model on* $\left(\mathcal{G}, q_0, (W_n)_{n\geq 1}\right)$. *We say that the graph* $\mathcal{G}$ *reaches a* (*distributed*) *consensus* *if* $\lim_{n\to\infty} W_n W_{n-1} ... W_1$ *exists and is a stable matrix. Set*

$$W_\infty = \lim_{n\to\infty} W_n W_{n-1} ... W_1$$

when $\lim_{n\to\infty} W_n W_{n-1} ... W_1$ *exists.*(W_∞ *is a real* $r \times r$ *matrix.*) *In particular, if*

$$W_\infty = e' \left(\frac{1}{r}, \frac{1}{r}, ..., \frac{1}{r} \right),$$

we say that the graph $\mathcal{G}$ *performs distributed averaging* ($e = e(r) = (1, 1, ..., 1)$; e' *is its transpose*).

Reaching a distributed consensus for the DeGroot model on distributed systems — this is the (*distributed*) *consensus problem* for/of the DeGroot model on distributed systems, and is similar to the consensus problem for the DeGroot model. When a distributed consensus is reached, it is reached in a finite or infinite time — obviously, the first case is the best. When distributed averaging is performed, it is performed in a finite or infinite time — the first case is also the best.

When the graph $\mathcal{G}$ performs distributed averaging, each vertex holds the average of initial values, $\frac{1}{r} \sum_{i=1}^{r} q_{0i}$ — an interpretation (see, *e.g.*, [30, p. 65]): if q_{0i} is the amount of a resource at the vertex i, $\forall i \in \langle r \rangle$, then the average is the fair or uniform allocation of the resource across the graph; an application (see, *e.g.*, [22, p. 228]): clock synchronization (by averaging) in distributed systems.

For more information concerning the above things, see, *e.g.*, [15], [27], and [30].

Recall that the DeGroot model and its consensus problem and the consensus problem in computer science have some things in common with the finite Markov chains — some references are given in the first paragraph before Theorem 2.2 (in Section 2).

Remark 3.2. (a) Concerning the (total) distributed consensuses, obviously, we have a theorem similar to Theorem 2.2 and a remark similar to Remark 2.3.

(b) Concerning the finite-time partial and total distributed consensuses (for reaching a partial or total distributed consensus in a finite time), only when the matrices at times $t+1$, $t+2$, ... are (see Definition 1.3) stochastic real (the matrices at times 1, 2, ..., t being (stochastic or nonstochastic) real), we have a theorem similar to Theorem 2.5 and remarks similar to Remarks 2.4(c) and 2.6(b)-(c).

(c) Concerning distributed averaging, if we construct/find t matrices, W_1, W_2, ..., W_t ($t \geq 1$), such that

$$W_t W_{t-1} ... W_1 = e' \left(\frac{1}{r}, \frac{1}{r}, ..., \frac{1}{r} \right),$$

it makes no sense to construct other matrices — however, if the reader wants to construct a sequence of matrices, $(W_n)_{n \geq 1}$, such that $W_\infty = W_t W_{t-1} ... W_1$, she/he can take, *e.g.*, $W_{t+1} = W_{t+2} = ... = I_r$, where I_r is the identity matrix of dimension r.

Below we give a result for finite-time (total) distributed averaging — an application of the G method (more precisely, of Theorem 1.10)...

Theorem 3.3. *Let* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *be a* (*nondirected simple finite*) *connected graph. Suppose that* $|\mathcal{V}| = 2^m$, $m \geq 1$. *Suppose that* $\mathcal{G}$ *has a spanning subgraph isomorphic to the* m*-cube graph,* $\mathcal{Q}_m$. *Then there exist the weight matrices* W_1, W_2, ..., W_m *such that distributed averaging is performed for* $\mathcal{G}$ *in* m *steps* (*for any initial value vector* q_0). (See Remark 3.2(c) again.)

Proof. It is sufficient to consider the case when $\mathcal{G} = \mathcal{Q}_m$ — Theorem 3.3 gives the best possible number of steps, m, for distributed averaging for this graph because the diameter of $\mathcal{Q}_m$ is equal to m. Further, we consider this case, and using the G method, we show that distributed averaging is performed in m steps.

Consider that $\mathcal{V} = \{0,1\}^m$ and

$$\mathcal{E} = \{\{x,y\} \mid x,y \in \{0,1\}^m \text{ and } H(x,y) = 1\},$$

where H is the Hamming distance (the edge set $\mathcal{E}$ is a set of 2-element subsets of $\{0,1\}^m$; $x = (x_1, x_2, ..., x_m)$ ...). Consider the stochastic matrices — stochastic matrices here! — $W_1, W_2, ..., W_m$ (see Remark 3.2(c) again),

$$(W_l)_{(x_1,x_2,...,x_m)(x_1,x_2,...,x_m)} =$$

$$= (W_l)_{(x_1,x_2,...,x_m)(x_1,x_2,...,x_{m-l},(x_{m-l+1}+1) \bmod 2, x_{m-l+2},...,x_m)} = \frac{1}{2},$$

$\forall l \in \langle m \rangle$, $\forall (x_1, x_2, ..., x_m) \in \{0,1\}^m$ ($(W_l)_{(x_1,x_2,...,x_m)(x_1,x_2,...,x_m)}$ is the entry $((x_1, x_2, ..., x_m), (x_1, x_2, ..., x_m))$ of W_l...; $x_{m-l+2}, x_{m-l+3}, , ..., x_m$ vanish when $l = 1$; $x_1, x_2, ..., x_{m-l}$ vanish when $l = m$; obviously, $W_1, W_2, ..., W_m$ are symmetric matrices). Let

$$U_{(x_1,x_2,...,x_v)} =$$

$$= \{(y_1, y_2, ..., y_m) \mid (y_1, y_2, ..., y_m) \in \{0,1\}^m \text{ and } y_1 = x_1, y_2 = x_2, ..., y_v = x_v\},$$

$\forall v \in \langle m \rangle$. Consider the partitions

$$\Delta_t = \left(U_{(x_1,x_2,...,x_{m-t+1})}\right)_{(x_1,x_2,...,x_{m-t+1})\in\{0,1\}^{m-t+1}}, \ \forall t \in \langle m \rangle ,$$

and

$$\Delta_{m+1} = (\{0,1\}^m)$$

of $\{0,1\}^m$. Obviously,

$$\Delta_1 = \left(U_{(x_1,x_2,...,x_m)}\right)_{(x_1,x_2,...,x_m)\in\{0,1\}^m} = (\{x\})_{x\in\{0,1\}^m} .$$

We have $W_l \in G_{\Delta_{l+1},\Delta_l}, \forall l \in \langle m \rangle$ — we prove this statement. Let $l \in \langle m \rangle$ and $K \in \Delta_{l+1}$. It follows that

$$K = \begin{cases} \{0,1\}^m & \text{if } l = m, \\ U_{(x_1,x_2,...,x_{m-l})} \text{ for some } (x_1,x_2,...,x_{m-l}) \in \{0,1\}^{m-l} & \text{if } l \in \langle m-1 \rangle . \end{cases}$$

Case 1. $l = m$. In this case, $\Delta_{m+1} = (\{0,1\}^m)$ and $\Delta_m = \left(U_{(0)}, U_{(1)}\right)$. Let $(x_1,x_2,...,x_m) \in \{0,1\}^m$. Since $\{0,1\}^m = U_{(0)} \cup U_{(1)}$, it follows that

$$(x_1,x_2,...,x_m) \in U_{(0)} \text{ or } (x_1,x_2,...,x_m) \in U_{(1)}.$$

If $(x_1,x_2,...,x_m) \in U_{(0)}$, then

$$((x_1+1) \bmod 2, x_2, ..., x_m) \in U_{(1)};$$

if $(x_1,x_2,...,x_m) \in U_{(1)}$, then

$$((x_1+1) \bmod 2, x_2, ..., x_m) \in U_{(0)}.$$

Further, by the definition of W_m, we have $W_m \in G_{\Delta_{m+1},\Delta_m}$.

Case 2. $l \in \langle m-1 \rangle$. In this case,

$$K = U_{(x_1,x_2,...,x_{m-l})} \text{ for some } (x_1,x_2,...,x_{m-l}) \in \{0,1\}^{m-l} .$$

Further, we have $K = U_{(x_1,x_2,...,x_{m-l},0)} \cup U_{(x_1,x_2,...,x_{m-l},1)}$, where, obviously, $U_{(x_1,x_2,...,x_{m-l},0)}, U_{(x_1,x_2,...,x_{m-l},1)} \in \Delta_l$. Let $(x_1,x_2,...,x_m) \in K$. It follows that

$$(x_1,x_2,...,x_m) \in U_{(x_1,x_2,...,x_{m-l},0)} \text{ or } (x_1,x_2,...,x_m) \in U_{(x_1,x_2,...,x_{m-l},1)}.$$

If $(x_1,x_2,...,x_m) \in U_{(x_1,x_2,...,x_{m-l},0)}$, then

$$x_{m-l+1} = 0 \text{ and } (x_1,x_2,...,x_{m-l},1,x_{m-l+2},...,x_m) \in U_{(x_1,x_2,...,x_{m-l},1)};$$

if $(x_1, x_2, ..., x_m) \in U_{(x_1,x_2,...,x_{m-l},1)}$, then

$$x_{m-l+1} = 1 \text{ and } (x_1, x_2, ..., x_{m-l}, 0, x_{m-l+2}, ..., x_m) \in U_{(x_1,x_2,...,x_{m-l},0)}.$$

Further, by the definition of W_l, we have $W_l \in G_{\Delta_{l+1},\Delta_l}$. The statement was proved. Further, since $W_l \in G_{\Delta_{l+1},\Delta_l}$, $\forall l \in \langle m \rangle$, by Theorem 1.10, $W_m W_{m-1} ... W_1$ is a stable matrix and

$$W_m W_{m-1} ... W_1 = e' W_m^{-+} W_{m-1}^{-+} ... W_1^{-+}.$$

Since

$$W_m^{-+} W_{m-1}^{-+} ... W_1^{-+} =$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & & \\ & & \frac{1}{2} & \frac{1}{2} \end{pmatrix} ... \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & & & & & & \\ & & \frac{1}{2} & \frac{1}{2} & & & & \\ & & & & \ddots & & & \\ & & & & & & \frac{1}{2} & \frac{1}{2} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{2^m} & \frac{1}{2^m} & ... & \frac{1}{2^m} \end{pmatrix},$$

we have

$$q_m = \left(W_m W_{m-1} ... W_1 q_0' \right)' = \left(\frac{q_{0(0,0,...,0)}}{2^m}, \frac{q_{0(0,0,...,0,1)}}{2^m}, ..., \frac{q_{0(1,1,...,1)}}{2^m} \right)$$

($\mathcal{V} = \{0,1\}^m$, so, $|\mathcal{V}| = 2^m$; $q_{0(0,0,...,0)}$ is the component $(0,0,...,0)$ of (initial value vector) q_0...). So, distributed averaging is performed in m steps. ∎

Example 3.4. To illustrate Theorem 3.3, we consider the 3-cube graph, $\mathcal{Q}_3$. In this case, we have $\mathcal{V} = \{0,1\}^3$ and

$$\mathcal{E} = \left\{ \{x, y\} \,\middle|\, x, y \in \{0,1\}^3 \text{ and } H(x, y) = 1 \right\} =$$

$$= \{\{(0,0,0), (0,0,1)\}, \{(0,0,0), (0,1,0)\}, ..., \{(1,1,0), (1,1,1)\}\}.$$

Further, we have

$$W_1 = \begin{array}{c} (0,0,0) \\ (0,0,1) \\ (0,1,0) \\ (0,1,1) \\ (1,0,0) \\ (1,0,1) \\ (1,1,0) \\ (1,1,1) \end{array} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & & & & & & \\ \frac{1}{2} & \frac{1}{2} & & & & & & \\ & & \frac{1}{2} & \frac{1}{2} & & & & \\ & & \frac{1}{2} & \frac{1}{2} & & & & \\ & & & & \frac{1}{2} & \frac{1}{2} & & \\ & & & & \frac{1}{2} & \frac{1}{2} & & \\ & & & & & & \frac{1}{2} & \frac{1}{2} \\ & & & & & & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

(the columns are labeled similarly, *i.e.*, using the lexicographic order from left to right),

$$W_2 = \begin{array}{c} (0,0,0) \\ (0,0,1) \\ (0,1,0) \\ (0,1,1) \\ (1,0,0) \\ (1,0,1) \\ (1,1,0) \\ (1,1,1) \end{array} \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & & & & \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & & & & \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & & & & \\ & & & & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ & & & & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ & & & & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ & & & & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix},$$

$$W_3 = \begin{array}{c} (0,0,0) \\ (0,0,1) \\ (0,1,0) \\ (0,1,1) \\ (1,0,0) \\ (1,0,1) \\ (1,1,0) \\ (1,1,1) \end{array} \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Further, we have

$$U_{(x_1,x_2,x_3)} = \{(x_1,x_2,x_3)\}, \ \forall (x_1,x_2,x_3) \in \{0,1\}^3,$$

$$U_{(0,0)} = \{(0,0,0),(0,0,1)\}, U_{(0,1)} = \{(0,1,0),(0,1,1)\},$$

$$U_{(1,0)} = \{(1,0,0),(1,0,1)\}, U_{(1,1)} = \{(1,1,0),(1,1,1)\},$$

$$U_{(0)} = U_{(0,0)} \cup U_{(0,1)} = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1)\},$$

$$U_{(1)} = U_{(1,0)} \cup U_{(1,1)} = \{(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}.$$

Further, we have

$$\Delta_1 = (\{(x_1,x_2,x_3)\})_{(x_1,x_2,x_3)\in\{0,1\}^3} = (\{(0,0,0)\},\{(0,0,1)\},...,\{(1,1,1)\}),$$

$$\Delta_2 = \left(U_{(0,0)}, U_{(0,1)}, U_{(1,0)}, U_{(1,1)}\right),$$

$$\Delta_3 = \left(U_{(0)}, U_{(1)}\right),$$

$$\Delta_4 = \left(\{0,1\}^3\right)$$

(Δ_4 is the improper (degenerate) partition of $\{0,1\}^3$ — Δ_4 has only one set). Further, we have $W_1 \in G_{\Delta_2,\Delta_1}$, $W_2 \in G_{\Delta_3,\Delta_2}$, $W_3 \in G_{\Delta_4,\Delta_3}$; further, we have

$$W_3^{-+} W_2^{-+} W_1^{-+} =$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & & \\ & & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & & & & & & \\ & & \frac{1}{2} & \frac{1}{2} & & & & \\ & & & & \frac{1}{2} & \frac{1}{2} & & \\ & & & & & & \frac{1}{2} & \frac{1}{2} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{2^3} & \frac{1}{2^3} & \cdots & \frac{1}{2^3} \end{pmatrix}.$$

By Theorem 1.10 ($W_3 W_2 W_1 = e' W_3^{-+} W_2^{-+} W_1^{-+}$) or by direct computation,

$$W_3 W_2 W_1 = \begin{pmatrix} \frac{1}{2^3} & \frac{1}{2^3} & \cdots & \frac{1}{2^3} \\ \frac{1}{2^3} & \frac{1}{2^3} & \cdots & \frac{1}{2^3} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{2^3} & \frac{1}{2^3} & \cdots & \frac{1}{2^3} \end{pmatrix}.$$

Finally, we have ($\mathcal{V} = \{0,1\}^3$, so, $|\mathcal{V}| = 2^3$; $q_{0(0,0,0)}$ is the component $(0,0,0)$ of q_0...)

$$q_3 = \left(W_3 W_2 W_1 q_0' \right)' = \left(\frac{q_{0(0,0,0)}}{2^3}, \frac{q_{0(0,0,1)}}{2^3}, ..., \frac{q_{0(1,1,1)}}{2^3} \right)$$

— distributed averaging is performed at time/step 3 (for any initial value vector q_0). To understand Theorem 3.3 and this example better, the reader, if she/he wants, can compute the value vectors q_1 and q_2. Note that the matrices W_1, W_2, and W_3 are symmetric, but by Theorem 1.16, W_2 and W_3 can be replaced with nonsymmetric ones.

Remark 3.5. (Research work — find the graphs.) Theorem 3.3 can be extended for any connected graph with $n_1 n_2 ... n_t$ vertices, where n_1, n_2, ..., $n_t \geq 2$ (preferably prime numbers), and having a spanning subgraph for which Theorem 1.10 can be applied for t stochastic matrices... — we will perform distributed averaging in t steps.

For the above remark, we consider the next example.

Example 3.6. (Suggested by Example 2.11 from [17], which refers to the uniform generation of permutations of order n.) Consider the graph $\mathcal{G}_3 = (\mathcal{V}_3, \mathcal{E}_3)$, where $\mathcal{V}_3 = \mathbb{S}_3$, $\mathbb{S}_3 =$ the set of permutations of order 3, and $\mathcal{E}_3 =$ the union of edge sets of nondirected graphs of matrices W_1 and W_2, where

$$W_1 = \begin{array}{c} (123) \\ (132) \\ (213) \\ (231) \\ (321) \\ (312) \end{array} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & & & & \\ \frac{1}{2} & \frac{1}{2} & & & & \\ & & \frac{1}{2} & \frac{1}{2} & & \\ & & \frac{1}{2} & \frac{1}{2} & & \\ & & & & \frac{1}{2} & \frac{1}{2} \\ & & & & \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

$$W_2 = \begin{array}{c} (123) \\ (132) \\ (213) \\ (231) \\ (321) \\ (312) \end{array} \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \end{pmatrix}.$$

The matrices W_1 and W_2 were obtained by the swapping method — for this method, see, *e.g.*, [8, pp. 645–646]. By Theorem 1.10 ($W_1 \in G_{\Delta_2,\Delta_1}$, $W_2 \in G_{\Delta_3,\Delta_2}$, where...) or by direct computation,

$$W_2W_1 = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \cdots & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \cdots & \frac{1}{6} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{6} & \frac{1}{6} & \cdots & \frac{1}{6} \end{pmatrix},$$

so, distributed averaging is performed, in this case, at time 2 ($3! = 6 = 2 \cdot 3$). Note that the graph $\mathcal{G}_3$ is a 3-regular graph and is not isomorphic to the triangular prism graph, which is also a 3-regular graph — $\mathcal{G}_3$ can be replaced with any graph with 3! vertices which has a spanning subgraph isomorphic to $\mathcal{G}_3$. For the triangular prism graph, using the G method, distributed averaging can also be performed at time 2 — an exercise for the reader.

The above example can be generalized for $\mathcal{G}_n = (\mathcal{V}_n, \mathcal{E}_n)$, $\mathcal{V}_n = \mathbb{S}_n$, $\mathbb{S}_n =$ the set of permutations of order n... Due to the swapping method, this graph must be an $\frac{n(n-1)}{2}$ - regular graph — a necessary but not sufficient condition (see the above example).

Remark 3.7. (a) For the DeGroot model on distributed systems, labelling the vertices of (involved) graphs is an important problem — a good/suitable labelling can lead to a good/fast algorithm (see Theorem 3.3 and its proof and Examples 3.4 and 3.6).

(b) Theorem 3.3 and its proof and Example 3.6 (see also Remark 3.5) lead to possible cases for the DeGroot submodel (from Section 2) on graphs/networks, in each case, a consensus being reached in a finite time — for the DeGroot model on graphs, see, *e.g.*, [9] and [12].

For the distributed consensuses (in particular, for distributed averaging), we could combine the G method with other methods, obtaining certain hybrid methods: we could use the G method for subgraphs, we could use the G method together with the flooding (for this method, see, *e.g.* [30, p. 65], see, *e.g.*, also [22, p. 228]), we could use a leader vertex or more... This idea can be developed — another research work —, but we give only an example, the next example.

Example 3.8. Consider the graph from Fig. 1 in [30] (this graph is also considered in [27, Fig. 1]) — the weights from there are not taken into account. This graph, say, $\mathcal{G}$, can be obtained from the wheel graph $\mathcal{W}_5$ and complete graph $\mathcal{K}_5$ by edge merging. Consider that the vertices of $\mathcal{K}_5$ are 1, 2, 3, 4, 5 and the vertices of $\mathcal{W}_5$ are 1, 2, 6, 7, 8, and that 8 is the universal vertex of $\mathcal{W}_5$, 6 is adjacent to 1 (6 is also adjacent to 8), and 7 is adjacent to 2 (7 is also adjacent to 6 and 8)— $\{1,2\}$ is the common edge of $\mathcal{W}_5$ and $\mathcal{K}_5$. Consider that the initial value vector is q_0. First, we consider the subgraph $\mathcal{W}_5$ of $\mathcal{G}$. For this subgraph, we consider that the initial value vector is $q_0^{(1)} = q_0 \mid_{\{1,2,6,7,8\}}$, $q_0 \mid_{\{1,2,6,7,8\}}=$ the restriction of q_0 to $\{1,2,6,7,8\}$. Consider the matrix $W_1^{(1)}$ with rows and columns 1, 2, 6, 7, 8,

$$W_1^{(1)} = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \cdots & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \cdots & \frac{1}{5} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{5} & \frac{1}{5} & \cdots & \frac{1}{5} \end{pmatrix}.$$

$W_1^{(1)}$ is a stable matrix, and, therefore, $W_1^{(1)} \in G_{\Delta_2^{(1)},\Delta_1^{(1)}}$, where $\Delta_1^{(1)} = (\{1\},\{2\},\{6\},\{7\},\{8\})$ and $\Delta_2^{(1)} = (\{1,2,6,7,8\})$ — a trivial case for the G method, more exactly, for Theorem 1.10. Further, from

$$\left(q_1^{(1)}\right)' = W_1^{(1)} \left(q_0^{(1)}\right)' = W_1^{(1)} \left(q_0 \mid_{\{1,2,6,7,8\}}\right)',$$

we obtain

$$q_1^{(1)} = \left(\frac{1}{5} \sum_{i\in\{1,2,6,7,8\}} q_{0i}, \frac{1}{5} \sum_{i\in\{1,2,6,7,8\}} q_{0i}, ..., \frac{1}{5} \sum_{i\in\{1,2,6,7,8\}} q_{0i}\right),$$

$q_1^{(1)}$ is the value vector at time 1 for the subgraph $\mathcal{W}_5$. Second, we consider the subgraph of $\mathcal{G}$ with vertices 1, 3, 4, 5 and isomorphic to the complete graph

$\mathcal{K}_4$. For this subgraph, we consider that the initial value vector is

$$\left(\frac{1}{5}\sum_{i\in\{1,2,6,7,8\}} q_{0i}, q_{03}, q_{04}, q_{05}\right) := q_0^{(2)}.$$

Consider the matrix $W_1^{(2)}$ with rows and columns 1, 3, 4, 5,

$$W_1^{(2)} = \begin{pmatrix} \frac{5}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{5}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{5}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{5}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \end{pmatrix}.$$

$W_1^{(2)}$ is also a stable matrix. Further, from

$$\left(q_1^{(2)}\right)' = W_1^{(2)}\left(q_0^{(2)}\right)',$$

we obtain

$$q_1^{(2)} = \left(\frac{1}{8}\sum_{i\in\langle 8\rangle} q_{0i},\ \frac{1}{8}\sum_{i\in\langle 8\rangle} q_{0i},\ \frac{1}{8}\sum_{i\in\langle 8\rangle} q_{0i},\ \frac{1}{8}\sum_{i\in\langle 8\rangle} q_{0i}\right),$$

$q_1^{(2)}$ is the value vector at time 1 for the above subgraph. Consider that the vertex 1 (it can be a leader) transmits the average $\frac{1}{8}\sum_{i\in\langle 8\rangle} q_{0i}$ to the vertices 2, 6, and 8 and, further, that the vertex 2 (this can also be a leader) or 6 transmits the above average to 7 — and thus distributed averaging is performed. Note that we can use $\mathcal{K}_5$ instead of the subgraph of $\mathcal{G}$ with vertices 1, 3, 4, 5 and isomorphic to $\mathcal{K}_4$...; finally, the vertices 1 and 2 (these can be leaders) will transmit the average $\frac{1}{8}\sum_{i\in\langle 8\rangle} q_{0i}$ to the vertices 6, 7, and 8. The study of this alternative way is left to the reader.

For the (nondirected simple finite) connected graphs with at least two vertices, distributed averaging based on the DeGroot model on distributed systems and the uniform (random) generation based on Markov chains of the vertices of a graph taking its edge set into account (when $\{i, j\}$ $(i \neq j)$ is not a edge of the graph, any transition matrix we use has the entries (i, j) and (j, i) equal to 0) are related problems/topics (...the stochastic matrices from Example 3.4 can be used to construct a Markov chain for the uniform generation of vertices of 3-cube graph...)

— *which of them is harder?...*

In this article, the framework for the DeGroot model and that for the DeGroot model on distributed systems we used are general, *i.e.*, homogeneous (when we use just one matrix at any time/step) and nonhomogeneous (when we use at least two different matrices at least two different times)

— *which of them are better*?

We believe that the nonhomogeneous frameworks are better — these frameworks are supported, among other things, by the DeGroot submodel, Theorem 3.3 (Example 3.4 is for $m = 3$), and Example 3.6 (we used nonhomogeneous products of matrices in these three places). For the homogeneous frameworks, see, *e.g.*, [7], [27], and [30].

References

[1] H. Attiya, J. Welch, *Distributed Computing*: *Fundamentals, Simulations and Advances Topics*, 2nd Edition. Wiley, Hoboken, NJ, 2004.

[2] R.L. Berger, *A necessary and sufficient condition for reaching a consensus using DeGroot's method*. J. Amer. Statist. Assoc. **76** (1981), 415−418.

[3] D.P. Bertsekas, J.N. Tsitsiklis, *Parallel and Distributed Computation*: *Numerical Methods*. Prentice-Hall, Englewood Cliff, NJ, 1989.

[4] S. Boccaletti, P. De Lellis, C.I. del Genio, K. Alfaro-Bittner, R. Criado, S. Jalan, M. Romance, *The structure and dynamics of networks with higher order interactions*. Phys. Rep. **1018** (2023), 1−64.

[5] S. Chatterjee, E. Seneta, *Towards consensus*: *some convergence theorems on repeated averaging*. J. Appl. Probab. **14** (1977), 89−97.

[6] G. Coulouris, J. Dollimore, T. Kindberg, G. Blair, *Distributed Systems*: *Concepts and Design*, 5th Edition. Addison-Wesley, Boston, 2012.

[7] M.H. DeGroot, *Reaching a consensus*. J. Amer. Statist. Assoc. **69** (1974), 118−121.

[8] L. Devroye, *Non-Uniform Random Variate Generation*. Springer-Verlag, New York, 1986.

[9] B. Golub, M.O. Jackson, *Naïve learning in social networks and the wisdom of crowds*. AEJ: Microeconomics **2** (2010), 112−149.

[10] M. Iosifescu, *Finite Markov Processes and Their Applications*. Wiley, Chichester & Ed. Tehnică, Bucharest, 1980; corrected republication by Dover, Mineola, N.Y., 2007.

[11] D.L. Isaacson, R.W. Madsen, *Markov Chains*: *Theory and Applications*. Wiley, New York, 1976; republication by Krieger, 1985.

[12] M.O. Jackson, *Social and Economic Networks*. Princeton University Press, Princeton, 2008.

[13] D.A. Levin, Y. Peres, *Markov Chains and Mixing Times*, 2nd Edition. AMS, Providence, RI, 2017. With contributions by E.L. Wilmer. With a chapter on "Coupling from the Past" by J.G. Propp and D.B. Wilson.

[14] S. Li, H. Du, X. Lin, *Finite-time consensus algorithm for multi-agent systems with double-integrator dynamics*. Automatica J. IFAC **47** (2011), 1706−1712.

[15] N.A. Lynch, *Distributed Algorithms*. Morgan Kaufmann Publishers, Inc., 1996.

[16] U. Păun, *General Δ-ergodic theory, with some results on simulated annealing.* Math. Rep. (Bucur.) **13(63)** (2011), 171−196.

[17] U. Păun, G_{Δ_1,Δ_2} *in action.* Rev. Roumaine Math. Pures Appl. **55** (2010), 387−406.

[18] U. Păun, *A hybrid Metropolis-Hastings chain.* Rev. Roumaine Math. Pures Appl. **56** (2011), 207−228.

[19] U. Păun, *G method in action*: *from exact sampling to approximate one.* Rev. Roumaine Math. Pures Appl. **62** (2017), 413−452.

[20] U. Păun, *Ewens distribution on* $\mathbb{S}_n$ *is a wavy probability distribution with respect to* n *partitions.* An. Univ. Craiova Ser. Mat. Inform. **47** (2020), 1−24.

[21] U. Păun, *G method in action*: *pivot*$^+$ *algorithm for self-avoiding walk.* ROMAI J. **20** (2024), no. 2, 81−98. With preprint: arXiv:2310.07564.

[22] M. Pease, R. Shostak, L. Lamport, *Reaching agreement in the presence of faults.* J. ACM **27** (1980), 228−234.

[23] W. Ren, R.W. Beard, E.M. Atkins, *A survey of consensus problems in multi-agent coordination.* In: Proceedings of the American Control Conference, 2005, pp. 1859−1864.

[24] L. Saloff-Coste, *Lectures on finite Markov chains.* In: E. Giné, G.R. Grimmett, and L. Saloff-Coste (Eds.), Lectures on Probability Theory and Statistics, Lecture Notes in Mathematics 1665, Springer, Berlin, 1997.

[25] L. Saloff-Coste, J. Zúñiga, *Merging and stability for time inhomogeneous finite Markov chains.* In: J. Blath, P. Imkeller, and S. Rœlly (Eds.), Surveys in Stochastic Processes, 127−151, EMS Ser. Congr. Rep. Eur. Math. Soc., Zürich, 2011.

[26] E. Seneta, *Non-negative Matrices and Markov Chains*, 2nd Edition. Springer-Verlag, Berlin, 1981; revised printing, 2006.

[27] S. Sundaram, C.N. Hadjicostis, *Finite-time distributed consensus in graphs with time-invariant topologies.* 2007 American Control Conference, New York, NY, USA, 2007, pp. 711−716.

[28] D.D. Šiljak, *Large-Scale Dynamic Systems*: *Stability and Structure.* North-Holland, New York, 1978.

[29] W. Xia, J. Liu, M. Cao, K.H. Johansson, T. Başar, *Generalized Sarymsakov matrices.* IEEE Trans. Automat. Control **64** (2019), 3085−3100.

[30] L. Xiao, S. Boyd, *Fast linear iterations for distributed averaging.* Systems Control Lett. **53** (2004) 65−78.